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SCHWINGER-DYSON EQUATIONS AND CURRENTS IN LATTICE GAUGE THEORY*

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ABSTRACT

After introducing appropriate derivatives, the structure of Schwinger-Dyson equations, currents and Ward-Takahashi identities (including the anomalous ones) on a finite lattice is completely clarified. A general relation between correlation functions without and with gauge fixing is given.

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The recent interest [1-4] in lattice Schwinger-Dyson equations concentrated on their relation to string equations. On the other hand, if suitably and generally derived for gauge theories, they could be a valuable tool in the nonperturbative analysis. In fact, the numerical calculations [5] of general features of QCD suggest to exploit the well-defined framework with a minimum of ingredients, consisting of a finite lattice with correlation functions defined by integrals [6], for analytical calculations too. An encouraging example within this respect is the general demonstration [7] of the interconnection between fermion-degeneracy regularization and axial anomaly.

Fermions have been included in lattice Schwinger-Dyson equations by Weingarten [3], who interprets the arising contributions in a string picture. From the present point of view one of these contributions must correspond to the current related to the equation of motion. Thus a detailed derivation can reveal its form. On the other hand, this current, as well as other ones, can be derived by transforming the integration variables, which leads to Ward-Takahashi identities. Thus there is the opportunity to find the precise forms of the mentioned relations and quantities and to check their consistency. All this is important for the envisaged calculations on a finite lattice.

In the case of Ward-Takahashi identities with eliminated currents, to make contact to usual continuum forms, gauge fixing is to be studied. This can be done in the present formulation in an unambiguous way.

In the present letter, gauge theory with fermions is considered on a lattice in four dimensions. To handle the non-Abelian fields, left and right derivatives are introduced, allowing to exploit the invariance of the integrations. Then Schwinger-Dyson equations and Ward-Takahashi identities (including the anomalous ones) are derived and seen to get a clear and workable form. The structure of these relations is richer than, but close to, that of the respective continuum ones. The currents are found to be related to links and to have a form which necessarily leads to the point splitting as introduced in continuum theory long ago [8,9]. Finally a general relation between correlation functions without and with gauge fixing is given, which shows that the choice of the gauge fixing function on a lattice needs some care.

The finite lattice to be used has 16 $N_1N_2N_3N_4$ sites in 4-dimensional Euclidean space. Periodicity for $n_\lambda \to n_\lambda + 2N_\lambda$ in the numbering of the variables is imposed as "boundary condition". The action is

$$S = v \sum_{n',n} \bar{\psi}_{n'} (\not{D} - R + M)_{n'n} \psi_n$$

$$+ v \sum_{n,\sigma,\lambda} \mathrm{Tr}(1 - U^\dagger_{\sigma n} U^\dagger_{\lambda,n+\sigma} U_{\sigma,n+\lambda} U_{\lambda n}) \,/\, (g a_\sigma a_\lambda)^2 \;, \qquad (1)$$

where $v = a_1a_2a_3a_4$, $\not{D} = \sum_\lambda \gamma_\lambda D_\lambda$, $R = -i\gamma_5 \sum_\lambda W_\lambda$ and $M_{n'n} = \delta_{n'n} m$, with D_λ and W_λ given by

$$D_{\lambda n'n} = (U^\dagger_{\lambda n'} \delta_{n'+\lambda,n} - U_{\lambda n} \delta_{n',n+\lambda}) \,/\, (2a_\lambda), \qquad (2)$$

$$W_{\lambda n'n} = (U^\dagger_{\lambda n'} \delta_{n'+\lambda,n} + U_{\lambda n} \delta_{n',n+\lambda} - 2\delta_{n'n}) \,/\, (2a_\lambda). \qquad (3)$$

R is the fermion-degeneracy regularization of Osterwalder and Seiler [10], which differs from the one of Wilson [11], $W = \sum_\lambda W_\lambda$, by a factor $-i\gamma_5$. R gives a simpler lattice fermion propagator and enables these authors to construct a Hilbert space with positive metric. R and

W, or more generally cR and cW, where c is a constant subject to $c \neq 0$, all give the same limit. In perturbation theory the crucial point is to ensure the correct limit for fermion loops. A nonperturbative criterion is that the triangle anomaly term [12] must arise correctly. This term turns out [7] to have a form which may be viewed as a generalization of the representation [8,13] $\mu \mathrm{tr}(\gamma_5 G_c)$, in which G_c is the continuum fermion propagator and μ its mass, the rôle of μ now being taken by the regularizations indicated above. This mechanism imposes no conditions on the coupling constant.

In (1) - (3) one further has $U_{\lambda n} = \exp\{iB_{\lambda n}\}$ with $B_{\lambda n} = \sum_\ell T^\ell B^\ell_{\lambda n}$ and the normalization $\mathrm{Tr}(T^\ell T^j) = \frac{1}{2}\,\delta_{\ell j}$. In the Abelian case the operation Tr in (1) is to be replaced by a factor ½ to conform with usual conventions. To make contact with continuum theory one has to put $B_{\lambda n} = g a_\lambda A_{\lambda n}$. A general correlation function is defined by

$$\langle P\rangle = \int e^{-S} P \Big/ \int e^{-S} \quad , \tag{4}$$

where $\int$ means $\int_U \int_\psi$, with $\int_\psi$ standing for the Grassmann-variable integrations $\prod_{n,\beta} \int d\,\psi_{n\beta} d\bar{\psi}_{n\beta}$ and $\int_U$ similarly for the invariant integrations over the gauge group.

For fermions left derivatives $\partial/\partial\psi_{n\beta}$ are abbreviated by $\vec{\partial}^{\psi}_{n\beta}$ and right derivatives by $\overleftarrow{\partial}^{\psi}_{n\beta}$. By using the general property

$$\int_\psi \vec{\partial}^{\psi}_{n\beta} Q = 0 \quad , \quad \int_\psi Q \overleftarrow{\partial}^{\psi}_{n\beta} = 0 \ , \tag{5}$$

with $Q = -e^{-S} P$ one obtains

$$-\int_\psi e^{-S}\Big((\vec\partial^\psi_{n\beta}S)P - \vec\partial^\psi_{n\beta}P\Big) = 0, \quad \int_\psi e^{-S}\Big(P(S\overleftarrow\partial^\psi_{n\beta}) - (P\overleftarrow\partial^\psi_{n\beta})\Big) = 0, \tag{6}$$

which by (4) gives Schwinger-Dyson equations. Analogous equations follow for $\vec\partial^{\bar\psi}_{n\beta}$ and $\overleftarrow\partial^{\bar\psi}_{n\beta}$. Since by (1) one has $\vec\partial^\psi_{n\beta}S = -S\overleftarrow\partial^\psi_{n\beta}$, and because from P only odd Grassmann elements contribute to the integrals, the second equation in (6) actually contains nothing new as compared to the first one.

To obtain similar relations for the gauge field, for a function Q of the gauge field variables, derivatives with respect to one of these are introduced by

$$\vec\partial^U_{\ell\lambda n}Q = \lim_{\varepsilon\to 0}\Big(Q(\ldots,\exp\{iT^\ell\varepsilon\}U_{\lambda n},\ldots) - Q(\ldots,U_{\lambda n},\ldots)\Big)/\varepsilon\ , \tag{7}$$

$$Q\overleftarrow\partial^U_{\ell\lambda n} = \lim_{\varepsilon\to 0}\Big(Q(\ldots,U_{\lambda n}\exp\{iT^\ell\varepsilon\},\ldots) - Q(\ldots,U_{\lambda n},\ldots)\Big)/\varepsilon\ . \tag{8}$$

From the invariance of the gauge group integrations, the property

$$\int_U \vec\partial^U_{\ell\sigma n}Q = 0 \quad , \quad \int_U Q\overleftarrow\partial^U_{\ell\sigma n} = 0 \tag{9}$$

follows. Inserting $Q = -e^{-S}P$ into (9) one gets

$$\int_U e^{-S}\Big((\vec\partial^U_{\ell\sigma n}S)P - \vec\partial^U_{\ell\sigma n}P\Big) = 0, \quad \int_U e^{-S}\Big(P(S\overleftarrow\partial^U_{\ell\sigma n}) - (P\overleftarrow\partial^U_{\ell\sigma n})\Big) = 0, \tag{10}$$

which by (4) again gives Schwinger-Dyson equations. In contrast to (6), in (10) the two equations have different content. In fact, evaluating the derivatives one obtains

$$\frac{1}{v}\overrightarrow{\partial}^{U}_{\ell\sigma n}S = \sum_{\lambda}\left(\mathscr{F}^{[4]\ell}_{\sigma\lambda,n} - \mathscr{F}^{[3]\ell}_{\sigma\lambda,n-\lambda}\right)/\,(ga_{\sigma}a_{\lambda})^{2} - \left(\overrightarrow{J}^{\ell}_{\sigma n} - \overrightarrow{j}^{\ell}_{\sigma n}\right)/\,a_{\sigma}, \tag{11}$$

$$\frac{1}{v}S\overleftarrow{\partial}^{U}_{\ell\sigma n} = \sum_{\lambda}\left(\mathscr{F}^{[1]\ell}_{\sigma\lambda,n} - \mathscr{F}^{[2]\ell}_{\sigma\lambda,n-\lambda}\right)/\,(ga_{\sigma}a_{\lambda})^{2} - \left(\overleftarrow{J}^{\ell}_{\sigma n} - \overleftarrow{j}^{\ell}_{\sigma n}\right)/\,a_{\sigma}, \tag{12}$$

with

$$\overrightarrow{J}^{\ell}_{\sigma n} = \frac{i}{2}\left(\bar{\psi}_{n}\gamma_{\sigma}U^{\dagger}_{\sigma n}T^{\ell}\psi_{n+\sigma} + \bar{\psi}_{n+\sigma}\gamma_{\sigma}T^{\ell}U_{\sigma n}\psi_{n}\right) \quad , \tag{13}$$

$$\overrightarrow{j}^{\ell}_{\sigma n} = \frac{1}{2}\left(\bar{\psi}_{n}\gamma_{5}U^{\dagger}_{\sigma n}T^{\ell}\psi_{n+\sigma} - \bar{\psi}_{n+\sigma}\gamma_{5}T^{\ell}U_{\sigma n}\psi_{n}\right) \quad , \tag{14}$$

and with $\overleftarrow{J}^{\ell}_{\sigma n}$ and $\overleftarrow{j}_{\sigma n}$ differing from (13) and (14), respectively, by having T^{ℓ} on the other side of $U_{\sigma n}$ and of $U^{\dagger}_{\sigma n}$. The $\mathscr{F}^{[\alpha]\ell}_{\sigma\lambda,n} = 2\mathrm{Tr}\left(T^{\ell}\mathscr{F}^{[\alpha]}_{\sigma\lambda,n}\right)$ are given by

$$\mathscr{F}^{[\alpha]}_{\sigma\lambda,n} = \left(\omega^{\dagger}_{p(\alpha)} - \omega_{p(\alpha)}\right)/\,(2i) \tag{15}$$

where $\omega_{n} = U^{\dagger}_{\sigma n}U^{\dagger}_{\lambda,n+\sigma}U_{\sigma,n+\lambda}U_{\lambda n}$ is the product starting with $U_{\lambda n}$ at point $p(1) = n$, and the other $\omega_{p(\alpha)}$ its cyclic permutations starting from $p(2) = n + \lambda$, $p(3) = n + \lambda + \sigma$, $p(4) = n + \sigma$, i.e., from the other corners around the plaquette. In the continuum limit $\mathscr{F}^{[\alpha]}_{\sigma\lambda,n}/(ga_{\sigma}a_{\lambda})$ for all α tends to $F_{\sigma\lambda}(x)$, $\overrightarrow{J}^{\ell}_{\sigma n}$ and $\overleftarrow{J}^{\ell}_{\sigma n}$ to $J^{\ell}(x)$, and $\overrightarrow{j}^{\ell}_{\sigma n}$ and $\overleftarrow{j}^{\ell}_{\sigma n}$ (the remainders from R) to zero, and one gets the usual classical equations. In the quantum case the limit needs some care, as can be seen, for example, in the context of the anomaly [7].

The products of gauge field factors around plaquettes in $\overrightarrow{\partial}^{U}_{\ell\sigma n}S$ and $S\overleftarrow{\partial}^{U}_{\ell\sigma n}$ have the link from n to n + σ in common and start from n + σ and n, respectively. Thus (10) can be immediately specialized to a Wilson loop by choosing $P = 2\mathrm{Tr}(T_\ell L_{n+\sigma})$ and $P = 2\mathrm{Tr}(T_\ell L_n)$, respectively, where L_n is a product of gauge field factors along a closed loop with starting point n. Then summing (10) over all ℓ, deformations of the loop show up in terms as, for example,

$$\sum_\ell \mathscr{F}^{[4]\ell}_{\sigma\lambda,n} P(\ell) = 2\mathrm{Tr}\left(\mathscr{F}^{[4]}_{\sigma\lambda,n} L_{n+\sigma}\right) \quad . \tag{16}$$

Further special cases are readily obtained by appropriate choices of P in (6) and (10). In addition to (6) and (10), Schwinger-Dyson equations involving repeated application of the derivatives can straightforwardly be derived.

In the Abelian case, where $\int_U$ becomes simply $\prod_{n,\lambda} \int_{-\pi}^{\pi} \frac{dB_{\lambda n}}{2\pi}$, the derivatives $\vec{\partial}^U_{\ell\sigma n}$ and $\overleftarrow{\partial}^U_{\ell\sigma n}$ in (9) and (10) are replaced by $\partial/\partial B_{\sigma n}$, and one has to be aware of the requirement $P(B) = P(B + 2\pi)$. Instead of (11) and (12) now $\frac{1}{v}\frac{\partial S}{\partial B_{\sigma n}} = \sum_\lambda (\sin f_{\sigma\lambda,n} - \sin f_{\sigma\lambda,n-\lambda})/(g a_\sigma a_\lambda)^2 - (J_{\sigma n} - j_{\sigma n})/a_\sigma$ occurs, where $f_{\sigma\lambda,n} = B_{\sigma n} + B_{\lambda,n+\sigma} - B_{\sigma,n+\lambda} - B_{\lambda n}$, and $J_{\sigma n}$ and $j_{\sigma n}$ are of form (13) and (14), respectively, with T^ℓ replaced by 1.

Next, by deriving Ward-Takahashi identities, currents are obtained in a more general way. First the transformation of variables $\psi'_n = V_n\psi_n$, $\bar{\psi}'_n = \bar{\psi}_n V^\dagger_n$ with $V_n = \exp\left\{i\sum_\ell T^\ell \alpha^\ell_n\right\}$ is performed in $\int_\psi e^{-S} IP$, in which for later convenience a factor I, invariant under the transformation, is included. Now derivatives $\vec{\partial}^V_{\ell n}$ and $\overleftarrow{\partial}^V_{\ell n}$ are introduced, which are related to V_n and α^ℓ_n in the same way as (7) and (8) are to $U_{\lambda n}$ and $B^\ell_{\lambda n}$. Then, by applying $\overleftarrow{\partial}^V_{\ell n}$ to $\int_{\psi'} e^{-S} IP$, one gets

$$\int_{\psi'} e^{-S} I\left(P(S\overleftarrow{\partial}^V_{\ell n}) - (P\overleftarrow{\partial}^V_{\ell n})\right) = 0. \tag{17}$$

By calculating the derivatives (the usual chain rules for Grassmann variables extend to $\overleftarrow{\partial}^{V}_{\ell n}$), and then going back to the variables ψ_n, $\bar{\psi}_n$, (17) gets the form

$$\int_\psi e^{-S_I}\Big[P\Big(\sum_\lambda(\overleftarrow{J}^{\ell}_{\lambda n} - \overrightarrow{J}^{\ell}_{\lambda,n-\lambda})/a_\lambda - \sum_\lambda(\overleftarrow{j}^{\ell}_{\lambda n} - \overrightarrow{j}^{\ell}_{\lambda,n-\lambda})/a_\lambda\Big)$$
$$- \frac{i}{v}\sum_\beta\Big((\overrightarrow{\partial}^{\psi}_{n\beta}P)(T^\ell\psi_n)_\beta + (\bar{\psi}_n T^\ell)_\beta\overrightarrow{\partial}^{\bar{\psi}}_{n\beta}P\Big)\Big] = 0. \tag{18}$$

With (18) one has the Ward-Takahashi identities for the currents introduced before (proceeding with $\overrightarrow{\partial}^{V}_{\ell n}$ instead of $\overleftarrow{\partial}^{V}_{\ell n}$ leads to (18) with T^ℓ replaced by $V^\dagger_n T^\ell V_n$, i.e., to nothing new).

The corresponding relations for the singlet current follow by using the transformation $\psi'_n = \exp\{i\alpha_n\}\psi_n$, $\bar{\psi}'_n = \bar{\psi}_n\exp\{-i\alpha_n\}$, in which case one has (17) with $\partial/\partial\alpha_n$ instead of $\overleftarrow{\partial}^{V}_{\ell n}$ and gets (18) with T^ℓ replaced by 1, for $J_{\sigma n}$ and $j_{\sigma n}$ given by (13) and (14) without T^ℓ.

The identities for the axial currents are similarly obtained. The transformation $\psi'_n = \exp\{i\gamma_5\sum_\ell T^\ell\alpha^\ell_n\}\psi_n$, $\bar{\psi}'_n = \bar{\psi}_n\exp\{i\gamma_5\sum_\ell T^\ell\alpha^\ell_n\}$, leads to

$$\int_\psi e^{-S_I}\Big[P\Big(\sum_\lambda (\overleftarrow{J}^{5\ell}_{\lambda n} - \overrightarrow{J}^{5\ell}_{\lambda,n-\lambda})/a_\lambda - 2im\bar{\psi}_n\gamma_5 T^\ell\psi_n\Big) \tag{19}$$
$$- \frac{i}{v}\sum_{n',\beta}\Big((\overrightarrow{\partial}^{\psi}_{n'\beta}P)(\delta_{n'n}\gamma_5 T^\ell\psi_n)_\beta - (\bar{\psi}_n\gamma_5 T^\ell\delta_{nn'})_\beta\overrightarrow{\partial}^{\bar{\psi}}_{n'\beta}P\Big)\Big] + X^\ell_n = 0,$$

where

$$\overrightarrow{J}^{5\ell}_{\lambda n} = \frac{i}{2}(\bar{\psi}_n\gamma_\lambda\gamma_5 U^\dagger_{\lambda n}T^\ell\psi_{n+\lambda} + \bar{\psi}_{n+\lambda}\gamma_\lambda\gamma_5 T^\ell U_{\lambda n}\psi_n) \tag{20}$$

and $\overleftarrow{J}^{5\ell}_{\lambda n}$ as (20) with T^ℓ on the other side of $U^\dagger_{\lambda n}$ and of $U_{\lambda n}$. Now the term

$$X^\ell_n = i\int_\psi e^{-S_I}P\sum_{n'}(\bar{\psi}_{n'}R_{n'n}\gamma_5 T^\ell\psi_n + \bar{\psi}_n\gamma_5 T^\ell R_{nn'}\psi_{n'}) \tag{21}$$

occurs, which contains the full R and therefore requires special care in the continuum limit [7]. By using (6) with $P\psi_n$ and $P\bar{\psi}_n$ inserted for P there and defining $G = (\not{D} - R + M)^{-1}$, (21) can be cast into the form

$$X_n^{\ell} = -\frac{i}{v}\mathrm{tr}\left(\gamma_5 T^{\ell}(GR + RG)_{nn}\right)\int_{\psi} e^{-S_I} P \qquad (22)$$

$$-\frac{i}{v}\int_{\psi} e^{-S_I}\sum_{n',\beta}\left((\vec{\partial}^{\psi}_{n'\beta}P)((GR)_{n'n}\gamma_5 T^{\ell}\psi_n)_{\beta} - (\bar{\psi}_n\gamma_5 T^{\ell}(RG)_{nn'})_{\beta}\,\vec{\partial}^{\bar{\psi}}_{n'\beta}P\right)$$

where tr refers to γ-matrices as well as to internal symmetry indices (while Tr applies only to the latter ones). Now (19) with (22) gives the Ward-Takahashi identities. Combining the terms with derivatives of P from (19) and (22), $\frac{1}{v}(\delta_{n'n} + (GR)_{n'n})$ and $\frac{1}{v}(\delta_{nn'} + (RG)_{nn'})$ occur which can be shown [7] to give, as well as $\frac{1}{v}\delta_{n'n}$ alone, $\delta^4(x' - x)$ in the continuum limit. For the trace term, adapting the demonstration for the singlet axial current [7], it follows that

$$\frac{1}{v}\mathrm{tr}\left(\gamma_5 T^{\ell}(GR + RG)_{nn}\right) \to \frac{g^2}{8\pi^2}\,\mathrm{Tr}T^{\ell}\sum_{\mu\nu} {}^*F_{\mu\nu}(x)F_{\mu\nu}(x) \qquad (23)$$

in the continuum limit. The r.h.s. of (23) is the usual continuum result [14].

For the singlet axial current the transformation $\psi'_n = \exp\{i\alpha_n\gamma_5\}\psi_n$, $\bar{\psi}'_n = \bar{\psi}_n\exp\{i\alpha_n\gamma_5\}$ is to be used, and one gets (19) with (22) where T^{ℓ} is replaced by 1, for $J^5_{\lambda n}$ given by (20) without T^{ℓ}. In (23) T^{ℓ} no longer occurs, and in the limit one has the Adler-Bell-Jackiw anomaly term [12].

The currents which have been obtained here on the basis of (1)-(3) are obviously related to the links of the lattice as the gauge fields are. Thus their nonlocality does not exceed the one already present due to the gauge fields. On the other hand, it has been known for a long time [8,9] that point-splitting forms of the currents are able to resolve some troublesome points in continuum theory. Therefore it appears rather satisfactory that the present lattice formulation provides such forms automatically. It seems then interesting to study, for example, current commutators along these lines; the details thereof, however, at the moment remain to be worked out.

Coming back to the transformation V_n it is to be noted that instead of the fermion variables one may transform the gauge field ones as $U'_{\lambda n} = V_{n+\lambda} U_{\lambda n} V^{\dagger}_n$, which gives a relation of form (17) with $\int_{\psi'}$ replaced by $\int_{U'}$. This can be evaluated by using $Q\overleftarrow{\partial}^{V}_{\ell n} = \sum_{\lambda} \left(Q\overleftarrow{\partial}^{U}_{\ell\lambda n} - \overrightarrow{\partial}^{U}_{\ell\lambda,n-\lambda} Q \right)$ which leads to

$$\int_U e^{-S_I} \Big[P \Big(- \sum_{\lambda} (\overleftarrow{J}^{\ell}_{\lambda n} - \overrightarrow{J}^{\ell}_{\lambda,n-\lambda})/a_{\lambda} + \sum_{\lambda} (\overleftarrow{j}^{\ell}_{\lambda n} - \overrightarrow{j}^{\ell}_{\lambda,n-\lambda})/a_{\lambda} \Big)$$

$$- \frac{1}{v} \sum_{\lambda} (P\overleftarrow{\partial}^{U}_{\ell\lambda n} - \overrightarrow{\partial}^{U}_{\ell\lambda,n-\lambda} P) \Big] = 0 \quad . \tag{24}$$

Hereby it has been used that $\mathscr{F}^{[\alpha']}_{\sigma\lambda,n} = - \mathscr{F}^{[\alpha]}_{\lambda\sigma,n}$ for $\alpha' = \alpha = 1$, for $\alpha' = \alpha = 3$, and for $\alpha' = 2$, $\alpha = 4$.

The combined transformation of ψ_n, $\bar{\psi}_n$ and $U_{\lambda n}$ under V_n, i.e., the gauge transformation, gives the Ward-Takahashi identities with eliminated currents

$$\int e^{-S_I} \Big[\frac{1}{v} \sum_{\lambda} (P\overleftarrow{\partial}^{U}_{\ell\lambda n} - \overrightarrow{\partial}^{U}_{\ell\lambda,n-\lambda} P)$$

$$+ \frac{i}{v} \sum_{\beta} \Big((\overrightarrow{\partial}^{\psi}_{n\beta} P)(T^{\ell}\psi_n)_{\beta} + (\bar{\psi}_n T^{\ell})_{\beta} \overrightarrow{\partial}^{\bar{\psi}}_{n\beta} P \Big) \Big] = 0. \tag{25}$$

To make contact to its usual continuum forms [15] gauge fixing is to be introduced.

The lattice counterpart of the usual gauge fixing procedure [16] can be derived in a general and well-defined way. Denoting the group integrations over the transformations by $\int_V$, for a given gauge fixing function f the invariant function ϕ is defined by $\phi(U)\int_V f(U') = 1$, from which it follows that $\int_U e^{-S} = \int_U e^{-S}\phi\int_V f(U') = \int_V\int_U e^{-S}\phi f(U)$. Then correlation functions with gauge fixing are defined by

$$\langle P\rangle_f = \int e^{-S}\phi f P \Big/ \int e^{-S}\phi f \quad . \tag{26}$$

Conversely one obtains $\langle P\rangle_f = \int_V\int e^{-S}\phi f P \Big/ \int_V\int e^{-S}\phi f =$ $\int e^{-S}\phi\int_V f(U')P(U',\psi',\bar{\psi}')\Big/\int e^{-S}$, i.e., the general relation between correlation functions with and without gauge fixing

$$\langle P\rangle_f = \left\langle\int_V f(U')P(U',\psi',\bar{\psi}')\Big/\int_V f(U')\right\rangle \quad . \tag{27}$$

According to (27) the effect of gauge fixing is to provide by f suitable factors for P, such that gauge-invariant terms arise which lead to non-trivial correlation functions.

Now, choosing $I = \phi$ and replacing P by fP in (25) by (26) gives the Ward-Takahashi identities with gauge fixing

$$\Big\langle f^{-1}\,\frac{1}{v}\sum_\lambda\Big(\big((fP)\overleftarrow{\partial}^U_{\ell\lambda n}\big) - \overrightarrow{\partial}^U_{\ell\lambda,n-\lambda}(fP)\Big)\Big\rangle_f$$

$$+\Big\langle\frac{i}{v}\sum_\beta\Big((\overrightarrow{\partial}^\psi_{n\beta}P)(T^\ell\psi_n)_\beta + (\bar{\psi}_n T^\ell)_\beta\overrightarrow{\partial}^{\bar{\psi}}_{n\beta}P\Big)\Big\rangle_f = 0 \quad . \tag{28}$$

It is, however, to be noted that according to (27) f is to be chosen with care. To illustrate this, the Abelian case with $P = \psi_{n'\beta'}\bar{\psi}_{n''\beta''}$ may be considered, where the first term in (28) becomes $\langle \frac{1}{v} \sum_\lambda \left(\frac{\partial \ell n f}{\partial B_{\lambda n}} - \frac{\partial \ell n f}{\partial B_{\lambda, n-\lambda}} \right) \psi_{n'\beta'} \bar{\psi}_{n''\beta''} \rangle_f$ (which corresponds to the divergence of the fermion-untruncated vertex function). Then looking for the lattice analogue of $\exp\left\{ -\frac{1}{2\alpha} \int d^4x \sum_\mu (\partial_\mu A_\mu)^2 \right\}$ it is seen that the naive choice [17] $f = \prod_n \exp\left\{ -(2\alpha g^2)^{-1} \sin^2 \left(\sum_\lambda (B_{\lambda n} - B_{\lambda, n-\lambda}) \right) \right\}$, because of its prescribed combinations of the $B_{\lambda n}$, does not allow to form gauge-invariant combinations as are necessary by (27) to avoid identically vanishing correlation functions. A choice with the same continuum limit, giving suitable contributions in (27), is

$$f = \exp\left\{ -(2\alpha g^2)^{-1} \left(\sum_{n,\lambda} 4(1-\cos B_{\lambda n}) + \sum_{n', \lambda', n, \lambda} \sin B_{\lambda' n'} \sin B_{\lambda n} \times \left((1-\delta_{\lambda' \lambda})(\delta_{n'n} + \delta_{n', n-\lambda'+\lambda}) - 2\delta_{n', n+\lambda} \right) \right) \right\} .$$

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