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CONSISTENT THREE PARTICLE EQUATIONS USING ONLY TWO PARTICLE OBSERVABLES*

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ABSTRACT

It is shown that the zero range limit of the Karlsson-Zeiger equations uniquely define unitary three particle amplitudes using only the observable phase shifts and binding energies of the two particle subsystems.

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If we understood the forces between two hadrons, we should at least be able to use this knowledge to calculate the behavior of three hadron systems. We know that currently we lack this much understanding. For example, the nucleon-nucleon scattering amplitudes are known up to, and in some cases well beyond, pion production threshold; but from this knowledge we cannot predict the binding energy of the triton or He^3, or their electromagnetic form factors. Even the n-p capture cross section at threshold differs by 10% from the model-independent[1] Bethe-Longmire prediction. The reason is, as we learned long ago from Wick,[2] that the coupling of the uncertainty principle to special relativity entails the creation of mesonic degrees of freedom at short distance. Nuclear physicists usually assume that these hidden degrees of freedom can be approximated by a "potential," but there is no unique way to define such a potential once the short range non-locality implied by the Wick-Yukawa mechanism is taken seriously.

Faced with this ambiguity, it is important to have clear experimental criteria for determining what new information is contained in three hadron observables which is not already predictable using two hadron observables. Starting from the "Fixed Past-Uncertain Future" interpretation of quantum mechanics,[3] it has been proposed that such a reference theory might be provided by calculating three particle amplitudes using only two particle on shell scatterings.[4] Once a way of doing this has been developed, the theoretically ambiguous mixture of mesonic effects -- designated but not defined by the terms "off-shell effects" and "three-body forces" -- could be uniquely parameterized by adding an on-shell three particle direct scattering term to the model.

A specific attempt to articulate this program[5] failed because it could not be proved unitary. Recently, it has been shown[6] that the zero range limit of the Karlsson-Zeiger equations[7] define a unique three particle theory of the type sought. Most requisite physical properties were established, including time reversal invariance, but not unitarity of the three particle amplitude. In this communication we remove this defect.

It has been shown[8] that the half off-shell two particle transition amplitude $t(q;\tilde{k}^2+i0^+) = t^+(\tilde{k}^2)\left[1+R^2(\tilde{q}^2-\tilde{k}^2)f_{k^2}(q)\right]$ where $t^{\pm}(\tilde{k}^2) = -e^{\pm i\delta_k}\sin\delta_k/\pi\mu k$ is the on-shell amplitude, μ the reduced mass, and the function f is the transformation to momentum space of the difference between the asymptotic wave function $e^{i\delta}(\sin ky+\delta)/k$ and the wave function inside the range of forces R; hence the "zero range limit" can be defined by taking $f=0$, which is equivalent to the zero range boundary condition $u'(y)/u(y)\Big|_{y=0^+} = k\,\mathrm{ctn}\delta_k$ in the two particle space. Because of this factorization, we find[6] that in the zero range limit the four KZ amplitudes can be written in terms of a single amplitude $F_{ab}(p_a,q_a;p_b',q_b';z) = t_a^+(\tilde{q}_a^2)Z_{ab}(p_a,p_b';z)t_b^+(\tilde{q}_b'^2)$, the elastic scattering or rearrangement amplitudes, breakup amplitude, and coalesence amplitude being simply the residues of the appropriate double or single poles in t. The function Z_{ab} is defined by the coupled integral equations

$$Z_{ab}(p_a,p_b;z) = -\bar{\delta}_{ab}\frac{1}{2}\int_{-1}^{1}d\xi\,\frac{1}{\tilde{p}_a^2+\tilde{q}_{ab}^{(2)2}-z}$$

$$-\sum_{c=a\pm}\bar{\delta}_{ac}\frac{1}{2}\int_{-1}^{1}d\xi\int_0^\infty p_c''^2\,dp_c''\,\frac{N_c^2 Z_{cb}(p_c'',p_b';z)}{\left(\tilde{q}_{ac}^{(2)2}+\tilde{K}_c^2\right)\left(\tilde{p}_c''^2-\tilde{K}_c^2-z\right)} \tag{1}$$

$$-\sum_{c=a\pm}\bar{\delta}_{ac}\frac{1}{2}\int_{-1}^{1}d\xi\int_0^\infty p_c''^2\,dp_c''\int_0^\infty q_c''^2\,dq_c''\,\frac{t_c^+(\tilde{q}_c''^2)\psi^-_{q_c''^2}(q_{ac}^{(2)})Z_{cb}(p_c'',p_b;z)}{\tilde{p}_c''^2+\tilde{q}_c''^2-z}$$

where

$$\psi^\pm_{k^2}(q) = \frac{\delta(q-k)}{qk} - \frac{t^\pm(\tilde{k}^2)}{\tilde{q}^2-\hat{k}^2\mp i\eta} = e^{\pm i\delta_k}\left[\cos\delta_k\,\frac{\delta(k-q)}{kq}+\frac{2\mathscr{P}}{\pi}\,\frac{\sin\delta_k}{q^2-k^2}\right] \tag{2}$$

and

$$q_{ac}^{(2)}(\xi)^2 = p_a^2+\left(\frac{m_a}{m_a+m_{a\mp}}\,p_{a\pm}\right)^2+\frac{2m_a p_a p_{a\pm}}{m_a+m_{a\mp}}\,\xi \quad . \tag{3}$$

Here we have restricted ourselves to the case when all angular momenta vanish ($\ell=0=L=0=\lambda$) since the generalization is obvious to three particle experts and uninteresting to others. Time reversal invariance follows immediately from the fact[7] that $\tilde{p}_a^2+\tilde{q}_{ab}^{(2)2} = \tilde{p}_b^2+\tilde{q}_{ab}^{(1)2}$, as previously explained.[6]

In order to demonstrate that the F_{ab} so defined is in fact unitary as well as time reversal invariant, we first use the algebraic identity $(1/xy) = [(1/x)-(1/y)]/(y-x)$ and Eq. (2) to rewrite the kernel as

$$\frac{1}{\tilde{p}_c^{\prime 2} + \tilde{q}_{ac}^{(2)2} - z}\left[t_c^{+}\left(\tilde{q}_{ac}^{(2)2}\right) + \int_0^\infty \frac{k^2|t_c(k^2)|^2 dk}{\tilde{k}^2 - \tilde{q}_{ac}^{(2)2} - i\eta} + \frac{N_c^2}{\tilde{q}_{ac}^{(2)2} + \tilde{K}_c^2}\right.$$

$$\left. - \int_0^\infty \frac{k^2|t_c(k^2)|^2 dk}{\tilde{p}^2 + \tilde{k}^2 - z} + \frac{N_c^2}{\tilde{p}_c^{\prime\prime 2} - \tilde{K}_c^2 - z}\right] \tag{4}$$

and by using the standard dispersion-theoretic representation for the on-shell two particle amplitude t

$$t^{\pm}(\tilde{q}^2) = -\int_0^\infty \frac{k^2|t(k^2)|dk}{k^2 - q^2 \mp i\eta} - \frac{N^2}{\tilde{q}^2 + \tilde{K}^2} + \int_{m_x^2}^\infty \frac{\rho(\sigma^2)d\sigma^2}{\tilde{\sigma}^2 + \tilde{q}^2}$$

$$\equiv \hat{t}^{\pm}(\tilde{q}^2) + \int_{m_x^2}^\infty \frac{\rho(\sigma^2)d\sigma^2}{\tilde{\sigma}^2 + \tilde{q}^2} \tag{5}$$

find that Eq. (1) can be rewritten as

$$Z_{ab}\left(p_a, p_b'; z\right) = -\bar{\delta}_{ab}\frac{1}{2}\int_{-1}^{1} d\xi \frac{1}{\tilde{p}_a^2 + \tilde{q}_{ab}^{(2)2}(\xi) - z}$$

$$-\sum_c \bar{\delta}_{ab}\int_0^\infty p_c^{\prime\prime 2}\, dp_c^{\prime\prime}\int_0^\infty q_c^{\prime\prime 2}\, dq_c^{\prime\prime}\frac{Z_{cb}\left(p_c^{\prime\prime}, p_b^{\prime\prime}; z\right)}{\tilde{p}_c^{\prime\prime\prime} + \tilde{q}_c^{\prime\prime 2} - z}$$

$$\times\frac{1}{2}\int_{-1}^{1} d\xi\frac{\delta\left(q_c^{\prime\prime} - q_{ac}^{(2)}\right)}{q_c^{\prime\prime}q_{ac}^{(2)}}\left[\hat{t}\left(z - \tilde{p}_c^{\prime\prime 2}\right) + \int_{m_x^2}^\infty\frac{\rho(\sigma^2)d\sigma^2}{\tilde{\sigma}^2 + \tilde{q}_c^{\prime\prime 2}}\right] \tag{6}$$

or more symbolically

$$Z_{ab} = -\bar{\delta}_{ab}G_0 - \sum_c \bar{\delta}_{ac}V_{ac}G_0\left[t_c\right]Z_{cb}$$

$$= -\bar{\delta}_{ab}G_0 - \sum_c Z_{ac}\left[t_c\right]G_0 V_{cb}\,\delta_{cb} \tag{7}$$

where the second line of Eq. (7) expresses the previously established time reversal invariance.

To complete the unitarity proof we first need to note that on-shell the KZ amplitude F_{ab} is related to the Faddeev amplitude M_{ab} by

$$M_{ab} = t_a \delta_{ab} + t_a Z_{ab} t_b \qquad , \tag{8}$$

and next that in establishing the unitarity relation for the amplitudes

$$M_{ab} - M^*_{ab} = - \sum_{cc'} M_{ac}\left(G_0 - G^*_0\right) V_{cc'} M^*_{c'b} \tag{9}$$

the energy-conserving δ-function $G_0 - G^*_0$ is proportional to $\delta(\tilde{p}^2 + \tilde{q}^2 - z)$. The final critical step is to note that under this restriction $\left[t_c\right]$ is the physical on-shell amplitude $t_c(z-\tilde{p}^2)$. Thus by substituting Eq. (8) into Eq. (9) and simplifying we find that the equation we need to prove is that

$$\begin{aligned} Z_{ab} - Z^*_{ab} = &- \sum_c Z_{ac} t_c \left(G_0 - G^*_0\right) Z^*_{ab} \\ &- \sum_{cc'} \bar{\delta}_{cc'} \left[\delta_{ac} + Z_{ac} t_c\right] \left(G_0 - G^*_0\right) V_{cc'} \left[\delta_{c'b} + t^*_c Z^*_{c'b}\right] \end{aligned} \tag{10}$$

The proof can now proceed by following the critical step in the Freedman-Lovelace-Namylowski proof[9] of the unitarity of the Faddeev equations, which is to use two particle on-shell unitarity to replace $t_c(G_0 - G^*_0) t^*_c$ by $-(t_c - t^*_c)$ in the diagonal term in Eq. (10). If we now invoke Eq. (7) in the appropriate order in the first two terms and simplify we find that Eq. (10) is equivalent to

$$0 = \sum_{c} Z_{ac} t_c \left[Z^*_{cb} + G_0 V_{cb} \bar{\delta}_{cb} - \left(G_0 - G^*_0 \right) V_{cb} \bar{\delta}_{cb} \right]$$

$$- \sum_{c'} \left[Z_{ac'} + \bar{\delta}_{ac'} V_{ac'} G^*_0 + \bar{\delta}_{ac'} \left(G_0 - G^*_0 \right) \right] t^*_{c'} Z_{c'b}$$

$$- \sum_{cc'} \bar{\delta}_{cc'} Z_{ac} t \left(G_0 - G^*_0 \right) V_{cc'} t^*_{c'} Z^*_{c'b} \tag{11}$$

which a second invocation of Eq. (7) readily verifies as an identity, completing the proof. The algebra is trivial, if tedious, and represents an explicit sequence of steps which have been carried through using the actual integral equations and the discrete contributions from the bound state poles, and independently checked.[10]

Although the proof has now been reduced to trivial algebra, and the rigorous consequences mentioned in the opening paragraphs now follow, some subtlties remain to be discussed. Had we simply taken the zero range limit in the Faddeev equations, the amplitude so defined would be proportional to $t(z-\tilde{p}^2)$, and in those circumstances where the left-hand cut $\rho(\sigma^2)$ is not zero would have a spurious singularity that does not occur in the original equations, a point emphasized by Lindsay Dodd.[11] We can see from Eq. (1) or Eq. (6) that the zero range limit of the KZ equations does not suffer from this disease. To locate the origin of this non-uniformity in the limit, we make use of the Low equation $t = V + VGV$ to express the fully off-shell amplitude $t(q,\bar{q};z-\tilde{p}^2)$ in terms of the half off-shell amplitudes using the spectral resolution theorem, and eliminate the potential V by invoking the time reversal invariance of t. This leads,[12] using the notation of this paper and some algebra, to the consistency condition

$$\int_{m_x^2}^{\infty} \frac{\rho(\sigma^2)d\sigma^2}{(\tilde{\sigma}^2+\tilde{q}^2)(\sigma^2+\tilde{\bar{q}}^2)} - R^2\left[f_{q^2}(q)\int_{m_x^2}^{\infty}\frac{\rho(\sigma^2)d\sigma^2}{\tilde{\sigma}^2+\tilde{q}^2} + f_{\bar{q}^2}(q)\int_{m_x^2}^{\infty}\frac{\rho(\sigma^2)d\sigma^2}{\tilde{\sigma}^2+\tilde{\bar{q}}^2}\right]$$

$$= R^2\int_0^{\infty} k^2|t(k^2)|^2 dk\left[\frac{f_{k^2}(\bar{q}) - f_{q^2}(\bar{q})}{\tilde{k}^2-\tilde{q}^2} + \frac{f_{k^2}(q) - f_{\bar{q}^2}(q)}{k^2-\tilde{\bar{q}}^2} + R^2 f_{k^2}(q) f_{k^2}(\bar{q})\right] \tag{12}$$

This is equivalent to the constraint studied by Baranger, Giraud, Mukhopadhyay and Sauer,[13] reduced to a non-singular form by invoking the on-shell dispersion relation given in Eq. (5). We see immediately that this relation cannot be maintained in the zero range limit in the presence of a left-hand cut. Thus we get into trouble by taking the zero range limit in the Faddeev equations in that case, but the KZ form allows the limit to be taken unambiguously, even when we come back as close as we can to the Faddeev form in Eq. (6). We conclude that we have found a route to a unique on-shell three particle theory, even though we cannot consistently define two particle off-shell amplitudes in a conventional way in a two particle space.

In order to actually solve these equations, it will be convenient to isolate the moving singularity in the kernel for $0 \le \tilde{p}^2 \le W$, and the coefficients of the primary singularities, from the non-singular parts which do not contribute to the asymptotic wave function. We do this by splitting Z into an exterior and interior piece by defining $Z = \theta(W-\tilde{p}^2)Z^E + \theta(\tilde{p}^2 - W)Z^I$. For Z^E we make a change of variable appropriate to the finite interval, e.g., $\tilde{p}^2 = W\sin^2\omega$, and expand in terms of an appropriate complete set, in this case $\sin 2n\omega/\sin 2\omega$. The logarithmic singularity in the kernel can then be integrated analytically, leaving a non-singular integral to be done to define the matrix coefficients in nn'; if we use empirical input for the two body observables, this last integral has to be done numerically in any case. Our splitting guarantees that the kernels for Z^I are non-singular. This leads to coupled equations of the form

$$Z_n^E = \sum_{n'} K_{nn'}^{EE} Z_{n'}^E + \sum_b K_{nb}^{EB} Z_b^E + \int_W^\infty K_n^{EI}(p') Z^I(p') \, d\tilde{p}'^2$$

$$Z_b^E = \sum_n K_{bn}^{BE} Z_n^E + \sum_{b'} K_{bb'}^{BB} Z_{b'}^E + \int_W^\infty K_b^{BI}(p') Z^I(p') \, d\tilde{p}'^2 \qquad (13)$$

$$Z^I(p) = \sum_n K_n^{IE}(p) Z_n^E + \sum_b K_b^{IB}(p) Z_b^E + \int_W^\infty K^{II}(p,p') Z^I(p') \, d\tilde{p}'^2$$

Here we exhibit the bound state indices b as a reminder that the value of Z at these singularities (elastic and rearrangement amplitudes) should be explicitly separated out, but have suppressed the Faddeev indices for simplicity. We see that for finite n the matrix for Z^E can be explicitly inverted and substituted into the continuum equation for Z^I making that equation also explicitly non-singular. This is one way to generalize a previous non-singular treatment of the two-body problem.[8] Since only the Z^E are physically observable in three particle systems, this two step process has the advantage that the solution for Z^I need only be good enough to guarantee the accuracy of the quadrature which occurs in the equations for Z^E. This is obviously a less stringent requirement than having to solve for the functional dependence on p.

Since these equations for Z are unique, there is no guarantee that they will agree with experiment. According to elementary particle theory, there will be additional effects generated by mesonic degrees of freedom at short distance. In conventional non-relativistic theories these are replaced by two-body off-shell effects arising from some assumed potential model and in some cases by three-body forces. Unfortunately there is no consensus as to how to separate these effects theoretically, and as has been pointed out[4] they are impossible to separate using only the two and three-body observables themselves. However, now that we have a <u>unique</u> reference theory based only on two particle observables, we do have a way of measuring the <u>combined</u> mesonic effects. One way to make this explicit is to introduce into the model a direct "zero range three particle" scattering amplitude. If this itself is unitary, like the two particle amplitudes, the equations remain unitary and provide via the parameters in this added amplitude a way to parameterize the discrepancy between the unique theory and experiment. Such a system need be inverted only once to obtain fitting formulae which can be used for data analysis, rather than requiring the solution of an integral equation each time a parameter is varied. Brayshaw[14] has demonstrated the practicality of this approach to data fitting in various relativistic situations. If we use relativistic kinematics our equations can be cast in covariant form and shown to be a special case of a general separable model given by Brayshaw,[14] as has been shown by Lindesay.[15] We have also found that minimal four particle equations can be obtained in a similar way, but will not pursue either of these applications of our approach here.

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