SLAC-PUB-2583
August 1980
(T/E)

QUANTUM CHROMODYNAMICS IN 2 + ε DIMENSIONS*

Michael Dine**, Christof Litwin†, and Larry McLerran††
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

ABSTRACT

We consider quantum chromodynamics without quarks, dimensionally continued to 2 + ε dimensions. Perturbation theory for this model is shown to be highly infrared singular, and the theory, order by order, is not smoothly connected to the two-dimensional theory. We show that with a certain selective resummation of diagrams, one can control these divergences, and obtain a structure very similar to that of the 't Hooft model for mesons. The model is then in fact equivalent to the large N limit of two-dimensional QCD with massless scalar particles in the fundamental representation. The two-gluon bound state equation which we obtain, however, is itself infrared singular and possesses no normalizable solutions. We discuss this problem in the context of an analog scalar model, and speculate on its possible resolution.

(Submitted to Physical Review D)

*Work supported by the Department of Energy under contract number DE-AC03-76SF00515.

**Address after September 1, 1980: Institute for Advanced Study, Princeton, New Jersey 08544.

†Work supported in part by a grant from the Danish Natural Science Research Council and a NATO fellowship. Address after September 1, 1980: Laboratory of Plasma Studies, Cornell University, Ithaca, New York 14853.

††Address after September 15, 1980: Physics Department, University of Washington, Seattle, Washington 98195.

I. INTRODUCTION

Quantum chromodynamics (QCD), the theory of quarks and gluons, is believed to completely describe the strong interactions.[1] A derivation of the properties of hadrons from QCD will require a detailed understanding of the interplay of quark and gluon dynamics. The pure Yang-Mills theory in the absence of quarks, however, is itself of great interest. Confinement of quarks, for example, is believed to arise from the properties of the pure gluon theory.[2] Moreover, the pure gluon theory should possess a perfectly sensible spectrum of states. In the real world, many of these states presumably mix strongly with quark-antiquark states, but some nearly pure "glueball" states may well exist in nature.[3]

The four-dimensional theory, even without quarks, is enormously complex. In this paper we consider a significantly simpler model, QCD, in the absence of fermions, dimensionally continued to $2 + \varepsilon$ dimensions, where ε is infinitesimal. This theory describes the interactions of colored gluons possessing ε degrees of freedom. While this model is obviously unrealistic, one might expect that its solution would exhibit many of the hoped-for attributes of the four-dimensional theory. For example, consider the problem of confinement. This problem can be studied by computing the energy of separation of external sources in the fundamental representation of the gauge group. In two dimensions this potential is linear, a fact that follows from dimensional analysis alone. In higher dimensions the situation is more complex. Before the gluon self-interaction is considered the potential is Coulombic: $V(r) \sim \ln(r)$ in $2 + 1$ dimensions and $V(r) \sim \frac{1}{r}$ in $3 + 1$ dimensions.

In four dimensions, it is widely believed that gluon interactions turn this Coulombic potential into a linear one, providing confinement of quarks. This is strongly suggested by work in lattice gauge theories.[2] In four or less dimensions, these theories presumably yield a linear potential between static sources.[4] These statements have recently received strong support from the Monte Carlo calculations of Creutz.[5] Polyakov has shown,[6] in the example of compact QED in 2 + 1 dimensions, that such a phenomenon can sometimes be observed in a continuum theory. Here the naive logarithmic potential was transformed by interactions into a linear one.

In $2 + \varepsilon$ dimensions, the leading term in the force law behaves as $E(r) \sim r^{1-\varepsilon}$. Because the theory possesses non-trivial interactions and is very sensitive to the infrared region, higher order effects could dramatically alter the force law (for example, giving higher powers of r). On the basis of our discussion above, one might expect that the force law which emerges should be strictly linear, $E(r) \sim r$. In this sense, one might hope to address the problem of confinement in an ε expansion.

The spectrum of this theory may also have features in common with the four-dimensional theory. In particular, one might expect that the spectrum consists of glueballs, color-singlet bound states of pure glue. The mass spectrum of these glueballs and the structure of their wave functions might be of interest in inferring the properties of the corresponding objects in the four-dimensional theory.

QCD in $2 + \varepsilon$ dimensions provides a simple model to study a number of other ideas which have been considered in the four-dimensional theories.

For example, the properties of the gluon propagator have been the subject of much study recently.[7] In particular, singular behavior of the gluon propagator in the infrared might be responsible for confinement. Paradoxes arise, however, when spectral decompositions are applied to the propagator. In $2 + \varepsilon$ dimensions, the gluon propagator has singular infrared behavior ab initio and some of these questions may be addressed in low orders of perturbation theory.

One might also hope to study the structure of the vacuum, and determine whether this structure plays a decisive role in the dynamics of the theory. Since in two dimensions the vacuum is trivial and contains no quanta, it may be possible to study the properties of the vacuum in an ε expansion. In particular the vacuum in $2 + \varepsilon$ dimensions might only mix weakly with glueball states. The implication of this mixing for confinement is surely of interest.

Order by order in perturbation theory, QCD in $2 + \varepsilon$ dimensions is plagued by severe infrared divergences and appears to be quite complex. As we shall see, however, it is possible to reorganize the perturbation expansion in such a way that a simple picture emerges. With this reordering, it becomes evident that creation and destruction of transverse gluons is suppressed. As a result, the perturbation expansion consists of only planar diagrams. The theory is then equivalent to the large N limit of QCD of ε scalar mesons in the fundamental representation of SU(N) in two dimensions (with an appropriate rescaling of the coupling constants). One can then easily write down a bound state equation for a color-singlet gluon pair.

The theory treated in this manner is plagued with difficulties, however. In particular, the bound state equation which we derive, while it does possess a simple physical interpretation, does not possess solutions. That the theory should have difficulties is perhaps not completely surprising. For example, the Yang-Mills Hamiltonian, when dimensionally continued below two dimensions is not manifestly positive.[8] It is not clear, however, that this is the source of the difficulties which we have discovered.

This paper is organized as follows: In section 2, we study one-loop perturbation theory for the gluon propagator. We find that at one-loop, the gluon propagator in $2 + \varepsilon$ dimensions is not smoothly connected to the propagator in two dimensions. This section also contains a review of the quantization of gauge theories in light-cone gauge, and exhibits the Feynman rules. In section 3, we find a solution to the Schwinger-Dyson equations for the gluon propagator in $2 + \varepsilon$ dimensions which is smoothly connected to the propagator in two dimensions. It is in close correspondence to the solution for the gluon and fermion propagators in the 't Hooft model.[9] In section 4, we derive a Bethe-Salpeter equation for scalar, color-singlet glueballs. The Feynman diagrams which contribute to this equation are planar, and in close correspondence to the diagrams which contribute to the bound-state equations for mesons in the 't Hooft model. We then proceed to demonstrate that this equation has no non-singular solutions. In section 5, we conclude by speculating on possible sources of the difficulties we have encountered.

II. ONE-LOOP PERTURBATION THEORY FOR THE GLUON PROPAGATOR

We define QCD in $2 + \varepsilon$ dimensions to be the sum of the Feynman diagrams obtained using the dimensional regularization procedure of 't Hooft and Veltman.[10] As an example of this procedure, and of the problems of an expansion around two dimensions, we evaluate the one-loop contribution to the gluon propagator in Landau gauge. The relevant Feynman diagrams are shown in Fig. 1. In d dimensions we obtain, for the gluon vacuum polarization tensor,

$$\Pi^{\mu\nu}(q) = (g^{\mu\nu}q^2 - q^\mu q^\nu)\,\frac{g^2 N_c(-q^2)^{d/2-2}}{(2\pi)^{d/2}2(d/2+1)}\,\frac{\Gamma(d/2)}{\Gamma(d)}\,\frac{\pi}{\sin(\frac{\pi d}{2})} \times \{7/2\, d^2 - 19/2\, d + 8\} \quad . \tag{2.1}$$

In this equation, g is the gluon coupling constant and N_c is the number of colors.

For $d = 2 + \varepsilon$, $\varepsilon << 1$, the one-loop polarization tensor becomes

$$\Pi^{\mu\nu}(q) = (g^{\mu\nu}q^2 - q^\mu q^\nu)\,\frac{3}{\varepsilon}\,\frac{g^2 N_c}{4\pi}\,\frac{1}{q^2} \quad . \tag{2.2}$$

The $1/\varepsilon$ in this expression is characteristic of the singular infrared behavior of this theory. Summing the bubble diagrams of Fig. 2 yields for the propagator

$$D^{\mu\nu}(q) = \left\{g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right\} \frac{1}{q^2 - \frac{3g^2 N_c}{4\pi\varepsilon}} \tag{2.3}$$

so that the gluon appears to have acquired a mass,

$$m^2 = \frac{3g^2 N_c}{4\pi\varepsilon} \quad . \tag{2.4}$$

This summation of a selected set of Feynman graphs is not the entire story, of course. $\Pi^{\mu\nu}$ is not a gauge-invariant quantity. Moreover, it is clear that because of the singular infrared behavior of this theory, we will have to examine many classes of diagrams before we can make statements about physical quantities. However, in addition to indicating that the perturbation expansion of the propagator may be singular near two dimensions, this result also suggests that some of the colored degrees of freedom may become very "massive" and, perhaps, decouple.

QCD in $2 + \varepsilon$ dimensions is, unfortunately, quite complex in Lorentz gauge. Instead of Lorentz gauge we shall employ light-cone gauge and light-cone coordinates in our analysis. The quantization of field theories in light-cone variables has been treated by Kogut and Soper.[11] The necessary formalism for non-Abelian gauge theories has been developed by Tomboulis.[12] Here we briefly review the results of their analyses. We use the metric conventions of the text by Bjorken and Drell.[13]

The light-cone coordinates are

$$\tau \equiv x^+ \equiv \frac{1}{\sqrt{2}}\,(x^o + x^1) \tag{2.5}$$

$$x \equiv x^- \equiv \frac{1}{\sqrt{2}}\,(x^o - x^1) \tag{2.6}$$

and the d-2 transverse coordinates are $\vec{x}_\perp$. In terms of these variables, the scalar product is

$$p \cdot q = p^+q^- + p^-q^+ - \vec{p}_\perp \cdot \vec{q}_\perp \quad . \tag{2.7}$$

With n a light like vector and A^a_μ the Yang-Mills potentials, the light-cone gauge condition is

$$n \cdot A^a = A^{+a} = 0 \quad . \tag{2.8}$$

This condition is imposed on the Yang-Mills Lagrange density

$$\mathscr{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \quad , \tag{2.9}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \quad . \tag{2.10}$$

Perturbation theory may be developed by the canonical procedure of Kogut and Soper.[11] The theory is quantized on equal-τ surfaces. The d-2 transverse components of $\vec{A}^a_\perp$ are treated as dynamical variables. The commutation relation of the dynamical variables may be extracted from the work of Tomboulis.[12] Since no derivatives of A^{-a} with respect to τ appear in the Lagrangian, A^{-a} is a constrained variable analogous to the Coulomb field of radiation gauge electrodynamics. In particular, A^{-a} satisfies a "Gauss" law

$$-\partial_-^2 A^{-a} = \partial_i \partial_- A^{ia} + j^{+a} \equiv \gamma^{+a} \tag{2.11}$$

where

$$j^{+a} = -g f^{abc} A^{ib}_\perp F^{i+c} \tag{2.12}$$

is the color current.

Since A^{-a} can be removed from the Lagrangian by Eq. (2.11), the existence of d-2 dynamical degrees of freedom is explicit. It is, therefore, worthwhile to briefly study the inversion of Eq. (2.11) for A^{-a}. This is accomplished by Fourier transforming to momentum space, with the result

$$A^{-a}(k^+,k^-,\vec{k}_\perp) = \frac{1}{(k^+)^2} \gamma^{+a}(k^+,k^-,\vec{k}_\perp) \quad . \tag{2.13}$$

In general, the resulting integrals over k^+ are ill-defined unless the singularity at $k^+ = 0$ is regulated. The regulation of this singularity has been extensively studied.[9,14] In the analysis below, we shall use the procedure of 't Hooft:[9]

$$\frac{1}{(k^+)^2} \rightarrow \begin{cases} \dfrac{1}{(k^+)^2} & |k^+| > \lambda \\ 0 & |k^+| < \lambda \end{cases} \qquad \lambda << g \quad . \tag{2.14}$$

In the 't Hooft model, this regularization prescription has no effect on color singlet states, and serves only to regulate intermediate steps in the computation of properties of these states. At the end of such computations, the limit $\lambda \rightarrow 0$ may be taken.

The Feynman rules may be derived using path integral methods, or using the techniques developed by Kogut and Soper. The propagator is

$$D^{\mu\nu} = \frac{1}{k^2 + i\varepsilon} \left\{ g^{\mu\nu} - \frac{k^\mu n^\nu + n^\mu k^\nu}{n \cdot k} \right\} \quad . \tag{2.15}$$

The individual components of the propagator, and our notation for them are indicated in Fig. 3. The vertices are indicated in Fig. 4.

Before attempting to find a solution of the theory, we first consider the one-loop contributions to the light cone gauge gluon propagator. One might expect, based on counting the number of intermediate states, that production of transverse gluons would be suppressed by powers of ε in perturbation theory. However, as this calculation illustrates, the phase space integrals over transverse gluon momenta are infrared singular and lead to compensating powers of $1/\varepsilon$. As a result, order by order in perturbation theory the theory is quite complex.

Consider, for example, the diagram of Fig. 5 in which a light-cone gluon splits into two transverse gluons, and the transverse gluons subsequently annihilate into a light-cone gluon. This diagram is given by the integral

$$\Pi_{ab}^{++} = -\frac{i\varepsilon g^2 N_c \delta^{ab}}{(2\pi)^d}\int dk^+ dk^- d^\varepsilon k_\perp \frac{(q^+ - 2k^+)^2}{(k^2 + i\varepsilon)((q-k)^2 + i\varepsilon)} \quad . \tag{2.16}$$

For definiteness, we shall assume that $q^+ > 0$, $\vec{q}_\perp = 0$, and $q^2 < 0$. The factor of ε in this equation comes from the polarization sum, as advertised. The k^- integral is easily carried out,

$$\Pi_{ab}^{++} = \frac{\varepsilon g^2 N_c \delta^{ab}}{2(2\pi)^{d-1}}\int_0^{q^+} dk^+ \int d^\varepsilon k_\perp \frac{(q^+ - 2k^+)^2}{[2q^- k^+(k^+ - q^+) + q^+ k_\perp{}^2]} \quad , \tag{2.17}$$

and upon performing the $k_\perp$ integration, we obtain

$$\Pi_{ab}^{++} = \frac{\varepsilon g^2 N_c \delta^{ab}}{2(2\pi)^{d-1}}\Gamma(2 - d/2)\int_0^{q^+} dk^+ \frac{(q^+ - 2k^+)^2}{q^+[2q^- k^+(k^+ - q^+)/q^+]^{1-\varepsilon/2}} \quad . \tag{2.18}$$

The endpoints of the k^+ integration, corresponding to low momentum transverse gluons, give a contribution of order $1/\varepsilon$. The naive expectation that this graph is of order ε is incorrect; it is of order one.

The one loop contributions to the transverse gluon propagator are shown in Fig. 6. In this figure, we have included appropriate combinatoric factors. For simplicity, we again take the incoming momenta, q, to have $q_\perp = 0$ and $q^2 < 0$. Angular averaging the $k_\perp$ integrals gives factors of $1/\varepsilon$ in the individual diagrams of Fig. 6(a). These factors cancel, however, in the sum. From these diagrams, we obtain

$$\Pi_{ab}^{ij} = \frac{-ig^2 N_c \delta^{ab}\delta^{ij}}{(2\pi)^d}\int d^d k \frac{k_\perp{}^2\{k^+ - 4q^+\}}{k^+ k^2 (k+q)^2} \quad . \tag{2.19}$$

The $k_\perp$ integration in Eq. (2.19) is ultraviolet divergent. We can regulate this divergence by introducing a cutoff at $|k_\perp| = \Lambda$. This procedure is plausible since gauge invariance guaranties that there are no ultraviolet divergences in this theory. In the sum of all diagrams for Π_{ab}^{ij}, the ultraviolet divergences must cancel. Moreover, since the divergence is only logarithmic, all cutoff procedures should yield the same result. With such a cutoff,

$$\int d^{\varepsilon}k_{\perp} \frac{k_{\perp}^{2}}{k^{2}(k+q)^{2}} \sim \varepsilon \quad , \tag{2.20}$$

and the diagrams contributing to Π_{ab}^{ij} which involve transverse gluon production are suppressed.

Alternatively, we can regulate the ultraviolet singularities dimensionally, using the rule

$$\int d^{\varepsilon}k_{\perp} \equiv 0 \quad . \tag{2.21}$$

We have verified that such a procedure yields the same results for Π_{ab}^{ij} as obtained with the cutoff even though the different procedures yield different contributions for individual diagrams. For simplicity we will use the cutoff procedure throughout the rest of this paper.

We must still analyze the contributions to Π_{ab}^{ij} coming from the diagrams of Fig. 6(b). We rewrite the light cone propagator appearing in the first of these graphs as

$$\frac{2k^{-}}{k^{2}k^{+}} = \frac{1}{(k^{+})^{2}} + \frac{k_{\perp}^{2}}{(k^{+})^{2}(k^{2}+i\varepsilon)} \quad . \tag{2.22}$$

The second term on the right-hand side yields a contribution very similar to that of Eq. (2.19). It is of order ε if we use our ultraviolet cut-off procedure. The remaining contribution from the diagrams of Fig. 6(b) is

$$\Pi^{ij}_{ab}(q) = i\,\frac{4g^2 N_c \delta^{ij}\delta^{ab}}{(2\pi)^2}\int dk^+ dk^- \frac{q^+(k^+ + q^+)}{(k^+)^2 (k+q)^2} + \mathcal{O}(\varepsilon) \quad . \tag{2.23}$$

(The $k_\perp$ integral has again been performed using our cutoff procedure.) This integral is precisely the integral for the one-loop contribution to the fermion self-energy in the 't Hooft model. Using 't Hooft's momentum space infrared cutoff, we obtain

$$\Pi^{ij}_{ab}(q) = \delta^{ij}\delta^{ab}\,\frac{g^2 N_c}{\pi}\left(\frac{|q^+|}{\lambda} - 1\right) \quad . \tag{2.24}$$

It is important to note that in the diagrams of Fig. 6(b), transverse gluons are neither created nor destroyed. We will shortly argue that in the solution of the full theory, only diagrams of this type survive.

III. A SELF-CONSISTENT SOLUTION OF THE SCHWINGER-DYSON EQUATIONS

We have seen that, order by order, the perturbation expansion of the gluon propagator is quite complicated. In particular, creation and absorption of transverse gluons is not suppressed. In this section we will show that if one first sums all contributions to the propagator which do not involve emission or absorption of transverse quanta, then contributions to the propagator involving such emissions <u>are</u> suppressed. Equivalently, we will show that there is a solution to the Schwinger-Dyson equation for which the creation or destruction of transverse gluons is down by powers of ε and λ.

It is convenient to demonstrate this result using the Schwinger-Dyson equations, assuming that they possess a solution of this form, and demonstrating that the solution derived with this assumption is self-consistent. Consider first the diagrams for $\Pi^{++}_{ab}(q)$ shown in Fig. 7. In each of these diagrams, the light cone gluon emits two transverse gluons. By assumption, then, these diagrams are suppressed, i.e.

$$\Pi^{++}_{ab}(q) = 0 \quad . \tag{3.1}$$

The contributions to $\Pi^{ij}_{ab}(q)$ are shown in Fig. 8. Only those contributions which do not involve the creation or destruction of gluons have been retained in this figure. In order to solve this equation, we must know the full light-cone propagator and the full light-cone-transverse gluon vertex. Using Eq. (3.1), it is straightforward to demonstrate that the light-cone propagator is the free propagator (up to terms of order $k_\perp^2$, which give vanishing contributions if we use the integration rules developed in Section II). Also, one can readily demonstrate that all higher order contributions to the vertex involve emission and absorption of transverse gluons, and thus vanish, by assumption. Thus the full vertex is just the bare vertex. The vanishing one-loop contributions to this vertex are shown in Fig. 9.

With the vertices and light-cone propagator replaced by bare vertices and propagators, the sum of contributions shown in Fig. 8 become those shown in Fig. 10. Upon using Eq. (3.1) for Π^{++}_{ab}, the transverse propagator is

$$D^{ij}_{ab}(q) = \delta^{ij}\delta^{ab}\,\frac{1}{q^2 - \Pi(q) + i\varepsilon} \quad , \tag{3.2}$$

where

$$\Pi^{ij}_{ab}(q) = \delta^{ij}\delta^{ab}\Pi(q) \quad . \tag{3.3}$$

Thus the contributions of Fig. 10 yield a closed, non-linear integral equation for $\Pi(q)$,

$$\Pi(q) = \frac{4ig^2N_c}{(2\pi)^2}\int d^2k \, \frac{q^+(k^+ + q^+)}{(k^+)^2[(k+q)^2 - \Pi(k+q) + i\varepsilon]} \quad . \tag{3.4}$$

This equation is almost identical to the equation for the fermion self-mass kernel in the 't Hooft model. Π is independent of q^-, as can be seen by a simple change of variables. After the k^- integral in Eq. (3.4) is performed the resulting k^+ integral is independent of Π, and yields

$$\Pi(q) = \frac{g^2N_c}{\pi}\left(\frac{|q^+|}{\lambda} - 1\right) \quad . \tag{3.5}$$

With this result, Eq. (3.5), in hand, one can now examine the diagrams which involve emission or absorption of transverse gluons and demonstrate that they are all suppressed by powers of λ or ε. For example, a simple calculation, using Eq. (3.5) and (3.2) yields for Π^{++}_{ab} (Fig. 7) a result proportional to λ. It is a straightforward matter to show that all the diagrams contributing to Π^{ij} which we have neglected up to now are in fact suppressed. This is a consequence of the fact that the transverse gluon is effectively very heavy. The verification of this result is somewhat tedious, and will not be repeated here. Two subtleties in the analysis are worthy of note, however. First, in the contribution of Fig. 11, one might worry that a vertex insertion of order λ could induce a finite result by convolution with a singular light-cone propagator. The integration regions $k^+ \to 0$ and $q^+ \to 0$ might give factors of $1/\lambda$, yielding a contribution of order one for the overall diagram. This does not in fact occur since as one can easily verify, the corresponding vertex insertion vanishes in these integration regions.

Similar dangerous diagrams appear in higher orders of perturbation theory, and we do not have a general proof that they do not give finite contributions. We have verified that no such finite terms are generated through two loops. If such a finite term did exist, it could generate a mass shift in the gluon propagator.

The second problem concerns the appearance of color-singlet bound states among the particles in intermediate states (Fig. 12). One expects that these bound states will have finite masses as $\lambda \to 0$ (proportional to g^2, the only scale in the problem). To determine whether such diagrams are suppressed by factors of λ, one must know something about the bound state spectrum and wave functions. It seems likely that any reasonable spectrum will lead to a rapid falloff of the appropriate form factors for large mass states, and that such diagrams will be suppressed by powers of λ. If this is not the case, the propagator could be much more complicated than suggested here. We will comment again on this possibility in the concluding section.

Several comments about the gauge invariance of our calculations are called for. Using an infrared cutoff as a measure of the contributions to Feynman diagrams is particularly dangerous since the infrared cutoff is itself gauge dependent. We believe that our calculations should be valid as long as all relevant orders in λ are retained in the calculation of gauge invariant quantities such as the properties of glueballs (to which we shall shortly turn). Calculations in another gauge would, of course, provide a useful check on our procedure, but such calculations appear to be quite difficult.

We should point out that the solution to the Schwinger-Dyson equations which we have found does satisfy the Ward-Takahashi-Slavnov identities. These identities are consequences of the underlying gauge invariance of the theory. The relevant identity here is

$$k^{\mu}\Gamma^{abc}_{\mu\nu\lambda}(k,q+k,q) = f^{abc}\{D^{-1}_{\nu\lambda}(q+k) - D^{-1}_{\nu\lambda}(q)\} \quad . \tag{3.6}$$

In particular, if all transverse momenta are set to zero, the requirement that the transverse gluon-2 light-cone gluon vertex is bare is equivalent to the requirement that Π^{ij}_{ab} is independent of q^-.

IV. THE BOUND STATE EQUATION FOR GLUEBALLS

In the previous section, we have made a conjecture for the gluon propagator, and have shown that it is self-consistent. We analyze here the bound state equation for a scalar, color-singlet gluon pair using our solution for the gluon propagator as input. We will see that, remarkably, a planar structure emerges which is similar to the planar structure in the 't Hooft model. This result follows from the suppression of creation and destruction of gluon pairs.

Our basic tool for the analysis will be the Bethe-Salpeter equation for a gluon pair, projected onto spin-zero color-singlet states. This equation is represented by the diagram of Fig. 13. The truncated kernel, K, is the s-channel two-particle-irreducible scattering amplitude. The wavefunction $\Psi(p)$ is the Fourier transform of

$$\Psi^{\mu\nu}_{ab}(x) = \langle 0|T(A^{\mu a}(x/2)A^{\nu b}(-x/2))|G\rangle \quad , \tag{4.1}$$

where T denotes ordering with respect to the light cone variable τ, and $|G\rangle$ denotes the two-gluon bound state. The total invariant four momentum of the glueball is $s = M_G^2$, where M_G is the glueball mass.

We first consider various contributions to the kernel, and verify the conjecture that only planar diagrams contribute. For simplicity, we will not be extremely careful about transverse momenta, $p_\perp$. We will assume that the wavefunction is essentially a delta function in $p_\perp$ centered around $p_\perp = 0$ and set all integrals over $p_\perp$ to 1 in the kernels. The errors which result from this approximation in the calculation of the kernel can be shown to be of order ε.

As we shall shortly see, the analysis of the various kernels simplifies enormously if for all the external momenta, $p_i^+ > 0$. This will in fact be the case if the kernel, K, is independent of q^-. One can demonstrate this easily by integrating both sides of the Bethe-Salpeter equation (Fig. 13) over p^- under this assumption. In the discussion which follows, we will proceed in a self-consistent manner. We will assume that the only surviving kernels in the Bethe-Salpeter equation are independent of q^-, and thus all external lines have $p^+ > 0$. With this assumption, we will in fact demonstrate that all q^- dependent kernels are suppressed by powers of ε and λ.

The first class of non-planar diagrams we consider are those which involve crossed ladders of Coulomb gluons. Consider for example, the diagram of Fig. 14. This contribution is

$$K_{(2)} \sim \int dk^- dk^+ \frac{P(k^-,p_i^+)}{(k^+)^2(k^+ + p_1^+ - p_1^{+'})^2\left[(p_1 + k)^2 - \frac{g^2}{\pi}\left(\frac{|p_1^+ + k^+|}{\lambda} - 1\right) + i\varepsilon\right]}$$
$$\times \frac{1}{\left[(p_2' + k)^2 - \frac{g^2}{\pi}\left(\frac{|p_2^{+'} + k^+|}{\lambda} - 1\right) + i\varepsilon\right]} \quad (4.2)$$

In this equation, P is a polynomial in k^+ and p_i^+. We have used equations 3.2 and 3.5 for the full gluon propagator. As discussed above, we take the momenta p_i^+ to be positive on external lines. Also without loss of generality, we will explicitly consider only the case where $p_2^{1'} > p_1^+$, although the analysis of the other case is similar.

Performing the k^- integral in Eq. 4.2, we obtain

$$K_{(2)} \sim \int_{-p_2'^{+}}^{-p_1^+} dk^+ \frac{P(k^+,p_i^+)}{(k^+)^2(k^+ + p_1^+ - p_1'^+)}$$

$$\frac{1}{\left[2(p_2^{-'} - p_1^-)(p_1^+ + k^+)(p_2^{+'} + k^+) - \frac{2g^2}{\pi}\frac{(p_1^+ + k^+)(p_2^{+'} + k^+)}{\lambda} + \frac{g^2}{\pi}(p_1^+ - p_2^{+'})\right]} \tag{4.3}$$

Because of our restrictions on the p_i^+, the singularities of the light-cone propagators are never encountered in the k^+ integral. This diagram is thus of order λ.

This result is simply understood physically. If we consider old-fashioned perturbation theory in light-cone variables, the Feynman diagram of Fig. 14 is the sum of several τ-ordered diagrams. One such τ-ordering is shown in Fig. 15. (Recall that the light-cone propagator is instantaneous in τ.) This contribution involves the production of transverse gluon pairs, and is suppressed by a power of λ. This argument can readily be extended to higher order kernels involving crossed (non-planar) light-cone gluon exchange.

Kernels involving exchanges of transverse quanta are even more simply dealt with. Consider, for example the diagrams shown in Fig. 16. These diagrams are immediately suppressed by a power of λ arising from the transverse gluon propagator. Kernels involving multiple transverse exchange are suppressed by additional factors of λ. (In addition, the contributions of Fig. 16 are suppressed by a power of ε.)

The above arguments take care of almost all the contributions to the kernel. However, just as in the case of the vacuum polarization tensor, there is one class of diagrams which looks particularly dangerous, involving vertex corrections to light-cone gluon exchange. Consider the diagram of Fig. 17. Again one might worry that even though the vertex correction is of order λ, the integral over the light-cone gluon yields a compensating $1/\lambda$, giving a finite contribution to the Bethe-Salpeter kernel. However, just as before, we are saved by the fact that the vertex correction vanishes as $q^+ \to 0$. As in the case of $\Pi^{\mu\nu}$, we have not been able to formulate a proof that there is never a pileup of singularities in higher-order diagrams which yields a finite contribution to the kernel. A careful study of some dangerous-looking higher order diagrams suggests that this is not the case, however.

Finally, one must worry about the problem of multimeson intermediate states (Fig. 18). It seems likely that these diagrams are suppressed by powers of ε, and it is not difficult to show this for some low order diagrams. In general, however, delicate cancellations are involved, and we have not been able to formulate a general argument that such diagrams are suppressed. For the rest of this discussion, we will assume that this is in fact the case. The reader should keep in mind the possibility, however, that some multimeson interactions are not down by powers of ε, and, as a result, the model is much more complex than our analysis suggests.

One may readily convince oneself that these arguments are sufficient to deal with any kernel in which transverse gluons are created or destroyed. We are therefore left with only planar diagrams, a result that follows from the absence of creation or destruction of transverse quanta. Thus the theory, in lowest non-trivial order in ε is a theory of ε conserved particles interacting by the exchange of instantaneous potentials in two dimensions.

The complete Bethe-Salpeter equation is now quite simple. It is shown diagramatically in Fig. 19. Projecting onto spin and color-singlet components, the Bethe-Salpeter equation is

$$\Psi(p,s) = \frac{-iN_c g^2}{(2\pi)^2} \int d^2k \frac{(2p^+ + k^+)(2s^+ - 2p^+ - k^+) + 2(k^+)^2}{(k^+)^2\left[(s-p)^2 - \frac{g^2}{\pi}\left(\frac{|s^+ - p^+|}{\lambda} - 1\right) + i\varepsilon\right]} \times \frac{1}{\left[p^2 - \frac{g^2}{\pi}\left(\frac{|p^+|}{\lambda} - 1\right) + i\varepsilon\right]} \frac{1}{(k^+)^2} \Psi(p+k,s) \tag{4.4}$$

This equation is almost identical to the meson equation of the 't Hooft model. It differs only in the different momentum dependent vertices coupling the transverse quanta to the light cone propagator, and the contact four-gluon coupling.

To find a solution to this equation, we define a "Schrodinger" wavefunction by

$$\varphi(p^+,s) = \int dp^- \Psi(p^-,p^+,s) \quad . \tag{4.5}$$

Integrating both sides of Eq. (4.4) with respect to p^-, and using Eq. (4.5), we obtain

$$\mu^2\varphi(x) = -\frac{1}{x(1-x)}\varphi(x) - \int \frac{dy}{(y-x)^2}\varphi(y) - \frac{1-2x}{2x(1-x)}\int \frac{dy}{y-x}\varphi(y) \quad . \tag{4.6}$$

In this equation

$$x \equiv \frac{p^+}{s^+} \quad , \tag{4.7}$$

$$\mu^2 = \frac{2\pi s^+ s^-}{g^2 N_c} \quad , \tag{4.8}$$

and $⨍$ means to integrate with a principle value prescription. As in the case of the equation for mesons in the 't Hooft model, all λ dependence has cancelled, a fact which is a reflection of the color neutrality of the system.

If one calls M^2 the operator on the right-hand side of Eq. (4.6), it is straightforward to show that M^2 is self-adjoint only if the scalar product of two wavefunctions Ψ and φ is defined as

$$\langle \Psi,\varphi \rangle = \int dx \; \Psi^*(x) x(1 - x)\varphi(x) \quad . \tag{4.9}$$

That this is the correct form of the scalar product can also be shown by obtaining the normalization condition for φ from the inhomogeneous Bethe-Salpeter equation, using standard techniques. Normalizability thus requires that φ be less singular than $1/x$ near the origin.

Armed with this information, it is now possible to study the bound state equation. Unfortunately, Eq. (4.6) is quite sick. This can be seen by studying the matrix elements of M^2 between functions of the form

$$\varphi_\varepsilon(x) = \frac{1}{x^{1/2 - \varepsilon}} \frac{1}{(1 - x)^{1/2 - \varepsilon}} \quad . \tag{4.10}$$

A straightforward calculation yields

$$\langle \varphi_\varepsilon, M^2 \varphi_\varepsilon \rangle = - \frac{1}{\varepsilon} + \mathcal{O}(1) \quad . \tag{4.11}$$

In other words, the "spectrum" of this theory contains an infinite well of tachyons.

A similar bound-state equation for glueballs has been obtained by Bardeen, Pearson, and Rabinovici in their studies of the transverse lattice.[15] Their equation differs from Eq. (4.6) in two respects. First, the contact term arising from gluon-gluon scattering is not included. This term, however, would only give a contribution of order one to Eq. (4.11), and thus its presence or absence does not alter our discussion above of the spectrum. The crucial difference between their equation and ours comes from the fact their transverse gluons are "massive." Such a mass term, they argue, will in general be induced by the presence of the lattice (its effects should disappear in the continuum limit.) The presence of this mass leads to the replacement

$$-\frac{1}{x(1-x)}\varphi(x) \to \frac{M^2}{x(1-x)}\varphi(x) \tag{4.12}$$

in the first term on the right-hand side of Eq. (4.6). For $M^2 \geq 0$, the bound state equation possesses a spectrum of states, without tachyons, very similar to that of the 't Hooft model. It is interesting to note that, for $M^2 = 0$, the equation has a zero mass solution,

$$\varphi(x) = x^{-1/2}(1-x)^{-1/2} \quad . \tag{4.13}$$

V. CONCLUSIONS

The problems which we have encountered can be at least partially understood by examining another two-dimensional model: the large N limit of QCD with scalar particles in the fundamental representation. This model, we will argue, is nearly identical to the model we have studied if the scalars are taken to be massless and a φ^4 term with an appropriate value for the quartic coupling is included. Explicitly, the analog model we consider here is described by the Lagrangian

$$\mathscr{L} = (\mathscr{D}_\mu\phi)^\dagger(\mathscr{D}^\mu\phi) - \lambda(\phi^*\phi)^2 \quad . \tag{5.1}$$

Here $\mathscr{D}_\mu$ is the usual covariant derivative,

$$(\mathscr{D}_\mu\phi)_i = (\delta^{ij}\partial_\mu - \frac{ig}{\sqrt{N}} T^a_{ij}A^a_\mu)\phi_j \quad . \tag{5.2}$$

Again we choose the light-cone gauge for the analysis.

Consider, first, the meson self-energy in this model. In the large N limit it is dominated by the diagrams shown in Fig. 20a. Explicitly this equation reads

$$-i\Pi(q) = \frac{1}{(2\pi)^2}\int \frac{d^2k}{(k^+)^2} \frac{g^2(k^+ - 2q^+)^2 + \lambda(k^+)^2}{[(k-q)^2 - \Pi(k-q)]} \quad . \tag{5.3}$$

Performing the k^- integral assuming, again that Π is independent of k^- yields

$$\Pi(q) = -\frac{1}{4\pi}\int_\lambda^{q^+} \frac{dk^+}{(k^+)^2} \frac{g^2(k^+ - 2q^+)^2 + \lambda(k^+)^2}{(k^+ - q^+)} \quad . \tag{5.4}$$

Note that we have again followed 't Hooft in cutting off the integral at $k^+ = \lambda$. The integral diverges as $k^+ \to q^+$ unless we choose $\lambda = -g^2$. This suggests that we define the theory with a negative sign for the quartic coupling. As a result, the Hamiltonian of the original field theory is not bounded from below. While the resulting expression for the scalar self-mass is identical to that for the fermions of the massless 't Hooft model, one cannot argue here that the positively of the orignal Hamiltonian insures the positively of the spectrum.

It is a straightforward matter, with this choice of coupling, to verify that the bound state equation (Fig. 20b) in this model is identical to the glueball equation, Eq. (4.6). Thus this model suffers from difficulties identical to those of $2 + \varepsilon$ dimensional QCD, at least with the approximations we have used. One should note that had we included a mass μ^2 for the scalars, with $\mu^2 \geq g^2$, the bound state equation would have yielded a sensible meson spectrum. It is clear, then that the quartic gluon coupling in $2 + \varepsilon$ dimensional QCD is, in effect, negative. The reader may readily verify, for example, that the contribution of this term to the vacuum energy is in fact negative (it is proportional to $\varepsilon^2 - \varepsilon$, which changes sign at $\varepsilon = 1$).

If one uses the dimensional regularization procedure very carefully as outlined in section II, the quartic coupling does not contribute to the gluon self-energy. However, when this procedure is employed, certain diagrams which are naively positive and ultraviolet divergent are rendered finite and negative, and the same results are obtained. The introduction of the regulator, in other words, while banishing the dangerous B^2 terms, itself destroys the naive positively arguments.

In the fermion theory, positivity of the Hamiltonian guarantees that the appearance of a tachyon at intermediate stages of the calculation can be viewed as an artifact of the approximations. This is not so in the case of bosons. Moreover, once one accepts the existence of tachyons, one can quickly see on "kinematic" grounds that a two-particle state can attain arbitrarily low mass by having both constituents nearly "on-shell" and letting k^+ for each tend to zero. This is precisely what happened in the calculation of Eq. (4.11). This doesn't happen for massless fermions because the boundary conditions for the bound-state equation in that case forbid it.

We seem, then, to be faced with the likely possibility that the theory we have studied possesses runaway solutions and, as such, does not exist as a physical model. Only if some dynamical means of stabilizing the theory can be found is it likely that QCD makes sense in the interval between two and three dimensions. Perhaps, for example, as we noted in the previous section, multimeson interactions are not suppressed by powers of ε. Or perhaps some non-perturbative mechanism generates a mass for the gluon, leading to a sensible bound state equation.

Acknowledgments

We wish to acknowledge useful conversations with Bill Bardeen, Richard Brower, Ken Johnson, Hossein Partovi, Bill Spence, and our colleagues at SLAC. This work was supported by the Department of Energy under contract DE-AC03-76SF00515.

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FIGURE CAPTIONS

1. Feynman diagrams for the 1-loop gluon propagator in Landau gauge. Wavy lines denote gluons. Dashed lines denote scalar ghosts.
2. Summing the bubble diagrams.
3. The components of the free gluon propagator.
4. The gluon vertices.
5. The one-loop modification of the light-cone gluon propagator.
6. The one-loop contributions to the transverse gluon propagator.
7. The Schwinger-Dyson equation for Π^{++}.
8. The contributions to Π^{ij} which do not involve the creation or destruction of transverse gluons.
9. The one-loop contribution to the 2-light-cone gluon-1-transverse gluon vertex.
10. The contributions to Π^{ij}.
11. A noteworthy contribution to Π^{ij}. k and q denote the momenta running through the light-cone propagator.
12. A contribution to Π^{ij} with a color-singlet particle in the intermediate state.
13. The Bethe-Salpeter equation for the glueball wavefunction.
14. A contribution to K involving crossed ladders of Coulomb gluons.
15. A τ-ordered contribution to $K_{(2)}$.
16. Contributions to the kernel involving transverse gluon exchange.
17. A frightening contribution to the kernel.
18. Diagrams involving two-meson intermediate states.
19. The Bethe-Salpeter equation for the glueball wavefunction.

20. Diagrams for scalar QCD in large-N limit in 2 dimensions: (a) propagator, (b) two-particle Bethe-Salpeter kernel. Solid lines denote mesons.

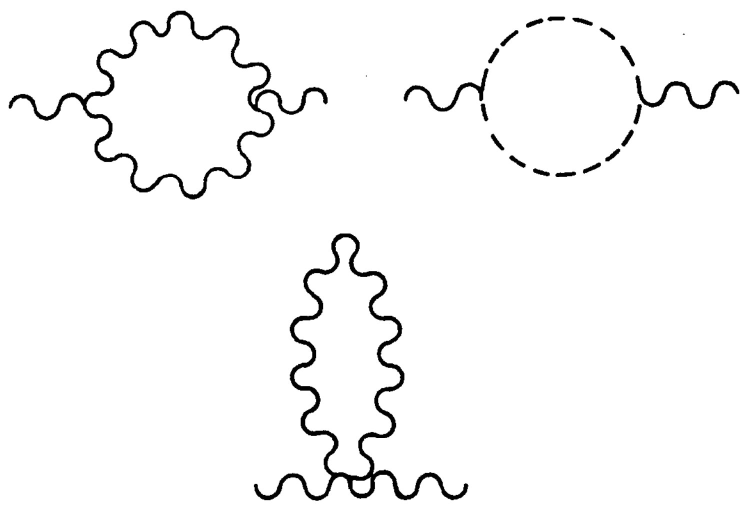

Fig. 1

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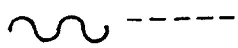
Fig. 2

$$D^{+\nu} = 0$$

$$D^{--} = \frac{1}{k^{+2}} + \frac{k_\perp^2}{k^{+2}(k^2 + i\epsilon)} =$$

$$D^{-i} = \frac{k^i}{k^+} \; \frac{1}{k^2 + i\epsilon} =$$



$$D^{ij} = \frac{\delta^{ij}}{k^2 + i\epsilon} =$$

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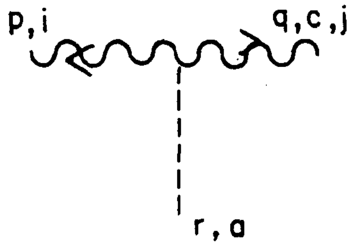
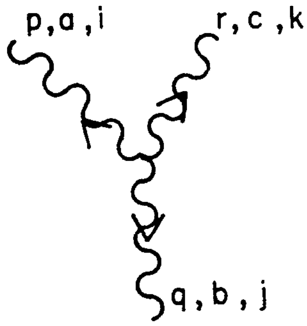
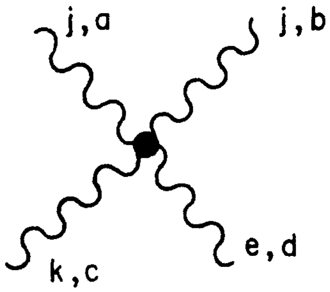
Fig. 3



$$= (q^+ - p^+)\ \delta^{ij}\ f^{abc}$$



$$= f^{abc}\left[(p-q)^k \delta^{ij} + (q-r)^i \delta^{jk} + (r-p)^j \delta^{ik} \right]$$



$$\begin{aligned} = -i\Big[& f^{abf} f^{cdf} \left(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}\right) \\ & + f^{acf} f^{bdf} \left(\delta^{ij}\delta^{kl} - \delta^{il}\delta^{jk}\right) \\ & + f^{adf} f^{cbf} \left(\delta^{ik}\delta^{jl} - \delta^{ij}\delta^{kl}\right)\Big] \end{aligned}$$

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Fig. 4

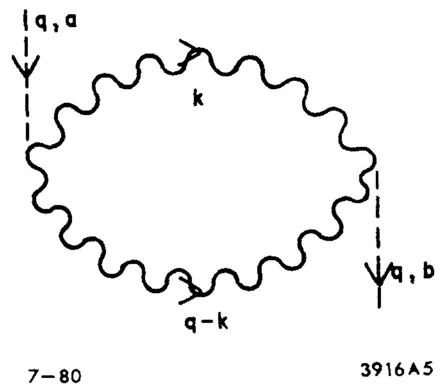

Fig. 5

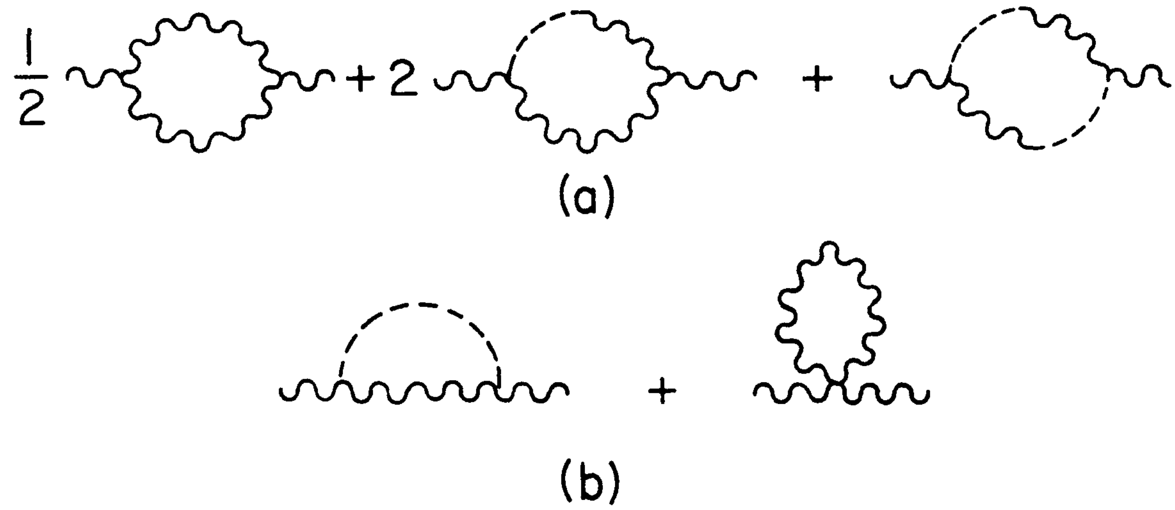

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Fig. 6

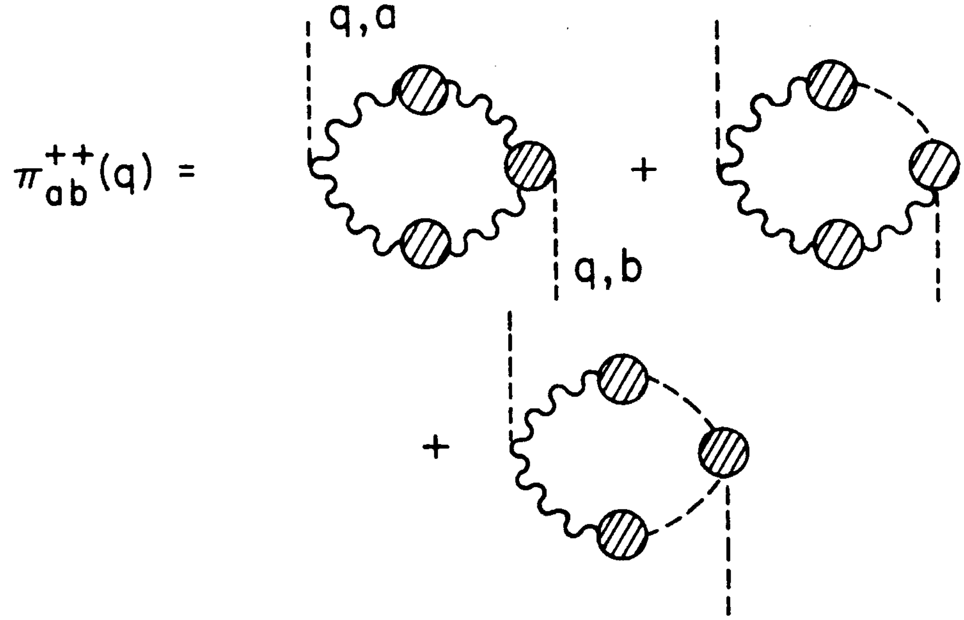

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Fig. 7

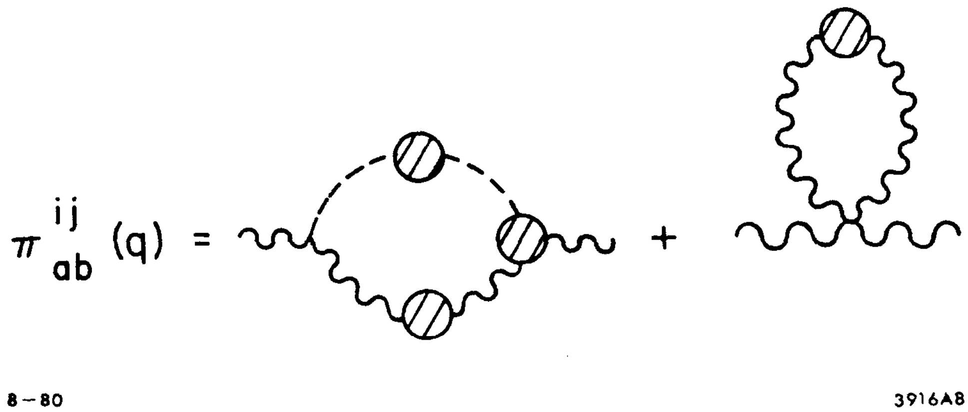

Fig. 8

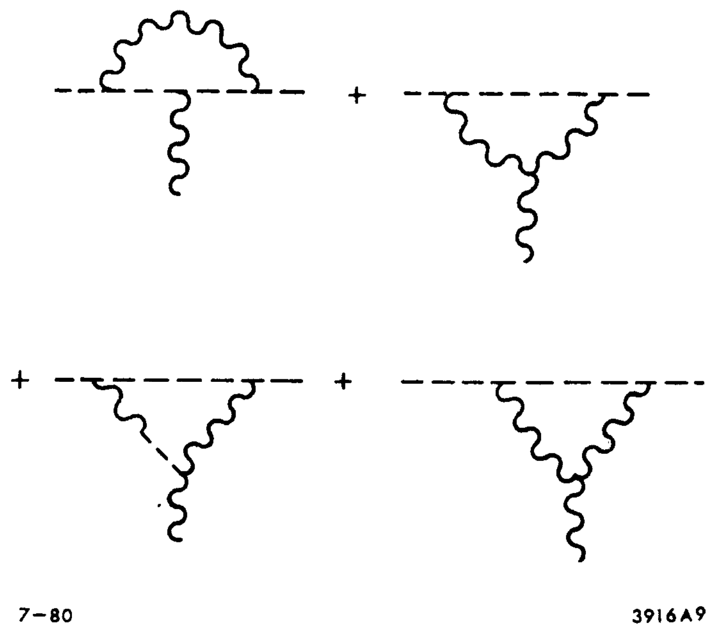

Fig. 9

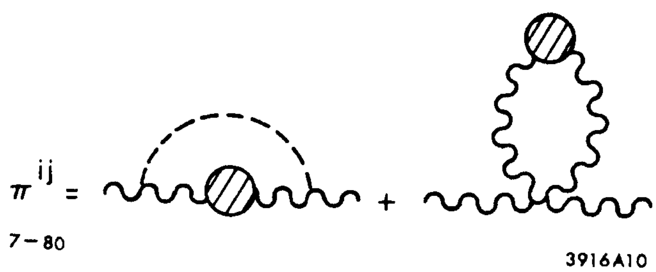

Fig. 10

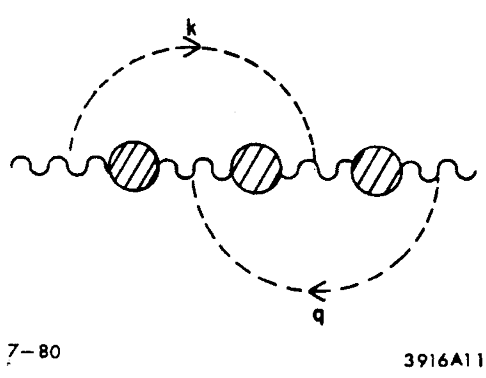

Fig. 11

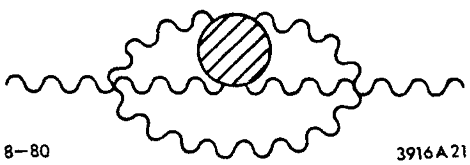

Fig. 12

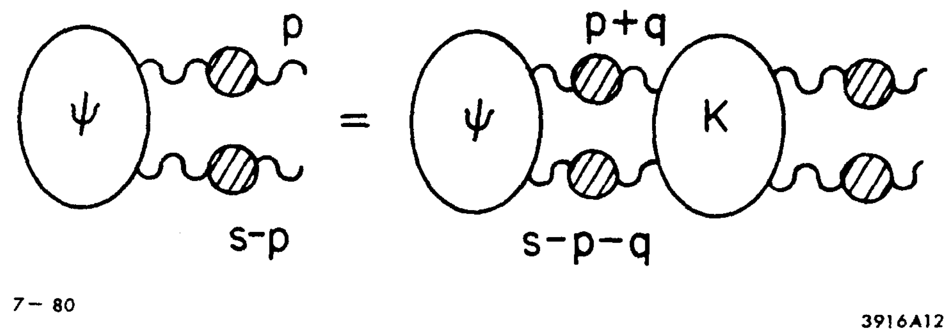

Fig. 13

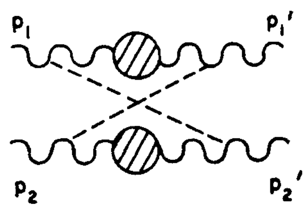

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Fig. 14

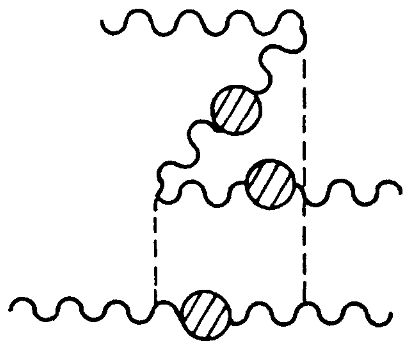

Fig. 15

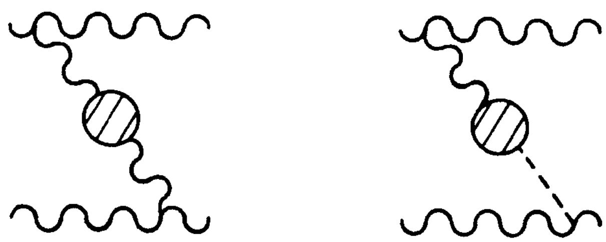

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Fig. 16

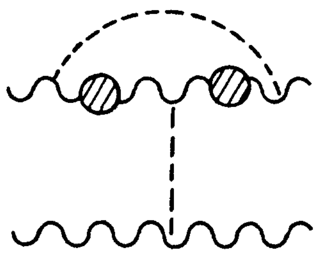

Fig. 17

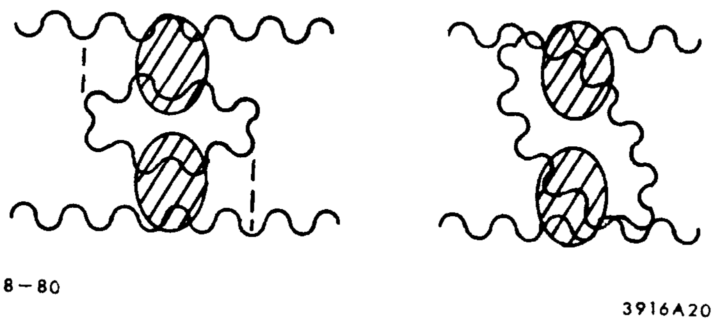

Fig. 18

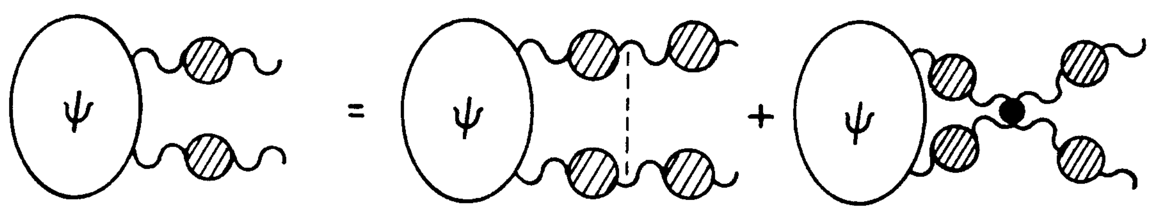

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Fig. 19

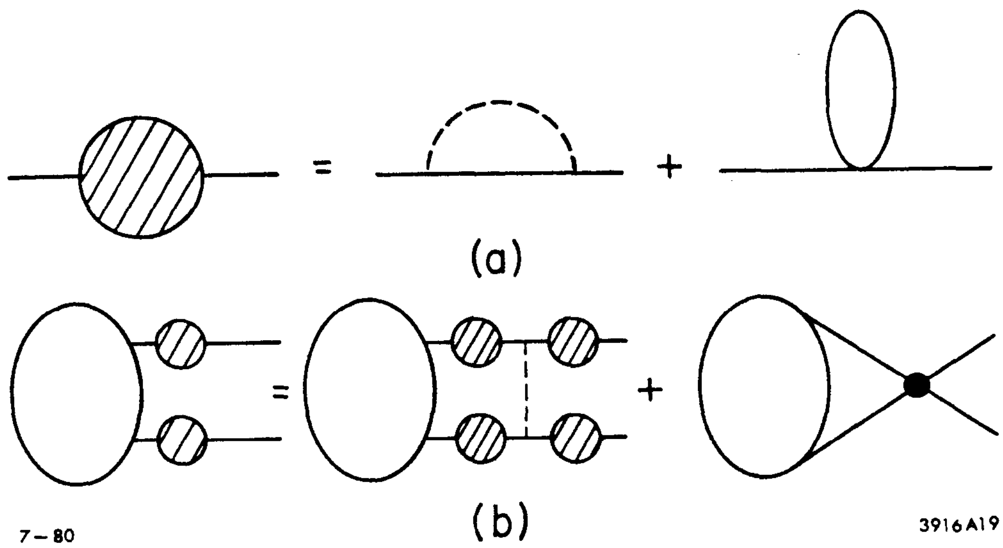

Fig. 20