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ELECTROMAGNETIC FIELDS IN AN AXIAL SYMMETRIC WAVEGUIDE WITH VARIABLE CROSS SECTION *

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Abstract — A new class of separable variables is found which allows one to find an approximate analytical solution of the Maxwell equations for axial symmetric waveguides with slow (but not necessarily small) varying boundary surfaces. Two examples of the solution are given. Possible applications and limitations of this approach are discussed.

Introduction

There are many problems the solution of which can be reduced to a much simpler one if only the eigenfunction of the corresponding equations were known. Numerous examples of such problems arise by investigation of the interaction of a bunch of particles with the electromagnetic field in an axial symmetric waveguide. Particularly in application to the accelerator theory, one needs to know how much energy is lost by the bunch in a vacuum chamber, its different tanks and cavities, the distribution of the higher mode fields induced by a bunch in the surroundings, the interaction between the particles in the same bunch or in different bunches through the wake fields induced by themselves and so on. All these problems can be reduced to some sort of summation over the field eigenfunctions for the given wall geometry of the vacuum chamber.

The only one known analytical solution of the Maxwell equations for the space inside an axial symmetric waveguide is the solution for the waveguide with a constant circular cross section. The solutions for a stepwise constant cross section waveguide can be obtained by properly matching the solutions for individual steps. This procedure is time and labor consuming. The example of such an approach for a stepwise periodic structure is given in work [1].

Exact analytical solution for a waveguide with variable cross section is not known. For the case when the corrugations of the cylindrical wall are smooth and small, the solution can be obtained by means of perturbation theory [2,3]. This method was recently applied to problems of finding the wake fields induced by a current moving on axis [4] and off axis [5] of a corrugated pipe.

In this paper another method is developed. It allows one to find an approximate analytical solution in a waveguide with quite arbitrary wall shape, satisfying certain conditions. The central idea of the method is a conform transformation to coordinates in which the boundary curve appears to be a straight line (Section 1). The limitations of this method and the area of its applicability are discussed in Section 2. The main part of the work is then devoted to the solution of the homogenous Maxwell equations in the new orthogonal curvilinear coordinates (Sections 3-6). As it is shown here in a certain approximation, the variables are separable even in the vector wave equation. Analogously to cylindric waveguide with constant cross section TM and TE modes are introduced.

Next step (Section 7) is then to satisfy the boundary conditions for the electromagnetic field inside the pipe and to find the eigenvalues (the frequencies and the propagation functions).

Two examples of the application of the derived method are given in Section 8. One can find here also the comparison of the calculations with the results of measurements taken from the paper [2].

I. Coordinates

Let us consider the following dependencies of the rectangular coordinates x, y, z on curvilinear coordinates ρ, θ, σ:

$$x = A\left(\rho + \sum_n a_n \operatorname{sh} n\rho \cos n\sigma\right)\cos\theta \tag{1.1}$$

$$y = A\left(\rho + \sum_n a_n \operatorname{sh} n\rho \cos n\sigma\right)\sin\theta \tag{1.2}$$

$$z = A\left(\sigma + \sum_n a_n \operatorname{ch} n\rho \sin n\sigma\right) + z_0 \tag{1.3}$$

From (1.1) and (1.2) we get for the radius in the plane $z = \text{const}$

$$r = A\left(\rho + \sum_n a_n \operatorname{sh} n\rho \cos n\sigma\right) \tag{1.4}$$

Symbol $\sum_n$ means either summation or integration over n. Generally speaking, one needs to distinguish between covariant and contravariant components of any vector $\vec{F}$. To avoid this, I will use only projections F_1, F_2, F_3 of any vector $\vec{F}$ on a base system of unit vectors $\vec{i}_1$, $\vec{i}_2$, $\vec{i}_3$. Figure 1 shows the three vectors $\vec{i}_i (i = 1,2,3)$ drawn in the directions of increasing coordinates ρ, θ and σ, such as to constitute a right-hand base system.

Using Equations (1.1, 1.2, 1.3), it is easy to find the metric tensor g_{ij}

$$g_{ij} = 0 \quad i \neq j \tag{1.5}$$

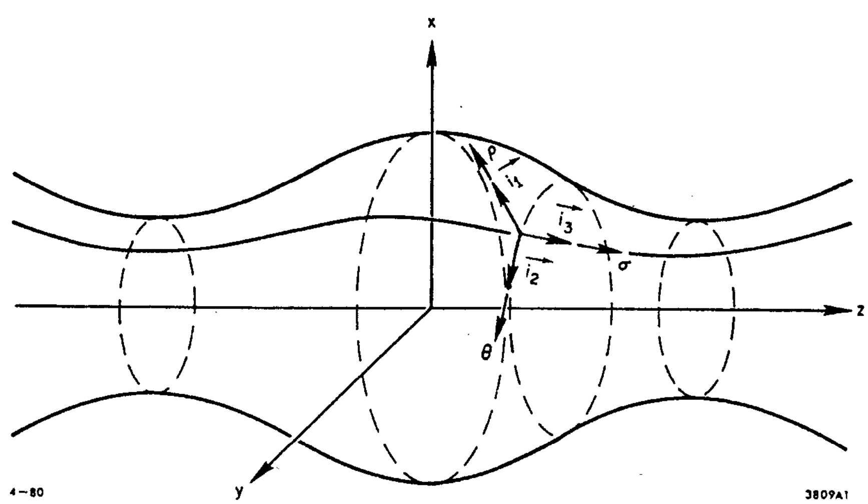

Fig. 1. The coordinate system ρ,θ,σ. $\vec{i}_1, \vec{i}_2$, and $\vec{i}_3$ are unit vectors constituting an orthogonal right-hand base system.

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$$g_{11} = g_{33} = A^2 \left[\left(1 + \sum_n a_n n \,\mathrm{ch}\, n\rho \cos n\sigma \right)^2 + \left(\sum_n a_n n \,\mathrm{sh}\, n\rho \sin n\sigma \right)^2 \right] \quad (1.6)$$

$$g_{22} = A^2 \left(\rho + \sum_n a_n \,\mathrm{sh}\, n\rho \cos n\sigma \right)^2 \quad (1.7)$$

Equation (1.5) proves the orthogonality of the coordinate system, ρ, θ, σ:

$$\vec{I}_i \vec{I}_j = \delta_{ij} \quad (1.8)$$

From (1.4) follows that $r = 0$ for $\rho = 0$. Constant z_0 in (1.3) allows the choice of any plane $z = \text{const}$ for placing there the origin $\rho = \sigma = 0$. If we put $z_0 = 0$, then the origin will be in the plane where the cross section has the maximum.

The constant A is an arbitrary scale factor with the dimension of the length. The ρ, θ and σ are dimensionless. In the limit $a_n \to 0$, ρ, θ and σ go into usual cylindrical coordinates r, θ, z. The constants a_n are dimensionless arbitrary quantities limited only by a natural condition $r > 0$ everywhere. The case $r = 0$ at some value of σ should be considered separately (it represents a closed axial symmetric cavity of an arbitrary shape).

II. Main Approximation

The metric coefficients $h_i = \sqrt{g_{ii}}$, which one finds from equations (1.6) and (1.7) are very complicated and do not give the possibility to solve the corresponding Maxwell equations. They became much simpler in the limit

$$N\rho < 1 \quad , \quad (2.1)$$

where N is the most sufficient harmonic number in the expansion (1.4). If (2.1) is fulfilled, then

$$h_1 = h_3 = h(\sigma) \quad , \quad (2.2)$$

$$h_2 = \rho h(\sigma) \quad , \quad (2.3)$$

where

$$h(\sigma) = A \left(1 + \sum_n a_n n \cos n\sigma \right) \quad (2.4)$$

For the coordinate transformation we get

$$r = \rho h(\sigma) \quad (2.5)$$

$$z = \int_0^\sigma h(\sigma_1)\, d\sigma_1 \quad (2.6)$$

or excluding σ from (2.5)

$$r = \rho f(z) \quad (2.6a)$$

If one chooses the function f(z) in such a way as to describe the shape of the wall in the plane $\theta = \text{const}$, then $\rho = \rho_{max} = \text{const}$ on the boundary.

The physical meaning of the condition (2.1) depends on the actual value of the coefficients a_n (or, in other words, on actual shape of the boundary). But, in the case where all a_n are small, the meaning can be understood easily. In this case, $z \simeq A\sigma$,

$$r_{boundary} = \rho_{max} f(z) \simeq \rho_{max} h(z/A) = A\rho_{max} \left(1 + \sum_n a_n n \cos n\, z/A \right) \quad (2.7)$$

and we see that $A = L/2\pi$, where L is the characteristic length of the change of the function f(z) (the period for a periodic function). Now, take the average of equation (2.7) over z;

$$\langle r \rangle_{boundary} \equiv b = A\rho_{max} = L\rho_{max}/2\pi \quad (2.8)$$

or

$$\rho_{max} = 2\pi b/L \quad , \quad (2.9)$$

where b has the meaning of the average pipe radius. The condition (2.1) now gives

$$b < L/2\pi N \quad (2.10)$$

Since ρ changes in the limits

$$0 \le \rho \le \rho_{max} \quad (2.11)$$

the condition (2.1) is fulfilled for all the values of ρ, and the approximation (2.2)-(2.4) is valid everywhere inside the pipe.

Before we go further, I will mention another limitation of this method. Since expansion (2.4) contains only cosines, not all shapes of the boundary can be treated in this way. Namely, we can consider only such boundary curves $r = r(z)$, which allow the representation in the form (2.5), (2.6) with $h(\sigma)$ symmetric

$$h(-\sigma) = h(\sigma) \quad (2.12)$$

For the sake of completeness, it is useful to mention also the restriction of the method by axial symmetric waveguides only.

III. The Maxwell Equations

Let us confine ourselves to the case when the media inside the waveguide is a vacuum. The generalization to a homogeneous dielectric media is straightforward. In the absence of charges and currents, the fields can be defined in terms of two vector functions, $\vec{\Pi}$ and $\vec{\Pi}^*$ (the Hertz vectors); each one of them satisfies homogeneous vector wave equation. Since the coefficients of this equation are constant it is possible to deal with each time Fourier harmonic of $\vec{\Pi}$ and $\vec{\Pi}^*$ separately:

$$\vec{\Pi} = \int \vec{\Pi}_\omega e^{-i\omega t} d\omega \quad (3.1)$$

$$\vec{\Pi}^* = \int \vec{\Pi}^*_\omega e^{-i\omega t} d\omega \quad (3.2)$$

Then, $\vec{\Pi}_\omega$ and $\vec{\Pi}^*_\omega$ satisfy the equations:

$$\nabla^2 \vec{\Pi}_\omega + k^2 \vec{\Pi}_\omega = 0 \quad , \quad (3.3)$$

$$\nabla^2 \vec{\Pi}^*_\omega + k^2 \vec{\Pi}^*_\omega = 0 \quad , \quad (3.4)$$

where

$$k = \omega/c \quad .$$

The differential operator ∇^2 here is defined by:

$$\nabla^2 \vec{\Pi}_\omega = \nabla \cdot \nabla \vec{\Pi}_\omega - \nabla \times \nabla \times \vec{\Pi}_\omega \quad (3.5)$$

In all the further formulae, I omit the subscript ω by Fourier components of the vector functions.

The fields can be found, provided $\vec{\Pi}$ and $\vec{\Pi}^*$ are known:

$$\vec{E}^{(1)} = \nabla \times \nabla \times \vec{\Pi} \quad (3.6)$$

$$\vec{H}^{(1)} = -ik \nabla \times \vec{\Pi} \quad (3.7)$$

$$\vec{E}^{(2)} = ik \nabla \times \vec{\Pi}^* \quad (3.8)$$

$$\vec{H}^{(2)} = \nabla \times \nabla \times \vec{\Pi}^* \quad (3.9)$$

Vectors $\vec{E}^{(1,2)}$ and $\vec{H}^{(1,2)}$ also satisfy equation analogous to (3.3).

In coordinates ρ, θ, σ and with the metric coefficients (2.2), (2.3) the projections of the equation (3.3) on axes $\vec{i}_1$ have the following form (all the appropriate formulae for differential operators, in these coordinates one can find in the Appendix):

$$\frac{1}{h^2}\frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\Pi_1 + \frac{1}{\rho^2 h^2}\frac{\partial^2\Pi_1}{\partial\theta^2} + \frac{1}{h^2}\frac{\partial}{\partial\sigma}\frac{1}{h}\frac{\partial}{\partial\sigma}h\Pi_1$$

$$+ k^2\Pi_1 + \frac{1}{h^2}\frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial\Pi_2}{\partial\theta} - \frac{1}{\rho^2 h^2}\frac{\partial}{\partial\rho}\rho\frac{\partial\Pi_2}{\partial\theta}$$

$$+ \frac{1}{h^4}\frac{\partial^2}{\partial\sigma\partial\rho}h^2\Pi_3 - \frac{1}{h^2}\frac{\partial^2\Pi_3}{\partial\sigma\partial\rho} = 0 \qquad (3.10)$$

$$\frac{1}{h^2}\frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\Pi_2 + \frac{1}{\rho^2 h^2}\frac{\partial^2\Pi_2}{\partial\theta^2} + \frac{1}{h^2}\frac{\partial}{\partial\sigma}\frac{1}{h}\frac{\partial}{\partial\sigma}h\Pi_2$$

$$+ k^2\Pi_2 + \frac{1}{h^2\rho^2}\frac{\partial^2\rho\Pi_1}{\partial\theta\,\partial\rho} - \frac{1}{h^2}\frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial\Pi_1}{\partial\theta}$$

$$+ \frac{1}{\rho h^4}\frac{\partial}{\partial\sigma}h^2\frac{\partial\Pi_3}{\partial\theta} - \frac{1}{\rho h^2}\frac{\partial^2\Pi_3}{\partial\sigma\partial\theta} = 0 \qquad (3.11)$$

$$\frac{1}{\rho h^2}\frac{\partial}{\partial\rho}\rho\frac{\partial\Pi_3}{\partial\rho} + \frac{1}{\rho^2 h^2}\frac{\partial^2\Pi_3}{\partial\theta^2} + \frac{1}{h}\frac{\partial}{\partial\sigma}\frac{1}{h^3}\frac{\partial}{\partial\sigma}h^2\Pi_3$$

$$+ k^2\Pi_3 + \frac{1}{\rho h}\frac{\partial}{\partial\sigma}\frac{1}{h}\frac{\partial}{\partial\rho}\rho\Pi_1 - \frac{1}{\rho h^3}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\sigma}h\Pi_1$$

$$+ \frac{1}{h\rho}\frac{\partial}{\partial\sigma}\frac{1}{h}\frac{\partial\Pi_2}{\partial\theta} - \frac{1}{\rho h^3}\frac{\partial^2}{\partial\theta\partial\sigma}h\Pi_2 = 0 \qquad (3.12)$$

The same system of equations holds also for the vector $\vec{\Pi}^*$.

IV. Cylindrically Symmetric Field

It is useful to find first the solution of the Maxwell equation for cylindrically symmetric case when neither $\vec{E}$ nor $\vec{H}$ depend on θ. From physical considerations it is clear that in this case, $\vec{H}$ should have only one component, H_2. The projections of the equation $\nabla^2\vec{H} + k^2\vec{H} = 0$ on the axes, $\vec{i}_1$ and $\vec{i}_3$, are zero. The second component gives:

$$\frac{\partial}{\partial\sigma}\frac{1}{h}\frac{\partial}{\partial\sigma}hH_2 + \frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho H_2 + k^2h^2H_2 = 0 \qquad (4.1)$$

The variables in this equation can be separated, and the solution is easy to find

$$H_1 = 0,\quad H_2 = -ik\lambda\, J_1(\lambda\rho)\, F_0(\sigma),\quad H_3 = 0 \qquad (4.2)$$

where $J_1(\lambda\rho)$ is the first order Bessel function of the first kind, λ is a separation constant, and $F_0(\sigma)$ is a solution of the following equation:

$$F_0'' + \frac{h'}{h}F_0' + F_0\left(\frac{h''}{h} - \frac{h'^2}{h^2} + k^2h^2 - \lambda^2\right) = 0 \qquad (4.3)$$

Prime here means differentiation with respect to σ. By means of the Maxwell equation, $\vec{E}=(i/k)\nabla\times\vec{H}$, one finds now the electric field:

$$E_1 = -\lambda J_1(\lambda\rho)\frac{(hF_0)'}{h^2},\quad E_2 = 0,\quad E_3 = \lambda_0^2 J_0(\lambda\rho)\frac{F_0}{h} \qquad (4.4)$$

J_0, J_1 here are the zero and the first order Bessel functions. The fields (4.2) and (4.4) satisfy also the boundary conditions.

$$E_2(\rho_{max}) = E_3(\rho_{max}) = H_2(\rho_{max}) = 0\ , \qquad (4.5)$$

if we choose the constant, λ, such that

$$J_0(\lambda\rho_{max}) = 0 \qquad (4.6)$$

We get infinite sequence of λ_{0k}:

$$\lambda_{0k} = \nu_{0k}/\rho_{max}\ , \qquad (4.7)$$

where ν_{0k} is the kth root of equation $J_0(x) = 0$.

V. TM Modes

We are now ready to solve equations (3.10)-(3.12). To do this, we look for the solution in the following form:

$$\Pi_1 = J_m'(\lambda\rho)\ F_1(\sigma)\ e^{im\theta} \qquad (5.1)$$

$$\Pi_2 = im\frac{J_m(\lambda\rho)}{\rho}\ F_2(\sigma)\ e^{im\theta} \qquad (5.2)$$

$$\Pi_3 = \lambda J_m(\lambda\rho)\ F_3(\sigma)\ e^{im\theta} \qquad (5.3)$$

A prime from now on means the differentiation with respect to ρ if it is applied to a function of ρ or with respect to σ if it is applied to a function of σ. $J_m(\lambda\rho)$ is the mth order Bessel function, λ is an arbitrary constant, and m is any positive or negative integer including zero.

The function F_1, F_2, F_3 is unknown up to this time and should be chosen in such a way that the functions (5.1)-(5.3) satisfy equations (3.10)-(3.12).

Substituting (5.1)-(5.3) into (3.10)-(3.12) we find, after some algebra, that the first two equations give $F_1 = F_2$, where F_1 is a solution of the following equation:

$$\frac{d}{d\sigma}\frac{1}{h}\frac{1}{d\sigma}hF_1 + (k^2h^2 - \lambda^2)\ F_1 + 2\frac{h'}{h}\lambda F_3 = 0 \qquad (5.4)$$

The third equation gives:

$$h\frac{d}{d\sigma}\frac{1}{h^3}\frac{d}{d\sigma}h^2F_3 + (k^2h^2 - \lambda^2)\ F_3 + \frac{2h'}{h}\lambda F_1 = 0 \qquad (5.5)$$

Equations (5.4) and (5.5) constitute a system of two coupled second-order ordinary differential equations for two functions, F_1 and F_3.

The Hertz vector $\vec{\Pi}$ (5.1)-(5.3) allows one to find the electric $\vec{E}^{(1)}$ and magnetic $\vec{H}^{(1)}$ fields from expressions (3.6) and (3.7). Using the expressions for differential operators in our coordinates ρ, θ, σ, (see Appendix) one gets:

$$E_1^{(1)} = J_m'\frac{(hF)'}{h^2}e^{im\theta} \qquad (5.6)$$

$$E_2^{(1)} = im\frac{J_m}{\rho}\frac{(hF)'}{h^2}e^{im\theta} \qquad (5.7)$$

$$E_3^{(1)} = \lambda^2 J_m\frac{F}{h}e^{im\theta} \qquad (5.8)$$

$$H_1^{(1)} = mk\frac{J_m}{\rho}F\,e^{im\theta} \qquad (5.9)$$

$$H_2^{(1)} = ik\,J_m' F\,e^{im\theta} \qquad (5.10)$$

$$H_3^{(1)} = 0 \qquad (5.11)$$

The function, F, is defined as follows;

$$F = \left(\lambda h F_3 - (hF_1)'\right)'/h^2 \tag{5.12}$$

First of all, we see that the third component of the magnetic field, $\vec{H}^{(1)}$ is zero. Analogously to cylindrical coordinates we can call the fields $\vec{E}^{(1)}$, $\vec{H}^{(1)}$ transverse magnetic modes (TM modes). One should remember, of course, that the field $\vec{H}^{(1)}$ is transverse to the direction $\vec{1}_3$, but it is not transverse to the axis of the waveguide $\vec{1}_z$.

Further, all the factors in $\vec{E}^{(1)}$, $\vec{H}^{(1)}$ depending on σ do not depend on m (see equations (5.4) and (5.5)). That means that the function F in expressions (5.6)-(5.11), which are valid also for the case $m = 0$, must be the same as the function F_0 in expressions (4.2), (4.4). (All the other factors coinside in corresponding expressions.) Hence $F = F_0$ and must satisfy equation (4.3). Indeed, substituting (5.12) into (4.3) after some algebra and with the help of equations (5.4) and (5.5) one finds that it is the case.

The field $\vec{E}^{(1)}$, $\vec{H}^{(1)}$ should satisfy also the boundary conditions (4.5). That can be achieved by proper choosing of the value of the constant λ. Namely, it implies:

$$J_m(\lambda\rho_{max}) = 0 \tag{5.13}$$

or

$$\lambda_{mk} = \nu_{mk}/\rho_{max} \tag{5.14}$$

where ν_{mk} is the kth root of equation $J_m(x) = 0$.

VI. TE Modes

The second independent set of electromagnetic fields inside the waveguide can be obtained from the same vector $\vec{\Pi}$ (5.1)-(5.3) by means of equations (3.8), (3.9). In this case, one gets:

$$E_1^{(2)} = mk \frac{J_m}{\rho} F e^{im\theta} \tag{6.1}$$

$$E_2^{(2)} = ik J_m' F e^{im\theta} \tag{6.2}$$

$$E_3^{(2)} = 0 \tag{6.3}$$

$$H_1^{(2)} = J_m' \frac{(hF)'}{h^2} e^{im\theta} \tag{6.4}$$

$$H_2^{(2)} = im \frac{J_m}{\rho} \frac{(hF)'}{h^2} e^{im\theta} \tag{6.5}$$

$$H_3^{(2)} = \lambda^2 J_m \frac{F}{h} e^{im\theta} \tag{6.6}$$

We can call these modes transverse electric modes (TE modes) since $E_3^{(2)} = 0$. To satisfy the boundary conditions for the TE modes, the value of the constant λ should be chosen in such a way that

$$J_m'(\lambda\rho_{max}) = 0 \tag{6.7}$$

or

$$\bar{\lambda}_{mk} = \bar{\nu}_{mk}/\rho_{max} \quad , \tag{6.8}$$

where $\bar{\nu}_{mk}$ is the kth root of equation $J_m'(x) = 0$.

VII. Propagation Function

Let us look more closely on the σ-dependence of the fields $\vec{E}$ and $\vec{H}$. It is described by the function F which satisfied equation (4.3) (the subscript 0 can be omitted from now on). Since, according to expression (2.6), z is the function of σ only, we can come back from the σ variable to the z variable,

$$dZ = h d\sigma \tag{7.1}$$

$$h' = h \frac{dh}{dz} \tag{7.2}$$

It is convenient at the same time to change the function $F(\sigma)$ to a new function u:

$$u(z) = F(\sigma) \cdot h(\sigma) \quad , \tag{7.3}$$

where it is understood that in the right-hand side of expression (7.3) σ should be substituted by the function $\sigma(z)$ from (2.6). Then, from the equation (4.3) we get the following equation for the function u:

$$d^2u/dz^2 + \left(k^2 - \nu^2/r_b^2\right) u = 0 \quad , \tag{7.4}$$

where $r_b = r_b(z)$ is the boundary curve. For TM modes, we get now:

$$E_1^{(1)} = \nu J_m' \frac{u'}{r_b} e^{im\theta} \quad , \tag{7.5}$$

$$E_2^{(1)} = im \frac{J_m}{r} u' e^{im\theta} \quad , \tag{7.6}$$

$$E_3^{(1)} = \nu^2 \frac{J_m}{r_b^2} u e^{im\theta} \quad , \tag{7.7}$$

$$H_1^{(1)} = mk \frac{J_m}{r} u e^{im\theta} \quad , \tag{7.8}$$

$$H_2^{(1)} = ik \frac{\nu}{r_b} J_m' u e^{im\theta} \quad , \tag{7.9}$$

$$H_3^{(1)} = 0 \quad , \tag{7.10}$$

and analogous expressions for TE modes. In expressions (7.4)-(7.10) $J_m = J_m(\nu r/r_b)$, $\nu = \nu_{mk}$ (See 5.14), $u = u(z)$ and prime means derivative over whole argument of corresponding function.

In the limit of the constant cross section waveguide $r_b = \rho_{max}$ and the solution of equation (7.4) is:

$$u = \exp\left(\pm \sqrt{k^2 - \lambda^2}\, z\right) \tag{7.11}$$

We can call u(z) a propagation function. When $r_b(z)$ is a periodic function, equation (7.4) becomes the Hill equation. The solution of it defines positions and widths of stopbands as well as the phase velocity of the wave in passbands. More detailed investigation of the behavior of the propagation function can be done only for a given function $r_b(z)$.

VIII. Examples

I present here two examples of the application of the derived theory. In the first one, the propagation function u(z) (See Section 7) is calculated for the wave guide with the following boundary:

$$r_b = b(1 + a\cos 2\pi z/L) \quad , \tag{8.1}$$

where b is the average radius of the pipe's cross section, a and L are the amplitude and the period of the boundary variation. The applicability of the theory limits the values of b and L by the condition $\rho_{max} = 2\pi b/L < 1$.

The general solution of equation (7.4) in regions of its existance ("passbands") can be expressed in terms of the corresponding β-function of this equation [6]:

$$u(z) = u_0\sqrt{\beta(z)}\, e^{i\phi(z) - i\omega t} \quad , \tag{8.2}$$

where

$$\phi(z) = \int_0^{2\pi z/L} d\xi/\beta(\xi) \qquad (8.3)$$

and the function $\zeta(z) = \sqrt{\beta(z)}$ is the solution of the following nonlinear equation:

$$\zeta'' + \left(k^2 - \nu^2/r_b^2\right)\zeta = 1/\zeta^3 \quad . \qquad (8.4)$$

Outside of the passbands there is no solution of equation (7.4) ("stopbands"). The electromagnetic wave with corresponding frequencies (or values k) can not propagate in the waveguide under consideration.

Inside one of the passbands we can determine the phase velocity v_{ph} of the wave. From (8.2) one finds

$$v_{ph}/c = (kL/2\pi)\ \beta(z) \quad . \qquad (8.5)$$

The phase velocity is not constant, but it is modulated with the frequency $2\pi/L$.

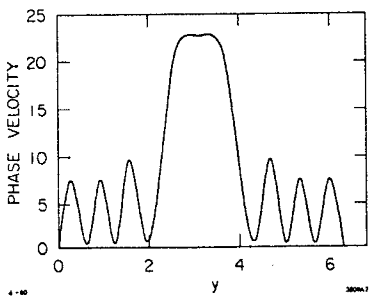


Fig. 2. The phase velocity of the TM_{01} (ν=2.405) mode versus longitudinal coordinate $y=2\pi z/L$. The waveguide parameters are ρ_{max}=0.3, a=0.111. The parameter K=9.02 (proportional to the field frequency ω) is chosen on the left side of the stopband (compare Fig. 8). v_{av} is the average relative phase velocity $\langle v_{ph}/c\rangle$=8.30.

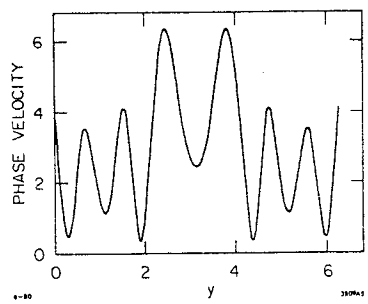


Fig. 5. The same as on Fig. 4, but for the value of K in between the stopbands (see Fig. 9). K=4.60, v_{av}=2.90.

Figures 2-7 illustrate the dependence of the phase velocity versus the longitudinal coordinate $y=2\pi z/L$ for different values of the parameters $K=kL/2\pi$, $\rho_{max}=2\pi b/L$ and a. For some problems the average (over z) phase velocity is of interest. This quantity is relevant for example to a problem of the coupling between the electromagnetic wave propagating in the waveguide and a particle moving along its axis. The relative average phase velocity $\langle v_{ph}/c\rangle$ is also given on these figures.

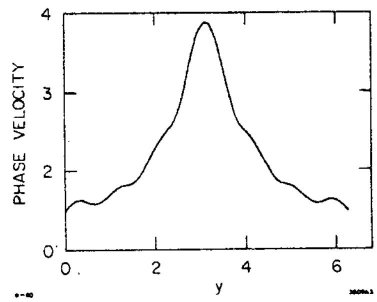


Fig. 3. The same as on Fig. 2, but the parameter K lies above the stopband (see Fig. 8). K=9.40, v_{av}=2.22.

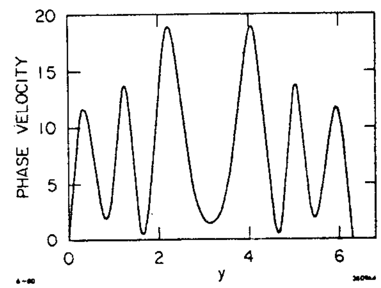


Fig. 4. The same as on Fig. 2, but for the waveguide with the parameters ρ_{max}=1.0, a=0.5. K lies below both stopbands (see Fig. 9). K=4.23, v_{av}=7.75.

Figs. 8-10 show dependencies of the average phase velocity on the parameter K as well as positions and widths of the stopbands.

The case ρ_{max}=2.87 is far outside the validity of the described method. It was calculated here for the sake of the comparison with known results. The crosses on Fig. 10 represent the results of calculations according to second order perturbation theory and of the measurements and are taken from the work [2].

Fig. 6. The same as on Fig. 4 but for the value of K above both stopbands (see Fig. 9). K=5.21, v_{av}=1.46.

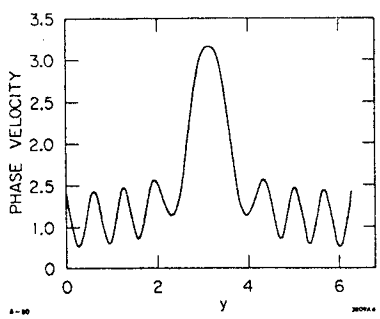


Fig. 7. The same as on Fig. 2 but for the waveguide with the parameters ρ_{max}=2.87, a=0.111. K is above the stopband shown on Fig. 10. K=1.03, v_{av}=3.04.

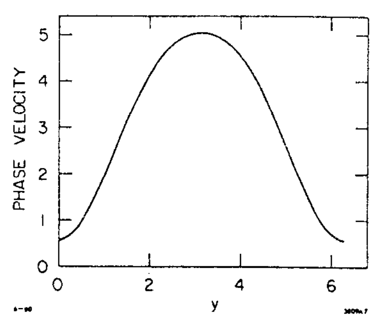


Fig. 8. The average relative phase velocity of the TM_{01} mode versus parameter $K=kL/2\pi$. The waveguide parameters are ρ_{max}=0.3, a=0.111. The position and the width of the stopband is also shown.

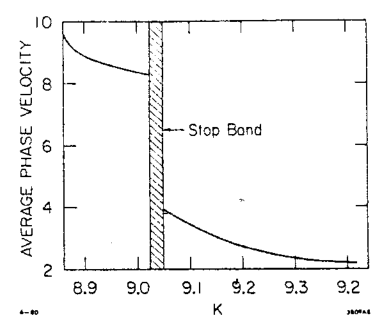
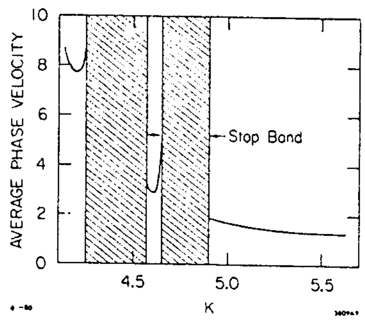


Fig. 9. The same as on Fig. 8 but for the waveguide with the parameters $\rho_{max} = 1.0$, $a = 0.5$. One can interpret this picture as widening of the stopband with increasing amplitude of the boundary modulation and appearing of a narrow passband inside it.



As the second example, I present here the approximate solution of the Maxwell equations for the cavity buildout of two hyperboloid surfaces:

$$r_b = b^2 / |z| \tag{8.6}$$

Equation (7.4) in this case looks like:

$$u'' + \left(k^2 - \frac{\nu^2}{b^4} z^2\right) u = 0 \tag{8.7}$$

This equation describes the wave function of a quantum mechanic harmonic oscillator and its solution is well known. The fields fade away at $|z| \to \infty$ only for the following values of the parameter k:

$$k_n = \sqrt{(2n+1)\nu}\,/b\ , \quad n = 0, 1, 2 \ldots \tag{8.8}$$

The corresponding solution $u_n(z)$ of (8.7) is

$$u_n = u_0\, e^{-\xi^2/2}\, H_n(\xi) \tag{8.9}$$

where $\xi = (\sqrt{\nu}) z/b$, H_n is the Hermite polynome of the nth order.

The main assumption $\rho_m < 1$ is not fulfilled here since $\rho_{max} = 1$ for (8.6). Hence the solution (8.8), (8.9) is only an approximate one.

Acknowledgments

I am grateful to Phil Morton, Alex Chao, Perry Wilson, Karl Bane and C. Yao for helpful discussions and criticism. The latter two helped to overcome statemates in the calculations.

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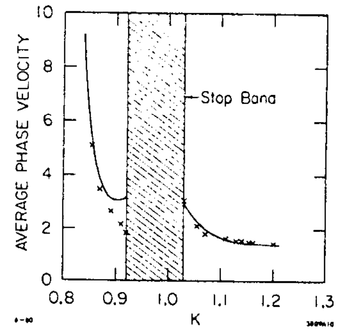
Fig. 10. The same as on Fig. 8 but for the waveguide with the parameters $\rho_{max} = 2.87$, $a = 0.111$. The crosses represent the results of the calculations and measurements of the work [2]. Although this case is outside the applicability region of this work method, the correspondence is not at all bad.



Appendix

Differential operators in coordinates ρ, θ, σ.
For any vector function $F(\rho, \theta, \sigma) = F_1\vec{i}_1 + F_2\vec{i}_2 + F_3\vec{i}_3$:

$$\nabla\vec{F} = \frac{1}{\rho h^3}\left[h^2 \frac{\partial}{\partial\rho}(\rho F_1) + h^2 \frac{\partial}{\partial\theta} F_2 + \rho \frac{\partial}{\partial\sigma}(h^2 F_3)\right] \tag{A.1}$$

$$(\nabla\times\vec{F})_1 = \frac{1}{\rho h^2}\left[h \frac{\partial F_3}{\partial\theta} - \rho \frac{\partial}{\partial\sigma}(hF_2)\right] \tag{A.2}$$

$$(\nabla\times\vec{F})_2 = \frac{1}{h^2}\left[\frac{\partial}{\partial\sigma}(hF_1) - h \frac{\partial F_3}{\partial\rho}\right] \tag{A.3}$$

$$(\nabla\times\vec{F})_3 = \frac{1}{\rho h}\left[\frac{\partial}{\partial\rho}(\rho F_2) - \frac{\partial F_1}{\partial\theta}\right] \tag{A.4}$$

$$(\nabla^2\vec{F})_1 = \frac{1}{h}\frac{\partial}{\partial\rho}\left\{\frac{1}{h\rho}\frac{\partial}{\partial\rho}(\rho F_1) + \frac{1}{\rho h}\frac{\partial F_2}{\partial\theta} + \frac{1}{h^3}\frac{\partial}{\partial\sigma}(h^2 F_3)\right\} - \frac{1}{\rho h^2}\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial F_2}{\partial\theta}\right) - \frac{1}{\rho}\frac{\partial^2 F_1}{\partial\theta^2} - \rho\frac{\partial}{\partial\sigma}\frac{1}{h}\frac{\partial}{\partial\sigma}(hF_1) + \rho\frac{\partial^2 F_3}{\partial\sigma\partial\rho}\right\} \tag{A.5}$$

$$(\nabla^2\vec{F})_2 = \frac{1}{\rho h}\frac{\partial}{\partial\theta}\left\{\frac{1}{h\rho}\frac{\partial}{\partial\rho}(\rho F_1) + \frac{1}{\rho h}\frac{\partial F_2}{\partial\theta} + \frac{1}{h^3}\frac{\partial}{\partial\sigma}(h^2 F_3)\right\} - \frac{1}{h^2}\left\{\frac{1}{\rho}\frac{\partial^2 F_3}{\partial\sigma\partial\theta} - \frac{\partial}{\partial\sigma}\frac{1}{h}\frac{\partial}{\partial\sigma}(hF_2) - \frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_2) + \frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial F_1}{\partial\theta}\right\} \tag{A.6}$$

$$(\nabla^2\vec{F})_3 = \frac{1}{\rho h}\frac{\partial}{\partial\theta}\left\{\frac{1}{h\rho}\frac{\partial}{\partial\rho}(\rho F_1) + \frac{1}{\rho h}\frac{\partial F_2}{\partial\theta} + \frac{1}{h^3}\frac{\partial}{\partial\sigma}(h^2 F_3)\right\} - \frac{1}{h^2}\left\{\frac{1}{\rho}\frac{\partial^2 F_3}{\partial\sigma\partial\theta} - \frac{\partial}{\partial\sigma}\frac{1}{h}\frac{\partial}{\partial\sigma}(hF_2) - \frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_2) + \frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial F_1}{\partial\theta}\right\} \tag{A.7}$$

For a scalar function $\phi(\rho, \theta, \sigma)$

$$\nabla\phi = \frac{\vec{i}_1}{h}\frac{\partial\phi}{\partial\rho} + \frac{\vec{i}_2}{\rho h}\frac{\partial\phi}{\partial\theta} + \frac{\vec{i}_3}{h}\frac{\partial\phi}{\partial\sigma} \tag{A.8}$$