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FROZEN AND BROKEN COLOR: A MATRIX SCHROEDINGER EQUATION
IN THE SEMICLASSICAL LIMIT*

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ABSTRACT

We consider a simple one-dimensional theory in which a colored spinless quark and antiquark are bound together by a confining, color-dependent potential. Our purpose is to investigate in more detail the dynamics underlying Lipkin's mechanism of hidden charge, and how his conclusions are modified in the presence of symmetry breaking.

We consider the case of "frozen color", i.e. where global color symmetry remains exact, but where colored states have a mass large compared to color-singlet mesons. Using semiclassical WKB formalism, we construct the spectrum of bound states. In order to determine the charge of the constituents, we then consider deep-inelastic scattering of an external probe (e.g. lepton) from our one-dimensional meson. We calculate explicitly the structure function, W , in the WKB limit and show how Lipkin's mechanism is manifested, as well as how scaling behavior comes about. The dominant physical process is one of excitation of a semiclassical state by the hard collision of the probe with the quark or antiquark.

We generalize these considerations to the case of broken color symmetry - but where the breaking is not so strong as to allow low-lying states to have a large amount of mixing with the colored states. In this case, the degeneracy of excited colored states can be broken. The WKB approximation again suffices to provide a description of the spectra. Again deep-inelastic scattering can be used to measure the charges of the constituents, and there will again be a distinct contribution from each type of "classical" state which can be excited by

the external probe. However, in the general case, the charge measured via excitation of a given state can be energy-dependent.

We find that local excitation of color guarantees global excitation of color; i.e. if at a given energy colored semiclassical states can be constructed with size comparable to that of the ground state wave function, colored states of that energy will also exist in the spectrum of the full theory and will be observed. However, global existence of color does not guarantee the excitation of colored-states via deep-inelastic processes: there may be no overlap of the wave functions of these colored states with the ground state wave function.

Finally, even in the absence of a direct physical application, we have examined how to implement the WKB method for bound-state problems in the presence of internal degrees of freedom. The methodology we have given may be of use in other semiclassical problems which have internal degrees of freedom.

I. INTRODUCTION

It is now generally believed that strongly interacting matter is composed of fractionally charged constituents. Alternatives to this view, notably those based on the Han-Nambu three-triplet model with integer charge quarks[1,2], have run into difficulty. As a consequence of the presumed integer charge of the constituent partons, a naive form of the three-triplet model predicted a ratio of the deep-inelastic electroproduction cross section to the corresponding neutrino induced cross section that is nearly a factor of two larger than that predicted by the fractional charge model. Experimental results conform the predictions of the latter.[3]

However, Lipkin[4] pointed out that, provided all presently observed hadron states are pure color singlets, the ratio predicted by the three-triplet theory reverts to that predicted by the fractionally charged model. This happens because the color-octet portion of the electromagnetic current (which distinguishes the two models in electromagnetic reactions) has zero matrix elements between such physical states. Subsequently Pati and Salam[5], as well as Rajasekharan and Roy[6], pointed out that if the Han-Nambu model were cast in the form of a (spontaneously broken) gauge theory, the color symmetry could be broken without effecting a large change in the electroproduction cross section. The contribution to the cross section from the electroproduction of colored states by virtual photon exchange destructively interferes with that from the exchange of a colored gauge boson (which mixes with the bare photon and therefore has a nonvanishing coupling to

the electron). These contributions give complete cancellation in the limit of momentum transfer large compared to the colored gauge-boson-mass. However, this means that this colored gauge boson \tilde{U} must not be too massive, and that it have a predictable leptonic width - in fact a rather large one. Electron-positron colliding-beam experiments rather decisively rule out this possibility.[7] Meanwhile, de Rujula, Giles, and Jaffe[8] suggested that the color symmetry of the fractional-charged quark model, as based on quantum chromodynamics (QCD), might be mildly broken, leading to unconfined, colored states of large mass. They argued, in fact, that as the symmetry-breaking (nonvanishing gluon mass, induced by some kind of Higgs mechanism) decreased, the mass and size of physical colored states would increase, tending to infinity as the symmetry was restored. Pati and Salam[9] invoked this mechanism ("Archimedes effect") to argue that the colored gluon \tilde{U} could have a very large physical mass without disrupting the color-cancellation mechanism they had previously proposed. This argument has been challenged by Okun, Voloshin, and Zakharov[10], who showed that if the color symmetry breaking is "soft" (important only in the infrared), then in the Han-Nambu integer charge quark scheme there will be observable effects. This happens even in pure leptonic processes such as $e^+e^- \rightarrow \mu^+\mu^-$, where at moderate energies modifications occur which are comparable in importance to the unacceptable \tilde{U} boson.

In addition, Chanowitz[11] has analyzed η and η' radiative decays, processes which are especially sensitive to the distinction between fractional and integer charge constituents. He concludes that fractional charge is rather strongly favored.

Thus the case for integer-charged quarks, whose full charge is hidden via the Lipkin mechanism of absence of colored states in the low-energy spectrum, looks very unlikely. Nevertheless, there remain large uncertainties surrounding the mechanism of quark confinement. For example, there is no general understanding of the nature of Higgs-broken QCD - even for fractionally charged quarks - when the Higgs-boson mass is less than or comparable to the confinement scale. Under such circumstances is confinement destroyed?[12] If there exist colored states, do their masses become large, a la de Rujula, Giles, and Jaffe, as the symmetry-breaking tends to zero?[13]

Therefore, there may be some reason remaining for studying the questions of broken color symmetry and/or hidden color degrees of freedom. And, even given that the likelihood of a direct application to the physics of quarks and hadrons is not large, it still remains the case that the quantum mechanics of the problem is interesting.

In this paper, following Lipkin, we begin with a very simple example of a one-dimensional theory in which a colored spinless quark and antiquark are bound together by a confining, color-dependent potential into either a color singlet or a color octet state.[14]. Our purpose is to investigate in more detail the dynamics underlying Lipkin's mechanism of hidden charge, and how his conclusions are modified in the presence of symmetry-breaking.

We consider in Section II the case of "frozen color", i.e. where global color symmetry remains exact, but where colored states have a mass large compared to color-singlet mesons. Using a semiclassical WKB formalism, we construct the spectrum of bound states. In order to

determine the charge of the constituents, we then consider deep-inelastic scattering of an external probe (e.g. lepton) from our one-dimensional meson. We calculate explicitly the structure function W in the WKB limit and show how Lipkin's mechanism is manifested, as well as how scaling behavior comes about. The dominant physical process is one of excitation of a semiclassical state by the hard collision of the probe with the quark or antiquark.

In Section III, we generalize these considerations to the case of broken color symmetry - but where the breaking is not so strong as to allow low-lying states to have a large amount of mixing with the colored states. In this case, the degeneracy of excited colored states can be broken. The WKB approximation again suffices to provide a description of the spectra. Again deep-inelastic scattering can be used to measure the charges of the constituents, and there will again be a distinct contribution from each type of "classical" state which can be excited by the external probe. However, in the general case, the charge measured via excitation of a given state can be energy-dependent.

In Section IV, we summarize our conclusions. We find that local excitation of color guarantees global excitation of color; i.e. if at a given energy colored semiclassical states can be constructed with size comparable to that of the ground state wave function, colored states of that energy will also exist in the spectrum of the full theory[15] and will be observed. However, global existence of color does not guarantee the excitation of colored-states via deep-inelastic

processes: there may be no overlap of the wave functions of these colored states with the ground state wave function.

II. FROZEN COLOR

We begin this section by introducing a simple example which will motivate, and later illustrate, our discussion of inelastic scattering from a target with frozen color degrees of freedom. The qualitative observations made for our simple example will then be made quantitative in a more general context.

The target particle for our example will be a meson whose states may be characterized by solutions of an equation involving only a single spacial dimension. We assume that these mesons are bound states of integer charge quark and antiquark via a Hamiltonian which allows color nonsinglet as well as color singlet states. For our quark-anti-quark system, this implies color-octet as well as color-singlet states. We will here consider the simplest situation, where (global) SU(3) color symmetry is unbroken.

For concreteness, consider the ordinary π^+ . We may write this using SU(3) states for flavor and color as $\pi_{\text{flavor}}^+ \times \eta'_{\text{color}} \equiv (\pi^+, \eta')$. This color singlet state is now supplemented by a (presumably more massive) degenerate octet $(\pi^+, \underline{8})$ as shown in Fig. 1. Other mesons will have a similar associated octet. Under our assumption that the Hamiltonian is invariant under rotations in color space, the $(\pi^+, \underline{1})$ and $(\pi^+, \underline{8})$ type of states will each have its own (confining) Hamiltonian, $H_1(P, x)$ and $H_8(P, x)$, ($P \equiv -i\hbar \frac{d}{dx}$). We will then have a level structure which can be sketched as shown in Fig. 2.

We can immediately qualitatively describe some factors which will determine the charge observed in deep inelastic scattering. The charge

observed will be determined by a matrix element of an operator, associated with charge, between initial and final states. Our first observation was made by Lipkin; if a piece of this operator transforms the initial state to one orthogonal (in internal space) to the final state, the corresponding charge will not be observed. Because of this, the Han-Nambu and fractional charge quark models will give the same charges for scattering from color-singlet to color-singlet states. We thus would not expect to see the charge associated with the octet states if we are exciting the target to an energy less than the minimum of the octet "potential." In addition to this energy, we would need an energy sufficient to excite the first level of the octet states. These statements describe what may be called color freezing.

We can also slightly extend this picture to include situations where the octet potential might have, for example, a repulsive core. In that case the overlap between the spacial parts of the initial and final wave functions may be so small that those octet states will contribute little to the observed charge, even well above the kinematic threshold.

We will now proceed to discuss these statements in a more quantitative way. In doing so, we will also describe scaling properties of the scattering cross section. Our discussion so far has indicated the need to find the spectrum of states and transition matrix elements of the target system. We will use a WKB approximation to determine these in terms of properties of the classical Hamiltonian corresponding to the quantum target Hamiltonian. Since it costs nothing, we will set up

the problem in an abstract setting. We assume the target consists of a pair of charged particles, whose physics can be described (by factoring out the center of mass) by an equivalent one-body wave function of a single variable. This $\underline{\Psi}(x)$ is vector valued with respect to the internal coordinates of the target.

We assume the binding of the target is described by a general diagonal matrix Hamiltonian:

$$\underline{H}(p,x) = \begin{bmatrix} H^{(1)}(p,x) & & & \\ & \cdot & & \\ & & \cdot & 0 \\ & 0 & & \cdot \\ & & & & H^{(L)}(p,x) \end{bmatrix} \quad (2.1)$$

where for all N , $H^{(N)}(p,x)$ is a "monotonic" infinite well with respect to p and x . Inserting the momentum operator $P = -i\hbar \frac{d}{dx}$, we may write the Schroedinger equation for $\underline{\Psi}(x)$:

$$\underline{H}(P,x)\underline{\Psi}(x) = E\underline{\Psi}(x) . \quad (2.2)$$

We emphasize that, except for the "well-like" restriction on $\underline{H}(p,x)$, it is completely general, e.g. not being restricted to the form $(p^2/2m)\underline{I} + \underline{V}(x)$.

The solution of this equation trivially reduces to the solution of the scalar equation

$$H^{(N)}(P,x)\psi^{(N)}(x) = E\psi^{(N)}(x) . \quad (2.3)$$

We hereafter drop the superscripts as superfluous until needed.

We have, in a companion paper[16], determined a semiclassical approximation for the solutions of this equation. They will be some linear combination (to be determined by turning point conditions) of solutions given as follows: $p_{\pm}(x,E) \equiv p_{\pm}(x)$ ($p_- < p_+$) are solutions of the classical equation $H(p_{\pm}(x),x) = E$. Also, $v_{\pm} \equiv (\partial H/\partial p)|_{p=p_{\pm}}$.

Then our approximate solutions are

$$\psi_{\pm}(x) = \frac{e^{\frac{i}{\hbar} \int_0^x p_{\pm}(x') dx'}}{\sqrt{v_{\pm}^{(N)}(x)}}. \quad (2.4)$$

As we shall show in our appendix, the full WKB solution will be

$$\psi(x) = \frac{e^{\frac{i}{\hbar} \int_0^x p_+(x') dx'}}{\sqrt{v_+(x)}} A_+ + \frac{e^{\frac{i}{\hbar} \int_0^x p_-(x') dx'}}{\sqrt{v_-(x)}} A_- \quad A_- \equiv e^{i\delta_R} A_+ \quad (2.5)$$

where the right-hand turning point (at $x=B$) condition tells us

$$A_- = e^{-\frac{i\pi}{2} \frac{i}{\hbar} \int_B^0 p_-(x') dx'} \frac{i}{\hbar} \int_0^B p_+(x') dx' A_+ \quad (2.6)$$

and the left-hand turning point (at $x=-B'$) condition tells us

$$A_- = e^{+\frac{i\pi}{2} \frac{i}{\hbar} \int_{-B'}^0 p_-(x') dx'} \frac{i}{\hbar} \int_0^{-B'} p_+(x') dx' A_+. \quad (2.7)$$

Equating the expressions from the right and left turning points, we obtain the Bohr-Sommerfeld formula

$$\frac{1}{\hbar} \left[\int_{-B'}^B p_+(x') dx' + \int_B^{-B'} p_-(x') dx' \right] = (2n+1)\pi. \quad (2.8)$$

This gives us the WKB approximation for the bound state energies.

Having obtained the bound state spectrum, we can obtain conditions for colored states to be excited. First, we must be at an energy above the minimum of $H^{(N)}(p,x)$. This, however, is not sufficient - if the well is very narrow, so that $\delta_N \equiv \frac{1}{\hbar} \left[\int_{-B}^B p_+(x') dx' + \int_B^{-B} p_-(x') dx' \right]$ slowly depends on energy, we might have to go up quite a bit higher than $H_{\min}^{(N)}(p,x)$ to excite the first colored state. Of course, both these observations come from misusing the WKB approximation to apply to the ground state of the colored channel. However, they do identify qualitative features we would expect to see, and can be used quantitatively for rough estimates.

With this formalism, we may now describe the bound-state spectrum for the colored "mesons" discussed earlier. For definiteness, we will again consider the flavor π^+ . We can now consider the spectrum in terms of the areas enclosed by classical orbits in phase space. We will have a nondegenerate singlet orbit and a degenerate octet orbit for a fixed energy. For a linear potential and relativistic kinematics, the orbit will be diamond shaped (Fig. 3a) while for a linear potential and nonrelativistic kinematics, we have a concatenation of two parabolas (Fig. 3b). In these figures, we have assumed the simplest mechanism for color freezing - that while the ranges of the potentials are comparable (e.g. "strings" of gluon field), the minimum of the octet Hamiltonian is much greater than the minimum of the singlet Hamiltonian (perhaps this is due to the extra self-energy associated with sources).

We may also consider the case where the color octet potential is of very short range. We could even have the minimum of the octet

Hamiltonian approximately that of the singlet Hamiltonian, in which case the absence of low-lying colored states is only a consequence of the additional zero-point energy. In such a case, the level density of colored states would be low. The color octet orbits would then enclose much less phase-space than color singlet orbits, even though they will have a similar extent in momentum space (Fig. 4).

Having gleaned what we can from the spectrum, we will consider excitation of these states using our WKB approximation. A direct way of exploring the internal structure of a bound system of the type we have been studying is to scatter a weakly interacting probe from it [17]. Determining the structure of the system means localizing the charge in space and time ("in time" means to measure the instantaneous charge distribution). Space localization requires the interaction of target and probe to have high momentum (hence high energy), while time localization requires poor energy resolution.

The classic example of such scattering is that of electron scattering from atoms, nuclei or nucleons. The most important component of this interaction is the instantaneous Coulomb force. If we use the Born approximation, with plane waves for incident and scattered probe wave functions, we may reduce the expression for the differential cross-section to a product of kinematic terms, independent of the internal dynamics of the target, and the structure function W , which contains the whole message the target can communicate to us via this type of inelastic scattering. The structure function is defined to be

$$W(\vec{q}, \nu) = \sum_f \left| \langle f | \sum_n Q_n e^{i\vec{q} \cdot \vec{x}_n} | i \rangle \right|^2 \delta(\nu - (E_f - E_i)) \quad , \quad (2.9)$$

where \vec{q} and ν are the momentum and energy transferred from probe to target; Q_n and \vec{x}_n are the charge and coordinate of the n^{th} target constituent, and E_i and E_f are the energies of the initial and excited target states.

We may extend this structure function formulation to more general interactions, replacing the scalar electromagnetic charges, Q_i , by matrix (over internal space) "charges" λ_i . Examples of this already exist in neutrino interactions, which involve transition operators in flavor space. It is conceivable that future examples might be provided by broken color, where external probes might couple to non-diagonal color degrees of freedom. Thus we shall proceed with the generalized structure function

$$W(\vec{q}, \nu) = \sum_f \left| \langle f | \sum_n \lambda_n e^{i\vec{q} \cdot \vec{x}_n} | i \rangle \right|^2 \delta(\nu - \Delta E) \quad , \quad (2.10)$$

$$(\Delta E \equiv E_f - E_i) \quad .$$

Before passing on to evaluate W , we can use its general form, along with the completeness sum over states $|f\rangle$, to establish a sum rule:

$$\begin{aligned}
 \int d\nu W(\vec{q}, \nu) &= \sum_f \left[\langle i | \sum_n e^{-\frac{i\vec{q}\cdot\vec{x}_n}{\hbar}} \lambda_n^\dagger | f \rangle \langle f | \sum_m e^{\frac{i\vec{q}\cdot\vec{x}_m}{\hbar}} \lambda_m | i \rangle \int d\nu \delta(\nu - \Delta E) \right] \\
 &= \langle i | \left(\sum_n e^{-\frac{i\vec{q}\cdot\vec{x}_n}{\hbar}} \lambda_n^\dagger \right) \left(\sum_m e^{\frac{i\vec{q}\cdot\vec{x}_m}{\hbar}} \lambda_m \right) | i \rangle .
 \end{aligned} \tag{2.11}$$

We note that the cross terms are proportional to nontrivial Fourier transforms of the ground state wave function, hence they vanish as $q \rightarrow \infty$. We thus obtain the sum rule

$$\int d\nu W(q, \nu) \xrightarrow{q \rightarrow \infty} \langle i | \sum_n \lambda_n^\dagger \lambda_n | i \rangle . \tag{2.12}$$

We now assume that we can use our one-dimensional meson results in the above form for W . This is not unreasonable if we can factor the center of mass target wave function out of W into another kinematic factor and if the wave functions and exponentials (from $e^{(i/\hbar)\vec{q}\cdot\vec{x}_n}$) that are left will be parametrized by a single coordinate x . Thus we will be looking at

$$W(q, \nu) = \sum_f \left| \langle f | \lambda_1 e^{\frac{i}{\hbar}qx} + \lambda_2 e^{-\frac{i}{\hbar}qx} | i \rangle \right|^2 \delta(\nu - \Delta E) . \tag{2.13}$$

It is obvious that to evaluate this, it will be necessary to calculate the excited state wave function in the region where the ground state wave function is non-negligible.

We have seen that, in the WKB limit, solutions of the wave equation must be of the form

$$\tilde{\Psi}^{(N)}(x) = \mathcal{N}^{(N)} \left[\frac{e^{\frac{i}{\hbar} \int_0^x p_+^{(N)}(x') dx'}}{\sqrt{v_+^{(N)}(x)}} + \frac{e^{\frac{i}{\hbar} \int_0^x p_-^{(N)}(x') dx'}}{\sqrt{v_-^{(N)}(x)}} e^{i\delta^{(N)}} \right] \tilde{e}^{(N)}$$

(2.14)

($\delta^{(N)} \equiv \delta_R^{(N)}$)

where $\tilde{e}^{(N)} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ Nth place .

We assume that the region in space where the ground state is non-negligible is microscopic; that is, classical quantities may be taken as constant.

We obtain the following plane wave approximation for the excited state wave function:

$$\tilde{\Psi}^{(N)}(x) = \mathcal{N}^{(N)} \left[\frac{e^{\frac{i}{\hbar} p_+^{(N)}(0) \cdot x}}{\sqrt{v_+^{(N)}(0)}} + \frac{e^{\frac{i}{\hbar} p_-^{(N)}(0) \cdot x}}{\sqrt{v_-^{(N)}(0)}} e^{i\delta^{(N)}} \right] \tilde{e}^{(N)}$$

(2.15)

$$\equiv \mathcal{N}^{(N)} \left[\frac{e^{\frac{i}{\hbar} p_{0,+}^{(N)} x}}{\sqrt{v_{0,+}^{(N)}}} + \frac{e^{\frac{i}{\hbar} p_{0,-}^{(N)} x}}{\sqrt{v_{0,-}^{(N)}}} e^{i\delta^{(N)}} \right] \tilde{e}^{(N)} .$$

To find $\mathcal{N}^{(N)}$ we demand that $\tilde{\Psi}^{(N)}$ be normalized properly. We assume that $\tilde{\Psi}^{(N)}$ is non-negligible only in the classically accessible region where the WKB approximation holds. We have (excising superfluous superscripts)

$$\begin{aligned}
 1 &= \int dx \Psi^\dagger(x) \Psi(x) \\
 &= |\mathcal{N}|^2 \int dx \left[\frac{1}{v_+} + \frac{1}{v_-} + \frac{1}{\sqrt{v_+ v_-}} \left(e^{\frac{i}{\hbar} \int (p_+ - p_-) dx' - i\delta} + e^{\frac{i}{\hbar} \int (p_- - p_+) dx' + i\delta} \right) \right]. \quad (2.16)
 \end{aligned}$$

The rapidly oscillating phase factors guarantee that the cross terms vanish, and

$$\frac{1}{|\mathcal{N}|^2} = \int dx \left[\frac{1}{v_+} + \frac{1}{v_-} \right] \cong \oint \frac{dx}{v} = \tau \quad (2.17)$$

where τ is the classical period of the motion. Thus,

$$|\mathcal{N}|^2 = \frac{1}{\tau}. \quad (2.18)$$

We have now calculated the excited state wave function in the region where the ground state wave function is non-negligible. We may take this (and integrate it against the ground state wave function over the whole region $x = -\infty$ to $x = \infty$ with impunity) to find the matrix element

$$\mathcal{M} \equiv \langle f | \lambda_1 e^{\frac{i}{\hbar} q x} + \lambda_2 e^{-\frac{i}{\hbar} q x} | i \rangle. \quad (2.19)$$

Barring a phase (which disappears when we find $|\mathcal{M}|^2$),

$$\begin{aligned}
 \mathcal{M} &= \frac{1}{\sqrt{\tau^{(N)}}} e^{(N)} \left[\frac{1}{\sqrt{v_{0,+}^{(N)}}} \left\{ \lambda_1 \tilde{\Psi}_0^{(N)}(p_{0,+} - q) + \lambda_2 \tilde{\Psi}_0^{(N)}(p_{0,+} + q) \right\} \right. \\
 &\quad \left. + e^{-i\delta^{(N)}} \frac{1}{\sqrt{v_{0,-}^{(N)}}} \left\{ \lambda_1 \tilde{\Psi}_0^{(N)}(p_{0,-} - q) + \lambda_2 \tilde{\Psi}_0^{(N)}(p_{0,-} + q) \right\} \right]. \quad (2.20)
 \end{aligned}$$

Here $\frac{\tilde{\Psi}_0(p)}{\sqrt{2\pi\hbar}}$ is the momentum space representation of the ground state wave function:

$$\tilde{\Psi}_0(p) \equiv \int dx e^{-\frac{i}{\hbar}px} \Psi_0(x) . \quad (2.21)$$

Because $\frac{\tilde{\Psi}_0(N)}{\sqrt{2\pi\hbar}}$ is the ground state wave function, it is reasonable to expect it to have the same form we assumed for $\Psi_0(x)$; that is, to have bell shaped behavior around some ground state region. We shall assume that we have set things up so that this region is "centered" about $p=0$. So $\tilde{\Psi}_0(p)$ will be significant only when its arguments $p \approx 0$. Now the classical orbits in phase space associated with $H^{(N)}(p^{(N)}, x) = E$ are really quite arbitrary, we could have, for instance, any of the situations shown in Fig. 5. For the sake of simplicity, we shall assume that the second case holds. Since $q > 0$, we see that the only non-negligible contribution to \mathcal{M} can come from the $\lambda_1 \tilde{\Psi}_0(p_{0,+}^{(N)} - q)$ or $\lambda_2 \tilde{\Psi}_0(p_{0,-}^{(N)} + q)$ terms, the other two terms in the matrix element may be safely neglected. Thus

$$\mathcal{M} = \frac{1}{\sqrt{v_{\tau}^{(N)}}} e^{(N)\dagger} \left[\frac{1}{\sqrt{v_{0,+}^{(N)}}} \lambda_1 \tilde{\Psi}_0(p_{0,+}^{(N)} - q) + e^{-i\delta^{(N)}} \frac{1}{\sqrt{v_{0,-}^{(N)}}} \lambda_2 \tilde{\Psi}_0(p_{0,-}^{(N)} + q) \right] . \quad (2.22)$$

We also note that there will be a qualitative difference between the case where $p_- \neq -p_+$ and the case where $p_- = -p_+$. In the former, for a fixed q , either the $\lambda_1 \tilde{\Psi}_0(p_{0,+}^{(N)} - q)$ or the $\lambda_2 \tilde{\Psi}_0(p_{0,-}^{(N)} + q)$ term - but not both - will contribute; hence, the structure function can be separated into a W_+ and W_- . On the other hand, in the $p_- = -p_+$ case, both terms will contribute and the structure function, given as the square of \mathcal{M} ,

will contain cross terms. Let us call these cases Case I and Case II respectively.

Then for Case I:

$$W(q \approx p_{0,\pm}^{(N)}) \equiv W_{\pm}^{(N)}(q, \nu) = \sum_{\mathbf{f}} \mathcal{M}^* \mathcal{M} \delta(\nu - \Delta E) \quad (2.23)$$

$$= \sum_{\mathbf{f}} \frac{1}{\tau^{(N)}} \tilde{\Psi}_0^{\dagger}(p_{0,\pm}^{(N)} \mp q) \lambda_{\mathbf{i}}^{\dagger} \frac{1}{\sqrt{\nu_{0,\pm}^{(N)}}} \underline{e}^{(N)} \underline{e}^{(N)\dagger} \frac{1}{\sqrt{\nu_{0,\pm}^{(N)}}} \lambda_{\mathbf{i}} \tilde{\Psi}_0(p_{0,\pm}^{(N)} \mp q) \delta(\nu - \Delta E).$$

Letting $P^{(N)} \equiv \underline{e}^{(N)} \underline{e}^{(N)\dagger}$,

$$W_{\pm}^{(N)}(q, \nu) = \sum_{\mathbf{f}} \frac{1}{\tau^{(N)} \nu_{0,\pm}^{(N)}} \tilde{\Psi}_0^{\dagger}(p_{0,\pm}^{(N)} \mp q) \lambda_{\mathbf{i}}^{\dagger} P^{(N)} \lambda_{\mathbf{i}} \tilde{\Psi}_0(p_{0,\pm}^{(N)} \mp q) \delta(\nu - \Delta E), \quad (2.24)$$

where we choose $i=1$ for $p_{0,+}^{(N)}$ and $i=2$ for $p_{0,-}^{(N)}$.

For Case II, we may extract an expression similar to those above plus some cross terms. We can write $W^{(N)} = W_{sq}^{(N)} + W_{cr}^{(N)}$ where $W_{sq}^{(N)}(q, \nu)$ contains the two parts of the previous case while $W_{cr}^{(N)}(q, \nu)$ contains cross terms which rapidly vary with energy,

$$\begin{aligned} W_{cr}^{(N)}(q, \nu) &= \sum_{\mathbf{f}} \frac{1}{\tau^{(N)} \sqrt{\nu_{0,+}^{(N)} \nu_{0,-}^{(N)}}} \left[\tilde{\Psi}_0^{\dagger}(p_{0,+}^{(N)} - q) \lambda_{\mathbf{1}}^{\dagger} P^{(N)} e^{-i\delta^{(N)}} \lambda_{\mathbf{2}} \tilde{\Psi}_0(p_{0,-}^{(N)} + q) + \right. \\ &\quad \left. + \tilde{\Psi}_0(p_{0,-}^{(N)} + q) \lambda_{\mathbf{2}}^{\dagger} e^{i\delta^{(N)}} P^{(N)} \lambda_{\mathbf{1}} \tilde{\Psi}_0(p_{0,+}^{(N)} - q) \right] \delta(\nu - \Delta E) \quad (2.25) \\ &\equiv \sum_{\mathbf{f}} \frac{1}{\tau^{(N)}} \left[e^{-i\delta^{(N)}} \Theta^{(N)*} + e^{i\delta^{(N)}} \Theta^{(N)} \right] \delta(\nu - \Delta E). \end{aligned}$$

We shall now proceed with our discussion by transforming the structure function into something more useful. The structure function we have is a sum of modulated δ -functions representing the excitation of individual states. If the spread of excitation energy, Δv , includes many states so that we may talk of a high level density, then we can "smooth out" the δ -functions by defining a coarse grained average of the structure function

$$\bar{W}(q, v) \equiv \frac{1}{\Delta v} \int_v^{v+\Delta v} dv' W(q, v') . \quad (2.26)$$

From our original definition of W , we see

$$\bar{W}(q, v) = \frac{1}{\Delta v} \sum_{f(v)}^{f(v+\Delta v)} \left| \langle f | \lambda_1 e^{\frac{i}{\hbar} q x} + \lambda_2 e^{-\frac{i}{\hbar} q x} | i \rangle \right|^2 . \quad (2.27)$$

We may pull all slowly varying (over the range Δv) quantities out of the integral in Eq. (2.26). Clearly this can be done for the constants λ 's and e 's as well as those quantities given in purely classical terms, the τ 's and v 's. Given the bell shaped nature of $\tilde{\Psi}_0$, even though it has \hbar dependence, it too may be taken as constant. So, the only quantities that cannot be pulled out will be the WKB phase shifts in the cross-term piece of the Case II structure function. Letting $\Delta f \equiv \sum_{f(v)}^{f(v+\Delta v)} 1$ = the number of states in the energy range $(v, v+\Delta v)$, we obtain:

For Case I:

$$\bar{W}_+^{(N)}(q, \nu) = \frac{1}{\tau^{(N)} \nu_{0,+}^{(N)}} \frac{\Delta f^{(N)}}{\Delta \nu} \tilde{\psi}_0^{(N)}(p_{0,+}^{(N)} - q) \lambda_1^{\dagger P^{(N)}} \lambda_1 \tilde{\psi}_0^{(N)}(p_{0,+}^{(N)} - q) \equiv \frac{\Delta f^{(N)}}{\tau^{(N)} \Delta \nu} \phi_+^{(N)}$$

$$\bar{W}_-^{(N)}(q, \nu) = \frac{1}{\tau^{(N)} \nu_{0,-}^{(N)}} \frac{\Delta f^{(N)}}{\Delta \nu} \tilde{\psi}_0^{(N)}(p_{0,-}^{(N)} + q) \lambda_2^{\dagger P^{(N)}} \lambda_2 \tilde{\psi}_0^{(N)}(p_{0,-}^{(N)} + q) \equiv \frac{\Delta f^{(N)}}{\tau^{(N)} \Delta \nu} \phi_-^{(N)} .$$

For Case II:

$$\bar{W}^{(N)} = \bar{W}_{sq}^{(N)} + \bar{W}_{cr}^{(N)}$$

$$\bar{W}_{sq}^{(N)} = \bar{W}_+^{(N)} + \bar{W}_-^{(N)} \quad (2.28)$$

$$\bar{W}_{cr}^{(N)} = \frac{1}{\tau^{(N)}} \frac{1}{\Delta \nu} \left[\theta^{(N)*} \sum_{f(\nu)} \frac{f(\nu + \Delta \nu)}{f(\nu)} e^{-i\delta^{(N)}} + \theta^{(N)} \sum_{f(\nu)} \frac{f(\nu + \Delta \nu)}{f(\nu)} e^{i\delta^{(N)}} \right] .$$

We note that the cross term can be neglected if

$$2\text{Re} \left[\theta^{(N)} \sum_{f(\nu)} \frac{f(\nu + \Delta \nu)}{f(\nu)} e^{i\delta^{(N)}} \right] \ll \Delta f^{(N)} \phi_{\pm}^{(N)} . \quad (2.29)$$

Letting

$$A_R^{(N)} \equiv \int_0^B p_+^{(N)}(x') dx' + \int_B^0 p_-^{(N)}(x') dx' \quad (2.30)$$

$$A_L^{(N)} \equiv \int_0^{-B'} p_-^{(N)}(x') dx' + \int_{-B'}^0 p_+^{(N)}(x') dx' ,$$

and using our Bohr-Sommerfeld formula, we see

$$\delta_R^{(N)} = \frac{A_R^{(N)}}{\hbar} = n\pi \frac{2A_R^{(N)}}{A_R^{(N)} + A_L^{(N)} - \hbar\pi} \quad (2.31)$$

Because the A's don't vary much over our small range Δv , we see

$$\begin{aligned} \sum_{f(v)}^{f(v+\Delta v)} e^{i\delta_R^{(N)}} &= \sum_{n=f}^{f+\Delta f} e^{in\pi X^{(N)}} \\ &= \frac{e^{i\pi X^{(N)}(f+\Delta f-1)} - e^{i\pi X^{(N)}(f+\Delta f)} - e^{i\pi X^{(N)}(\Delta f-1)} + e^{i\pi X^{(N)}}}{2(1-\cos(\pi X^{(N)}))} \end{aligned} \quad (2.32)$$

where we have defined

$$X^{(N)} \approx \frac{2A_R^{(N)}}{A_R^{(N)} + A_L^{(N)} - \hbar\pi}$$

We see that there is no way $W_{cr} \approx W_{sq}$ unless

$$\frac{1}{1-\cos(\pi X^{(N)})} \approx \Delta f^{(N)} \gg 1 \quad (2.33)$$

Thus there will be no problem unless the potential is extremely asymmetric. We shall assume that such an asymmetry does not occur and hence are justified in taking

$$\bar{W}(q, v) = \bar{W}_{sq}(q, v) \quad (2.34)$$

To complete our final form of the coarse-grained structure function, we shall evaluate $\Delta \mathcal{N}^{(N)} / \Delta v$ and show that $\bar{W}(q, v)$ is independent of the size of the energy range Δv we use.

Using the Bohr-Sommerfeld rule, and the fact that

$p_+^{(N)}(x) = p_-^{(N)}(x)$ at the classical turning point, we obtain

$$\frac{\Delta \mathcal{N}^{(N)}}{\Delta E} = \frac{1}{2\pi\hbar} \left[\int_{-B'}^B \frac{dx'}{\partial E / \partial p_+^{(N)}} + \int_B^{-B'} \frac{dx'}{\partial E / \partial p_-^{(N)}} \right] = \frac{1}{2\pi\hbar} \tau^{(N)}. \quad (2.35)$$

Using this, and noting that once we have banished the cross terms, the distinction between the two cases vanishes; we obtain:

$$\begin{aligned} \bar{W}^{(N)}(q, \nu) = \frac{1}{2\pi\hbar} & \left[\frac{1}{v_{0,+}^{(N)}(\nu)} \tilde{\psi}_0^+(p_{0,+}^{(N)}(\nu) - q) \lambda_1^+ P^{(N)} \lambda_1 \tilde{\psi}_0^-(p_{0,+}^{(N)}(\nu) - q) \right. \\ & \left. + \frac{1}{v_{0,-}^{(N)}(\nu)} \tilde{\psi}_0^+(p_{0,-}^{(N)}(\nu) + q) \lambda_2^+ P^{(N)} \lambda_2 \tilde{\psi}_0^-(p_{0,-}^{(N)}(\nu) + q) \right]. \end{aligned} \quad (2.36)$$

We may now turn to the physics embodied in these \bar{W} 's. Let us first take up scaling. Treating the $\bar{W}^{(N)}$'s as functions of ν with q held fixed, we see that the $\bar{W}^{(N)}$'s will only be significant for those values of ν such that $p_{0,+}^{(N)}(\nu)$ or $-p_{0,-}^{(N)}(\nu)$ are close to q . Using this observation, we may derive a scaling variable as follows. We recall that $p_{0,+}^{(N)}(\nu)$ was defined by $\nu = H^{(N)}(p_{0,+}^{(N)}(\nu), 0)$. Letting

$$w_{0,+}^{(N)}(q) \equiv H^{(N)}(p, 0) \Big|_{p=q} \quad \text{and} \quad u_{0,+}^{(N)}(q) \equiv \frac{\partial H^{(N)}(p, 0)}{\partial p} \Big|_{p=q} \quad (2.37)$$

and expanding $\nu(p)$ in a Taylor series about $p=q$, we see that

$$\nu \approx w_{0,+}^{(N)}(q) + u_{0,+}^{(N)}(q) (p_{0,+}^{(N)} - q), \quad (2.38)$$

Thus

$$P_{0,+}^{(N)} \approx y_+^{(N)}(q, \nu) \equiv \frac{v - w_{0,+}^{(N)}(q)}{u_{0,+}^{(N)}(q)} \quad (2.39)$$

and hence

$$v_{0,+}^{(N)}(\nu) \bar{w}_+^{(N)}(q, \nu) \approx \frac{1}{2\pi\hbar} \tilde{\Psi}_0^+(y_+^{(N)}(q, \nu)) \lambda_1^+ P^{(N)} \lambda_1 \tilde{\Psi}_0(y_+^{(N)}(q, \nu)). \quad (2.40)$$

Similarly, with $y_-^{(N)} \equiv (v - w_{0,-}^{(N)}(q)) / u_{0,-}^{(N)}(q)$

$$v_{0,-}^{(N)}(\nu) \bar{w}_-^{(N)}(q, \nu) \approx \frac{1}{2\pi\hbar} \tilde{\Psi}_0^+(y_-^{(N)}(q, \nu)) \lambda_2^+ P^{(N)} \lambda_2 \tilde{\Psi}_0(y_-^{(N)}(q, \nu)). \quad (2.41)$$

We can thus see how the total coarse-grained structure function $\bar{W}(q, \nu) = \sum_N \bar{w}^{(N)}(q, \nu)$ behaves. It will be large only for branches in q - ν space characterized by the equation $\nu = w_{\pm}^{(N)}(q)$, and the domain where it will be non-negligible can be described for each branch by a single variable $y_{\pm}^{(N)}(q, \nu)$: the momentum spread of the ground state wave function. These curves $\nu = H^{(N)}(q, 0)$ correspond to classical impulse scattering of either of the two constituent in the corresponding classical channels. This must come, not from our use of the WKB approximation which only reduces the quantum to a classical problem, but from the use of the Born approximation.

We close this section by evaluating the sum rule using our approximation of the coarse-grained structure function. For simplicity, let us again concentrate on $w_+^{(N)}(q, \nu)$. We are interested in $\int d\nu w_+^{(N)}(q, \nu)$, which an easy calculation shows is equal to $\int d\nu \bar{w}_+^{(N)}(q, \nu)$. Thus we are led to evaluate

$$\int dv \bar{w}_+^{(N)}(q, v) = \frac{1}{2\pi\hbar} \int \frac{dv}{v_+^{(N)}(v)} \tilde{\psi}_0^\dagger(y_+^{(N)}(q, v)) \lambda_{1P}^{(N)} \lambda_{1\tilde{\psi}_0}(y_+^{(N)}(q, v)). \quad (2.42)$$

Now $y_+^{(N)}(q, v) = (v - w_+^{(N)}(q)) / u_+^{(N)}(q)$ so

$$dy_+^{(N)} = \frac{dv}{u_+^{(N)}(q)}. \quad (2.43)$$

Also, because we are considering functions which are only significant for $v = w_+^{(N)}(q)$, we may take

$$u_+^{(N)}(q) = v_+^{(N)}(v). \quad (2.44)$$

With similar reasoning for $\bar{w}_-^{(N)}(q, v)$, we obtain

$$\int dv W^{(N)}(q, v) = \frac{1}{2\pi\hbar} \int dy^{(N)} \left[\tilde{\psi}_0^\dagger(y^{(N)}) \lambda_{1P}^{(N)} \lambda_{1\tilde{\psi}_0}(y^{(N)}) + \tilde{\psi}_0^\dagger(y^{(N)}) \lambda_{2P}^{(N)} \lambda_{2\tilde{\psi}_0}(y) \right]. \quad (2.45)$$

Since $W = \sum_{N, \text{available}} W^{(N)}$, we see explicitly how each channel contributes to the charge, and that this contribution depends on the ground state wave function. This corresponds to local excitation of color. Finally we note that if q is large enough, all channels will be excited, so

$$\begin{aligned} \int dv W(q, v) &= \frac{1}{2\pi\hbar} \int \tilde{\psi}_0^\dagger \sum_{i=1}^2 \lambda_i^\dagger \sum_N P^{(N)} \lambda_{i\tilde{\psi}_0} dy \\ &= \frac{1}{2\pi\hbar} \int \tilde{\psi}_0^\dagger \sum_i \lambda_i^\dagger \lambda_{i\tilde{\psi}_0} dy = \int \tilde{\psi}_0^\dagger(x) \sum_i \lambda_i^\dagger \lambda_{i\tilde{\psi}_0}(x) dx. \end{aligned} \quad (2.46)$$

This result, taken from our approximation for the coarse-grained structure function, is identical to the exact result derived earlier, which leads us to believe our approximation should not neglect any significant contributions to the structure function.

In order to better understand this result, let us apply this formalism to our example of the (π^+, η') . Factoring out the spatial parts of the wave function gives

$$\int d\nu W^{(N)}(q, \nu) = \langle \psi_0 | \sum_i \lambda_i^{\dagger P(N)} \lambda_i | \psi_0 \rangle = \sum_i |\langle \psi^{(N)} | \lambda_i | \psi_0 \rangle|^2 \quad (2.47)$$

where $\langle \psi_0 |$ and $\langle \psi^{(N)} |$ evidently are internal wave functions only.

The internal wave function of (π^+, η') is

$$\psi(\pi^+, \eta') = \frac{1}{\sqrt{3}} (u_R \bar{d}_R + u_Y \bar{d}_Y + u_B \bar{d}_B) \quad (2.48)$$

where the charges of the quarks are given in Table I. The color-

Table I

	R	Y	B
U	1	1	0
D	0	0	-1
S	0	0	-1

singlet classical orbit will have the same internal wave function as the ground state; hence the matrix element of the piece of the charge operator acting on u-quarks will give a factor

$$\langle \pi^+, \eta' | Q_u | \pi^+, \eta' \rangle = (1/\sqrt{3})^2 [1+1+0] = 2/3 \quad (2.49)$$

just as in the fractional-charge model. In a similar way the matrix element of the charge operator for the anti-down quark is

$$\langle \pi^+, \eta' | Q_{\bar{d}} | \pi^+, \eta' \rangle = -(1/\sqrt{3})^2 [0+0+(-1)] = 1/3 . \quad (2.50)$$

Upon summing over all states which contribute to the color-singlet (π^+, η') classical orbit we see that we obtain

$$\text{color-singlet} \int d\nu W(q, \nu) = (2/3)^2 + (1/3)^2 = 5/9 . \quad (2.51)$$

orbits

Another way of saying this is that the charge matrix Q exhibited in Table 1 is a sum of two pieces Q_1 and Q_8 .

$$Q = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix} = Q_1 + Q_8 = \begin{pmatrix} 2/3 & 2/3 & 2/3 \\ -1/3 & -1/3 & -1/3 \\ -1/3 & -1/3 & -1/3 \end{pmatrix} + \begin{pmatrix} 1/3 & 1/3 & -2/3 \\ 1/3 & 1/3 & -2/3 \\ 1/3 & 1/3 & -2/3 \end{pmatrix} \quad (2.52)$$

(R) (Y) (B)

where Q_1 is color singlet and flavor octet, while Q_8 is flavor singlet and color octet. Q_1 has nonvanishing matrix elements only between the ground-state and color-singlet (π^+, η') states, while Q_8 has nonvanishing matrix-elements only between the ground state and the (π^+, η)

states. (It is clear from Eq. (2.52) that Q_8 commutes with color-isospin and therefore cannot couple to (π^+, π^0) or (π^+, K^0) states.)

The internal wave function of the (π^+, η) states is

$$\psi(\pi^+, \eta) = 1/\sqrt{6} [u_R \bar{d}_R + u_Y \bar{d}_Y - 2u_B \bar{d}_B] \quad (2.53)$$

and the transition-matrix elements of the (total) charge-operator are now

$$\begin{aligned} \langle \pi^+, \eta | Q_u | \pi^+, \eta' \rangle &= 1/\sqrt{3} \cdot 1/\sqrt{6} [1+1+0] = \sqrt{2}/3 \\ \langle \pi^+, \eta | Q_d | \pi^+, \eta' \rangle &= -1/\sqrt{3} \cdot 1/\sqrt{6} [0+0-(2)(-1)] = -\sqrt{2}/3 \end{aligned} \quad (2.54)$$

The resultant contribution to the structure function is

$$\int_{\text{color octet orbit}} dvW(q, \nu) = (\sqrt{2}/3)^2 + (-\sqrt{2}/3)^2 = 4/9 \quad (2.55)$$

and the total sum is

$$\int dvW(q, \nu) = (2/3)_{\text{u contribution}} + (1/3)_{\text{d contribution}} = 1 \quad (2.56)$$

as must be the case for the integer-charge case.

III. BROKEN COLOR

The example of the previous section considered the Hamiltonian as color conserving, i.e. invariant under rotations in color space giving pure color octet and singlet states. However, it is easy to envisage a more complicated situation, where the Hamiltonian is not invariant, and where the energy eigenstates will not be pure color singlet or octet states. Then we are forced to consider a nondiagonal matrix Schroedinger equation over the (color) internal variable.

As in the previous case, we consider first the abstract problem. It is set up as before, with the target being described by a wave function $\underline{\Psi}(x)$ satisfying

$$\underline{H}(P,x)\underline{\Psi}(x) = E\underline{\Psi}(x) . \quad (3.1)$$

However, the matrix $\underline{H}(p,x)$ is no longer diagonal, rather

$$\underline{H}(p,x) = \underline{R}(p,x) \begin{bmatrix} H_D^{(1)}(p,x) & & 0 \\ & \cdot & \\ 0 & & H_D^{(L)}(p,x) \end{bmatrix} \underline{R}^{-1}(p,x) \quad (3.2)$$

where, for all N , $H_D^{(N)}(p,x)$ is a "monotonic" infinite well with respect to p and x .

The main message of this section is that, in the semiclassical limit, all the results of the previous section survive; with modulations which are classical and not quantum mechanical. These modulations can be found from the classical matrix Hamiltonian. We proceed to demonstrate this.

The general solutions of our matrix Schroedinger equation will be some linear combination (to be determined by turning point conditions) of solutions $\tilde{\psi}^{(N)}(x)$ given as follows: the momenta $p_{\pm}^{(N)}(x, E) \equiv p_{\pm}^{(N)}(x)$ ($p_-^{(N)} < p_+^{(N)}$) are now solutions of

$$H_D^{(N)}(p_{\pm}^{(N)}(x), x) = E. \quad (3.3)$$

Also,

$$v_{\pm}^{(N)} \equiv \left. \frac{\partial H_D^{(N)}}{\partial p} \right|_{p=p_{\pm}^{(N)}}. \quad (3.4)$$

Then it can be shown[16] that

$$\tilde{\psi}_{\pm}^{(N)}(x) = \tilde{R}_{\pm}^{(N)}(x) \frac{e^{\frac{i}{\hbar} \int_0^x p_{\pm}^{(N)}(x') dx'}}{\sqrt{v_{\pm}^{(N)}(x)}} \begin{bmatrix} 0 \\ \vdots \\ \tilde{c} \\ 0 \\ \vdots \\ 0 \end{bmatrix} (p_{\pm}^{(N)}, x) \quad (3.5)$$

$$(\tilde{R}_{\pm}^{(N)}(x) \equiv \tilde{R}(p_{\pm}^{(N)}, x))$$

where $\tilde{c}(p_{\pm}^{(N)}, x) \equiv \tilde{c}_{\pm}^{(N)}$ is a vector over the "classically degenerate" subspace upon which $H_D^{(N)}(p, x)$ acts. The vector $\tilde{c}^{(N)}$ satisfies the equation

$$-\frac{d\tilde{c}^{(N)}}{dx} = \left[R^{-1} \frac{\partial R}{\partial x} + \frac{1}{2v^{(N)}} \left\{ R^{-1} \frac{\partial R}{\partial p} (H_D - H_D^{(N)}) R^{-1} \frac{\partial R}{\partial x} - R^{-1} \frac{\partial R}{\partial x} (H_D - H_D^{(N)}) R^{-1} \frac{\partial R}{\partial p} \right\} \right] \tilde{c}^{(N)} \equiv -\tilde{M}^{(N)}(x) \tilde{c}^{(N)}. \quad (3.6)$$

where \tilde{Z} is the projection of the matrix Z over the "classically degenerate" subspace.

As advertized earlier, the form of our semiclassical wave function shows us that it may be split into two distinct parts, the scalar piece

$$\psi_{sc}^{(N)} = \frac{e^{\frac{i}{\hbar} \int_0^x p_{\pm}^{(N)}(x') dx'}}{\sqrt{v_{\pm}^{(N)}(x)}} \quad (3.7)$$

that we saw before and the vector piece:

$$\tilde{R}_{\pm}^{(N)} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \tilde{c}_{\pm}^{(N)} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (3.8)$$

The vector piece, which describes how the wave function vector rotates in internal space as a function of position, is given only in terms of classical quantities and so must be slowly varying over microscopic distances.

Now that we have the pieces that compose the target state wave function, we must find turning point conditions which will enable us to find the bound states. For simplicity, we will treat the first "classically degenerate" subspace and see what a general solution $\tilde{\psi}^{\{1\}}(x)$ must be.

We can write

$$\underline{\psi}_+^{\{1\}}(x) = \underline{R}_+^{\{1\}}(x) \frac{e^{\frac{i}{\hbar} \int_0^x p_+^{\{1\}}(x') dx'}}{\sqrt{v_+^{\{1\}}(x)}} \begin{bmatrix} \underline{Q}_+^{\{1\}}(x) \underline{A}^{\{1\}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (3.9)$$

$$\underline{\psi}_-^{\{1\}}(x) = \underline{R}_-^{\{1\}}(x) \frac{e^{\frac{i}{\hbar} \int_0^x p_-^{\{1\}}(x') dx'}}{\sqrt{v_-^{\{1\}}(x)}} \begin{bmatrix} \underline{Q}_-^{\{1\}}(x) \underline{B}^{\{1\}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (3.10)$$

where

$$\underline{Q}_\pm^{\{1\}}(x) = x_{\Omega_0} \underline{M}_\pm^{\{1\}}(x') \quad (3.11)$$

Here $x_{\Omega_0} \underline{Z}(x')$ denotes the convergent Dyson series:

$$x_{\Omega_0} \underline{Z}(x') = \underline{I} + \int_0^x dx_1 \underline{Z}(x_1) + \int_0^x dx_1 \underline{Z}(x_1) \int_0^{x_1} dx_2 \underline{Z}(x_2) + \dots \quad (3.12)$$

Solving the turning point problem, generalizing the corresponding scalar problem, will give us $\underline{B}^{\{1\}}$ in terms of $\underline{A}^{\{1\}}$.

Let's look closely at the right-handed turning point region as shown in Fig. 6. Region I is a microscopic region (of the order of a few wavelengths (where $\lambda(B-d)$ is considered as a wavelength) inside which the WKB approximation begins to fail.

Thus, in Region $A \cap I'$, the WKB approximation is good.

We pick A to be microscopic, i.e. $\underline{R}(p,x)$ and $\underline{Q}(p,x)$ may be considered constant across A. (For example we could take $\Delta = 7\delta$, we know δ will be microscopic in the semiclassical limit.)

We can therefore see that we are completely justified, in region G,
in writing

$$\underline{\psi}_{\sim+}^{\{1\}}(x) = \underline{R}_{\sim+}^{\{1\}}(B) \frac{e^{\frac{i}{\hbar} \int_{O}^{x} p_{\sim+}^{\{1\}}(x') dx'}}{\sqrt{v_{\sim+}^{\{1\}}(x)}} \begin{bmatrix} \underline{\phi}_{\sim+}^{\{1\}}(B) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (3.13)$$

$$\underline{\psi}_{\sim-}^{\{1\}}(x) = \underline{R}_{\sim-}^{\{1\}}(B) \frac{e^{\frac{i}{\hbar} \int_{O}^{x} p_{\sim-}^{\{1\}}(x') dx'}}{\sqrt{v_{\sim-}^{\{1\}}(x)}} \begin{bmatrix} \underline{\phi}_{\sim-}^{\{1\}}(B) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where

$$\begin{aligned} \underline{\phi}_{\sim+}^{\{1\}}(B) &\equiv B_{\Omega} \underline{M}_{\sim+}^{\{1\}}(x') \underline{A}^{\{1\}} \\ \underline{\phi}_{\sim-}^{\{1\}}(B) &\equiv B_{\Omega} \underline{M}_{\sim-}^{\{1\}}(x') \underline{B}^{\{1\}} \end{aligned} \quad (3.14)$$

We further note that since $p_{\sim+}^{(N)}(B) = p_{\sim-}^{(N)}(B)$ (thus $\underline{R}_{\sim+}^{(N)}(B) = \underline{R}_{\sim-}^{(N)}(B) \equiv \underline{R}^{(N)}(B)$) we may write

$$\underline{\psi}^{\{1\}}(x) = \underline{\psi}_{\sim+}^{\{1\}}(x) + \underline{\psi}_{\sim-}^{\{1\}}(x) = \underline{R}^{\{1\}}(B) \left\{ \begin{aligned} &e^{\frac{i}{\hbar} \int_{O}^{x} p_{\sim+}^{\{1\}}(x') dx'} \begin{bmatrix} \underline{\phi}_{\sim+}^{\{1\}}(B) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ &+ e^{\frac{i}{\hbar} \int_{O}^{x} p_{\sim-}^{\{1\}}(x') dx'} \begin{bmatrix} \underline{\phi}_{\sim-}^{\{1\}}(B) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{aligned} \right\} \quad (3.15)$$

This suggests the conjecture that throughout region A we may express the functional dependence of $\tilde{\Psi}^{\{1\}}(x)$ on x as:

$$\tilde{\Psi}^{\{1\}}(x) = \tilde{R}^{\{1\}}(B) \left\{ F_+^{\{1\}}(x) \begin{bmatrix} \tilde{\phi}_+^{\{1\}}(B) \\ 0 \\ \vdots \\ 0 \end{bmatrix} + F_-^{\{1\}}(x) \begin{bmatrix} \tilde{\phi}_-^{\{1\}}(B) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\} \quad (3.16)$$

where $F_{\pm}^{\{1\}}(x)$ are scalar functions, which satisfy the equation

$$\begin{aligned} H_D^{\{1\}}(P, x) \left[F_+^{\{1\}}(x) \phi_+^{\{1\}, (N)}(B) + F_-^{\{1\}}(x) \phi_-^{\{1\}, (N)}(B) \right] \\ = E \left[F_+^{\{1\}}(x) \phi_+^{\{1\}, (N)}(B) + F_-^{\{1\}}(x) \phi_-^{\{1\}, (N)}(B) \right] \end{aligned} \quad (3.17)$$

where the $\phi_{\pm}^{\{1\}, (N)}$'s are the components of $\tilde{\phi}_{\pm}^{\{1\}}$.

In accordance with our theme advertized earlier, we have "factored out" the scalar part of the turning point problem, now we only need to solve the scalar turning point problem for

$$H_D^{\{1\}}(P, x) \psi^{\{1\}}(x) = E \psi^{\{1\}}(x) . \quad (3.18)$$

Using the same result we used in the previous section, we see that

$$\begin{aligned} \tilde{\phi}_-^{\{1\}}(B) &= e^{i\delta^{\{1\}}} \tilde{\phi}_+^{\{1\}}(B) = \\ &= e^{\frac{i}{\hbar} \left[\int_{0P_+}^B \{1\}(x') dx' + \int_{BP_-}^0 \{1\}(x') dx' \right]} - \frac{i\pi}{2} \tilde{\phi}_+^{\{1\}}(B) \end{aligned} \quad (3.19)$$

or

$$\overline{B}_{\Omega_0 \overline{M}}^{\{1\}}(x') \overline{B}^{\{1\}} = \tag{3.20}$$

$$e^{\frac{i}{\hbar} \left[\int_{0^+}^B \{1\}(x') dx' + \int_{B^-}^0 \{1\}(x') dx' \right]} - i \frac{\pi}{2} \overline{B}_{\Omega_0 \overline{M}}^{\{1\}}(x') \overline{A}^{\{1\}} .$$

We may express this in an aesthetic form by noting that

$$\left[\overline{B}_{\Omega_0 \overline{M}}^{\{1\}}(x') \right]^{-1} = \overline{B}_{\Omega_0 \overline{M}}^{\{1\}}(x') . \tag{3.21}$$

Then

$$\overline{B}^{\{1\}} = e^{-i \frac{\pi}{2} \frac{i}{\hbar} \int_{B^-}^0 \{1\}(x') dx'} e^{\frac{i}{\hbar} \int_{0^+}^B \{1\}(x') dx'} \overline{B}_{\Omega_0 \overline{M}}^{\{1\}}(x') \overline{B}_{\Omega_0 \overline{M}}^{\{1\}}(x') \overline{A}^{\{1\}} . \tag{3.22}$$

A similar expression for the left-handed turning point (at $-B'$) is seen, by inspection, to be:

$$\overline{B}^{\{1\}} = e^{+i \frac{\pi}{2} \frac{i}{\hbar} \int_{-B', P_-}^0 \{1\}(x') dx'} e^{\frac{i}{\hbar} \int_{0^+}^{-B'} \{1\}(x') dx'} \overline{B}_{\Omega_{-B'} \overline{M}}^{\{1\}}(x') \overline{B}_{\Omega_{-B'} \overline{M}}^{\{1\}}(x') \overline{A}^{\{1\}} . \tag{3.23}$$

Equating the expressions from the right and left handed turning points allows us to proceed towards the matrix analogue of the Bohr-Sommerfeld formula. Let us define the following scattering functions:

$$s_R^{\{1\}} \equiv e^{-i\frac{\pi}{2} \frac{i}{\hbar} \int_{B^-}^0 \{1\} dx'} e^{\frac{i}{\hbar} \int_0^{P_+} B^{\{1\}} dx'}$$

$$\bar{s}_R^{\{1\}} \equiv \Omega_{\Omega_{B^-} \bar{M}^{\{1\}}(x')} B \Omega_{\Omega_{0^+} \bar{M}^{\{1\}}(x')} \quad (3.24)$$

$$[s_L^{\{1\}}]^{-1} \equiv e^{+i\frac{\pi}{2} \frac{i}{\hbar} \int_{-B}^0 \{1\} dx'} e^{\frac{i}{\hbar} \int_0^{-B^+} \{1\} dx'}$$

$$[\bar{s}_L^{\{1\}}]^{-1} \equiv \Omega_{\Omega_{-B} \bar{M}^{\{1\}}(x')}^{-B^+} \Omega_{\Omega_{0^+} \bar{M}^{\{1\}}(x')} .$$

Then equating the turning point conditions gives

$$[s_L^{\{1\}}]^{-1} [\bar{s}_L^{\{1\}}]^{-1} \bar{A}^{\{1\}} = s_R^{\{1\}} \bar{s}_R^{\{1\}} \bar{A}^{\{1\}} , \quad (3.25)$$

so

$$\bar{A}^{\{1\}} = s_L^{\{1\}} s_R^{\{1\}} \bar{s}_L^{\{1\}} \bar{s}_R^{\{1\}} \bar{A}^{\{1\}} . \quad (3.26)$$

Before going on to characterize this more complicated version of the Bohr-Sommerfeld condition, it might be timely to make a few observations. In line with our theme, we have factored the scalar and matrix parts of the problem. We note that s's carry all the quantum contributions which vary quickly with respect to energy contributions; while the \bar{s} 's carry only classical, slowly varying quantities. We further note that \bar{U}_{LR} , the matrix which we will use to diagonalize $\bar{s}_L \bar{s}_R$ will also be a classical, slowly varying quantity. Finally we may note that our last equation may be written in the suggestive way:

$$\bar{A}^{\{1\}} = e^{-i\pi \frac{i}{\hbar} \int_{P_+}^0 \{1\} dx'} \bar{\Omega}_{\bar{M}^{\{1\}}(x')} \bar{A}^{\{1\}} . \quad (3.27)$$

In words, we have a matrix analogue of De Broglie's formulation, evolving the wave function about a complete orbit must bring it back to the value it had originally. (We ignore the $e^{-i\pi}$ factor.)

Returning to Eq. (3.26), we note that both $\bar{S}_L^{\{1\}}$ and $\bar{S}_R^{\{1\}}$ are unitary - because $\frac{1}{i}\bar{M}_\pm^{\{1\}}$ is hermitian, the \bar{S} 's which involve products of $\Omega i(\frac{1}{i}\bar{M})$ must be unitary. Hence there is a unitary $\bar{U}_{LR}^{\{1\}}$ which diagonalizes $\bar{S}_{LR}^{\{1\}} \equiv \bar{S}_L^{\{1\}}\bar{S}_R^{\{1\}}$, i.e.

$$\bar{S}_{LR}^{\{1\}} = \bar{U}_{LR}^{\{1\}} \begin{bmatrix} \bar{S}_{LR}^{D(1)} & & & \\ & 0 & & \\ & & \ddots & \\ 0 & & & \ddots \end{bmatrix} [\bar{U}_{LR}^{\{1\}}]^{-1} \equiv \bar{U}_{LR}^{\{1\}} \bar{S}_{LR}^{D(1)} [\bar{U}_{LR}^{\{1\}}]^{-1} . \quad (3.28)$$

Letting

$$\bar{A}^{\{1\}} \equiv \bar{U}_{LR}^{\{1\}} \bar{A}^{D(1)} , \quad (3.29)$$

we obtain

$$\bar{A}^{D(1)} = s_L^{\{1\}} s_R^{\{1\}} \bar{S}_{LR}^{D(1)} \bar{A}^{D(1)} . \quad (3.30)$$

This equation can only have solutions

$$\bar{A}^{D(N)} = \mathcal{N}^{(N)} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ in the } N^{\text{th}} \text{ place} \equiv \mathcal{N}^{(N)} \bar{\lambda}^{(N)} \quad (3.31)$$

when

$$s_L^{\{1\}} s_R^{\{1\}} \bar{S}_{LR}^{D(N)} = e^{2n\pi i} . \quad (3.32)$$

Letting $S_{LR}^{D(N)} \equiv e^{i\Delta_N}$ we find

$$e^{-\frac{i\pi}{2}} \frac{i}{\hbar} \int_{B'P_-}^0 \{1\}(x') dx' \frac{i}{\hbar} \int_0^{BP_+} \{1\}(x') dx' \quad (3.33)$$

$$e^{-\frac{i\pi}{2}} e^{-\frac{i}{\hbar} \int_{-B'P_-}^0 \{1\}(x') dx'} e^{-\frac{i}{\hbar} \int_0^{-B'P_+} \{1\}(x') dx'} e^{i\Delta_N} = e^{2n\pi i},$$

thus

$$\frac{1}{\hbar} \left[\int_{-B'P_+}^B \{1\}(x') dx' + \int_B^{-B'P_-} \{1\}(x') dx' \right] + \Delta_N \equiv \delta_N = (2n+1)\pi. \quad (3.34)$$

This is the matrix version of the Bohr-Sommerfeld rule, which gives us the bound state energies.

Before discussing the implications of this formulation for the bound state spectrum, we should make one observation concerning degeneracy. We have called the Hamiltonian $H(p,x)$ "classically degenerate" if some of the eigenvalues of $H(p,x)$, the $H_D^{(N)}(p,x)$'s, are identical. This does not imply that the associated $\psi^{(N)}(x)$'s are degenerate in the familiar sense of having the same eigenvalues. In fact, we have seen that that sort of degeneracy depends on whether the matrix $\overline{S_L S_R}$ is degenerate. We have assumed, in writing $\overline{A}^{D(N)} = \mathcal{N}^{(N)} \overline{\lambda}^{(N)}$ that $\overline{S_L S_R}$ was not degenerate; i.e. that the Δ_N 's were distinct. We shall continue to do so.

Having obtained the bound state spectrum, we can obtain conditions for colored channels to be excited. These are identical in form to those presented in the last section. First we must be at an energy above the minimum of $H_D^{(N)}(p,x)$. This, however, is not sufficient—if the well is very narrow, so that δ_N slowly depends on

energy, we might have to go up quite a bit higher than $H_{D,\min}^{(N)}$ to excite the colored state. Again, both these observations come from misusing the WKB approximation to apply to the ground state of the colored channel; but they do identify qualitative features we would expect to see and can be used quantitatively for rough estimates.

As in the previous section, we can now look at our π^+ example from the perspective of classical orbits in phase space. The results of the previous section survive with the modification that the $(\pi^+, 8)$ orbits now split. For example, color breaking terms which, at the quark level, depend on the color matrix $\lambda_8^{(c)}$ or invariantly couple color to flavor $\tilde{\lambda}^{(c)} \cdot \tilde{\lambda}^{(f)}$ (thereby allowing explicit flavor breaking interactions such as strange quark mass to also break color) will leave only color-isospin as a good symmetry. The (π^+, η) splits from (π^+, π) and (π^+, K) . The quadruple degeneracy of $(\pi^+, 8)$ orbits with charge $Q=1$ will be broken into a doublet $[(\pi^+, K^0)$ and $(\pi^+, \bar{K}^0)]$ and two singlets (π^+, π^0) and (π^+, η) as shown in Fig. 7. Notice that, provided that symmetry-breaking is not as large as the octet-singlet splitting, we may expect that there will remain a distinct low-energy orbit, the remnant of the singlet orbit, despite the existence of mixing of octet with singlet configurations. This single low-energy orbit will give a similar set of levels as in the unbroken theory.

Before leaving our orbits, we must note that there is, in principle, an additional complication possible in describing the semiclassical limit of this, or other, matrix problems. The problem is that classically nondegenerate orbits may approach each other at high energy.

For example, for nonrelativistic octet and singlet orbits described by the classical equations

$$\frac{p^2}{2m} + V_1(x) = E \quad \frac{p^2}{2m} + V_8(x) = E, \quad (3.35)$$

The difference $\Delta p \equiv p_1 - p_8$ in the momenta at a given x is

$$\Delta p \approx \sqrt{\frac{m}{2E}} \Delta V \xrightarrow{E \rightarrow \infty} 0 \quad (\Delta V \equiv V_8 - V_1); \quad (3.36)$$

and hence, at sufficiently high E , large segments of the orbits will be degenerate; i.e. have negligible different action J . For a fixed interval (x_1, x_2) far from the turning points,

$$\frac{\Delta J}{\hbar} = \frac{1}{\hbar} \int_{x_1}^{x_2} \Delta p(x) \approx \frac{1}{\hbar} \sqrt{\frac{m}{2E}} \int_{x_1}^{x_2} dx \Delta V(x) \xrightarrow{E \rightarrow \infty} 0. \quad (3.37)$$

The perturbing terms mixing the orbits became important. [Note that this phenomenon does not occur if the nonrelativistic kinetic energy $p^2/2m$ is replaced by the relativistic $|p|$.] The orbits (with non-relativistic kinematics) at high energy are shown in Fig. 8.

Returning to our general discussion, we have so far determined that

$$\begin{aligned} \tilde{\Psi}^{(N)}(x) = \mathcal{N}^{(N)} & \left[\tilde{R}_+^{(N)}(x) \frac{e^{\frac{i}{\hbar} \int_0^x p_+^{(N)}(x') dx'}}{\sqrt{v_+^{(N)}(x)}} Q_+(x) + \right. \\ & \left. \tilde{R}_-^{(N)}(x) \frac{e^{\frac{i}{\hbar} \int_0^x p_-^{(N)}(x') dx'}}{\sqrt{v_-^{(N)}(x)}} Q_-(x) s_R^{(N)} \tilde{S}_R^{(N)} \right] U_{LR} e^{(N)} \end{aligned} \quad (3.38)$$

The $\underline{Q}(x)$'s and \underline{S} 's are block diagonal concatenations of their degenerate space counterparts. A rewriting of our semiclassical probability arguments of the previous section gives the same result for the normalizing factor:

$$|\mathcal{N}^{(N)}|^2 = \frac{1}{\tau^{(N)}} \quad . \quad (3.39)$$

We may proceed, as in the previous section, to evaluate the structure function. We have to carry along the extra luggage of \underline{R} 's, \underline{S} 's and \underline{U} 's, but they appear as classical quantities and may be taken as constant in microscopic regions. We obtain, retracing our previous footsteps, with our previous observations,

$$\begin{aligned} \bar{w}^{(N)}(q, \nu) = \frac{1}{2\pi\hbar} & \left[\frac{1}{v_{0,+}^{(N)}(\nu)} \tilde{\Psi}_0(p_{0,+}^{(N)}(\nu)-q) \lambda_{1+}^{\dagger} P_+^{(N)}(\nu) \lambda_{1-} \tilde{\Psi}_0(p_{0,+}^{(N)}(\nu)-q) \right. \\ & \left. + \frac{1}{v_{0,-}^{(N)}(\nu)} \tilde{\Psi}_0(p_{0,-}^{(N)}(\nu)+q) \lambda_{2-}^{\dagger} P_-^{(N)}(\nu) \lambda_{2+} \tilde{\Psi}_0(p_{0,-}^{(N)}(\nu)+q) \right] \quad . \end{aligned} \quad (3.40)$$

However, here

$$P_+^{(N)} = \tilde{R}_{0,+}^{(N)} \tilde{U}_{LR} e^{(N)} e^{(N)\dagger} \tilde{U}_{LR}^{\dagger} \tilde{R}_{0,+}^{(N)\dagger}$$

and

$$P_-^{(N)} = \tilde{R}_{0,-}^{(N)} \tilde{S}_R \tilde{U}_{LR} e^{(N)} e^{(N)\dagger} \tilde{U}_{LR}^{\dagger} \tilde{S}_R^{\dagger} \tilde{R}_{0,-}^{(N)\dagger} \quad . \quad (3.41)$$

These are the projection operators into the diagonal channel bases of the classical Hamiltonian at the origin with momentum $p_{0,+}^{(N)}$ and $p_{0,-}^{(N)}$ respectively.

We may see this more concretely by returning to our example of the previous section. Let us assume that the ground state of the target is pure color singlet (π^+, η') , but that the excited (π^+, η') and (π^+, η) classical orbits mix with mixing angle θ . (In other words, we need a two-by-two orthogonal matrix $R(\theta)$ to diagonalize the classical Hamiltonian in the (π^+, η') - (π^+, η) sector.) Then the matrix elements of Q_u and $Q_{\bar{d}}$ from the ground state to the mixed orbits are given by Table II where

$$\begin{aligned} (\pi^+, I) &= (\pi^+, \eta') \cos \theta + (\pi^+, \eta) \sin \theta \\ (\pi^+, VIII) &= (\pi^+, \eta) \cos \theta - (\pi^+, \eta') \sin \theta \end{aligned} \tag{3.42}$$

are the internal wave functions for the "eigenorbits."

Table II

	(π^+, I)	$(\pi^+, VIII)$
Q_u	$\frac{2}{3} \cos \theta + \frac{\sqrt{2}}{3} \sin \theta$	$\frac{\sqrt{2}}{3} \cos \theta - \frac{2}{3} \sin \theta$
$Q_{\bar{d}}$	$\frac{1}{3} \cos \theta - \frac{\sqrt{2}}{3} \sin \theta$	$-\frac{\sqrt{2}}{3} \cos \theta - \frac{1}{3} \sin \theta$

Now when we measure the charges via the structure functions W we obtain a different result. For the contribution from the I orbit we get

$$\begin{aligned} \int_I dvW(v,q) &= \left(\frac{2}{3} \cos\theta + \frac{\sqrt{2}}{3} \sin\theta \right)^2 + \left(\frac{1}{3} \cos\theta - \frac{\sqrt{2}}{3} \sin\theta \right)^2 \\ &= \frac{5}{9} \cos^2\theta + \frac{2\sqrt{2}}{9} \cos\theta \sin\theta + \frac{4}{9} \sin^2\theta \end{aligned} \quad (3.43)$$

and from the VIII orbit we find

$$\begin{aligned} \int_{VIII} dvW(v,q) &= \left(\frac{\sqrt{2}}{3} \cos\theta - \frac{2}{3} \sin\theta \right)^2 + \left(-\frac{\sqrt{2}}{3} \cos\theta - \frac{1}{3} \sin\theta \right)^2 \\ &= \frac{4}{9} \cos^2\theta - \frac{2\sqrt{2}}{9} \cos\theta \sin\theta + \frac{5}{9} \sin^2\theta . \end{aligned} \quad (3.44)$$

Again, of course, we recover the general sum rule, Eq. (2.56) upon summing over both orbits. However, the contribution of orbit I can vary greatly depending upon the mixing angle θ . For example, for $\theta=0$, $\int_I dvW(v,q) = \frac{5}{9} = 0.56$ while for $\theta=-45^\circ$, it is $\frac{1}{2} - \frac{\sqrt{2}}{9} \approx 0.34$. We again remind the reader that this piece can as well be energy-dependent, even without inclusion of the contribution of the "octet" VIII orbit.

It is obvious from the form of the structure function that our discussion of scaling and sum rules survives intact. There is, however, one modification in our scaling discussion. Because of the possible ν dependence of $\underline{R}(p(\nu))$ and $\underline{U}(\nu)$ the projection operators can be functions of energy. Then the individual channel contributions to the coarse-grained structure functions, $\nu^{(N)} \overline{W}^{(N)}$ will contain an explicit energy dependence.

IV. CONCLUSIONS

In the introduction, it was emphasized that the existing evidence does not at all favor an interpretation of hadron structure in terms of integer-charge quarks as underlying constituents. Nor is there theoretical motivation for introduction of even a slightly broken color symmetry; such an introduction creates difficult theoretical issues to overcome.

Nevertheless, it does not seem prudent to accept too readily exact color symmetry as an absolute law of nature. And if color symmetry is for some reason not exact, the Han-Nambu integer-charge scheme (or some similar variant) might conceivably be, despite the difficulties, a viable alternative. Given this possibility, it would seem worthwhile to explore to some extent the consequences of such a scenario.

In this spirit, we investigated (in one spacial dimension) the properties of a Han-Nambu π^+ meson built of an integer-charge quark and antiquark, which possesses "hidden" color, i.e. low-lying excited states are (up to a small amount of impurity) color singlets. The spectrum of such a meson was examined in the WKB limit, and the "deep-inelastic" excitation of the ground-state meson by weakly interacting external probes was studied. The main results which we found are as follows:

(1) The low-lying spectrum, even when color-symmetry breaking is introduced, has essentially the same properties as in the simple unbroken case: the classifications of the levels are unmodified, and only small admixtures of color-octet states will be found in the low-energy spectrum. (However, this does not mean such admixtures are unobservable; see point (4) below.)

(2) In deep-inelastic processes, what matters for the structure function W are the properties of the excited-state wavefunction in only the small region of space where the ground-state wavefunction is nonvanishing. (In the WKB limit these wavefunctions are in fact just free plane-wave ("parton") states.) We showed that in this potential-scattering model, a sufficient condition for global excitation of color was local excitation of color. That is, if color-octet states could be excited with an external probe of energy ν within a small region of order of the size occupied by the ground-state wavefunction, then these color states would be found in the spectrum of the full Hamiltonian at excitation energies of the same energy ν . That is, they would be observable. However, we pointed out that the converse need not be true; there may be low-lying color states in the spectrum which are not excited by deep-inelastic scattering (until extremely high energies). This can happen if the color-octet potential is strongly repulsive at short distances, thereby excluding the excited state color wavefunctions from the spacial region occupied by the ground state wavefunction.

(3) The way all this is implemented mathematically is that in the presence of L internal color degrees of freedom, the WKB solution to the corresponding time-independent Schroedinger equation has L channels corresponding to L different classical orbits. These orbits can be excited by deep inelastic probes, and they each can contribute significant fractions to the sum-rules satisfied by the structure functions W . For example, in our example of π^+ , two distinct orbits contribute to the

familiar sum rule for the charges

$$\int d\nu W(\nu, q^2) = \sum_i Q_i^2 \quad (4.1)$$

of the constituents, as determined via deep-inelastic electromagnetic scattering. The first is the (essentially) color-singlet (π^+, η') and the second the (essentially) color-octet (π^+, η) .

In the absence of color-symmetry breaking, the (π^+, η') orbit contributes the same amount to the sum rule as the fractional charge quark model, while the other (π^+, η) color octet contributes the remainder necessary to count the integer charges of the Han-Nambu quarks.

(4) In the presence of color-symmetry breaking, the sum rule still holds. Furthermore, just as before the sum is saturated by the two distinct contributions from the two orbits. However, because color-symmetry is no longer exact, the relative balance between these two contributions can be different, and in particular can be energy dependent. That is, the excited local ("parton") plane-wave states can be mixed, and this mixing can depend upon the excitation energy ν . Hence even before the color threshold is reached, the charge as measured by excitation of ostensibly color singlet states could be dependent upon excitation energy ν .

While these conclusions have been obtained within the context of a very simplified dynamical system, we are optimistic that they hold under more general circumstances, in particular in higher dimensions and for many-particle systems. Our reasoning is that once one makes coarse-grained averages over the deep-inelastic structure-functions,

the detailed properties of the spectrum of the full Hamiltonian do not seem to matter. Only those of the local Hamiltonian are relevant, and those are well-approximated by an essentially free-particle system.

However, there are probably more essential complications in a field theory context, especially in connection with pair creation and vacuum polarization, which are beyond the scope of the discussion in this paper. However, one may anticipate that if one excludes light quarks u , d , s from the discussion (thereby largely suppressing pair-creation and polarization effects), the dynamical picture we have sketched may not be completely unrealistic.

Finally, even in the absence of a direct physical application, we have examined how to implement the WKB method for bound-state problems in the presence of internal degrees of freedom. The methodology we have given may be of use in other semiclassical problems which have internal degrees of freedom.

APPENDIX - THE GENERALIZED SCALAR WKB TURNING POINT PROBLEM

Our scalar equation is

$$H(P,x)\psi(x) = E\psi(x) . \quad (A.1)$$

$H(p,x)$ is a general well with respect to p and x , so, in phase space, a solution to the equation $H(p,x) = E$ might look like Fig. 9.

The solution to the turning point problem stems from the observation that this curve, near $x = B$, looks just like the horizontal parabola we would get from the ordinary nonrelativistic Hamiltonian. And we know the connection formulae for that problem. We proceed in more detail as follows: we think that we may generally write

$$H(P,x) = H(p_0,x_0) + \sum_{\substack{M \text{ or } R \\ > 0}} \frac{1}{2} a_{MR}(p_0,x_0) \left\{ (P-p_0)^M (x-x_0)^R + \text{h.c.} \right\} \quad (A.2)$$

where

$$a_{MR}(p_0,x_0) = \frac{1}{M!} \frac{1}{R!} \left(\frac{\partial}{\partial p} \right)^M \left(\frac{\partial}{\partial x} \right)^R H(p,x) \Big|_{\substack{x=x_0 \\ p=p_0}} \quad (A.3)$$

We assume that if we are close to the turning point (close with respect to macroscopic quantities), we may truncate $H(P,x)$ to the first few terms ($M + R \leq 2$). Thus

$$\begin{aligned} H(P,x) \approx H'(P,x) \equiv & H(p_0,x_0) + \frac{1}{2} a_{01}(p_0,x_0) \left\{ 1 \cdot (x-x_0) + (x-x_0) \cdot 1 \right\} \\ & + \frac{1}{2} a_{10}(p_0,x_0) \left\{ (P-p_0) \cdot 1 + 1 \cdot (P-p_0) \right\} \\ & + \frac{1}{2} a_{11}(p_0,x_0) \left\{ (P-p_0) (x-x_0) + (x-x_0) (P-p_0) \right\} \\ & + \frac{1}{2} a_{20}(p_0,x_0) \left\{ (P-p_0)^2 \cdot 1 + 1 \cdot (P-p_0)^2 \right\} \\ & + \frac{1}{2} a_{02}(p_0,x_0) \left\{ 1 \cdot (x-x_0)^2 + (x-x_0)^2 \cdot 1 \right\} . \end{aligned} \quad (A.4)$$

If we are expanding about the turning point $x_0 = B$, $p_0 = B$, then

$$a_{10} = \left. \frac{\partial H}{\partial p} \right|_{\substack{x=x_B \\ p=p_B}} = v(p_B, B) = 0 . \quad (\text{A.5})$$

We may also associate a magnitude $p(x)-p_0$ to the operator $P-p_0$ (where $p(x)$ is defined by $H(p(x), x) = E$). If we pick our region in phase space such that $p(x)-p_0 \sim \epsilon$ and $x-x_0 \sim \epsilon^2$, it is reasonable to expect that, to $\theta(\epsilon^2)$

$$\begin{aligned} H'(P, x) &\approx H''(P, x) \equiv H(p_0, x_0) + a_{01}(p_0, x_0)(x-x_0) \\ &\quad + a_{20}(p_0, x_0)(P-p_0)^2 \\ &\equiv \frac{1}{2\mu_B} (P-p_0)^2 + \alpha_B(x-B) + E_B . \end{aligned} \quad (\text{A.6})$$

We are almost in the familiar linear approximation for the turning point problem, we only have to make a trivial adjustment of the kinetic term: letting

$$\psi(x) \equiv \left(e^{\frac{i}{\hbar} P_B x} \psi_{\text{red}}(x) \right), \quad (\text{A.7})$$

$$(P-p_B)^2 \left(e^{\frac{i}{\hbar} P_B x} \psi_{\text{red}} \right) = e^{\frac{i}{\hbar} P_B x} P^2(\psi_{\text{red}}), \quad (\text{A.8})$$

therefore $\psi_{\text{red}}(x)$ satisfies

$$\left(\frac{1}{2\mu_B} P^2 + \alpha_B(x-B) + E_B \right) \psi_{\text{red}}(x) = E_B \psi_{\text{red}}(x) . \quad (\text{A.9})$$

We may now apply the familiar joining conditions; thus in region G (i.e. before we reach the turning point)

$$\psi_{\text{red}}(x) \propto \cos \frac{1}{\hbar} \int_x^B \Delta p(x') dx' - \frac{\pi}{4} \quad (\text{A.10})$$

$$\Delta p(x') \equiv +\sqrt{2\mu_B \alpha_B (B-x')} \approx p_+(x') - p_B \approx p_B - p_-(x') .$$

Paralleling the usual method we can now derive the turning point phase shift. We have just seen that, in region G,

$$\psi(x) = e^{\frac{i}{\hbar} p_B x} \psi_{\text{red}}(x) e^{\frac{i}{\hbar} p_B x} \frac{1}{2} \left[e^{i \left[\frac{1}{\hbar} \int_x^B \Delta p(x') dx' - \frac{\pi}{4} \right]} + e^{-i \left[\frac{1}{\hbar} \int_x^B \Delta p(x') dx' - \frac{\pi}{4} \right]} \right]. \quad (\text{A.11})$$

Our WKB approximation, using phase shift notation is:

$$\psi_{\text{WKB}}(x) = e^{\frac{i}{\hbar} \int_0^x p_+(x') dx'} + e^{i\delta} e^{\frac{i}{\hbar} \int_0^x p_-(x') dx'} \quad (\text{A.12})$$

$$= e^{\frac{i}{\hbar} \left[\int_0^B p_+(x') dx' - \int_x^B p_+(x') dx' \right]} + e^{i\delta} e^{\frac{i}{\hbar} \left[\int_0^B p_-(x') dx' - \int_0^x p_-(x') dx' \right]}$$

= (because, in region G $p_{\pm}(x') = p_B \pm \Delta p(x')$) =

$$\begin{aligned} &= e^{\frac{i}{\hbar} \int_0^B p_+(x') dx' - \frac{i}{\hbar} p_B (B-x) - \frac{i}{\hbar} \int_x^B \Delta p(x') dx'} \\ &+ e^{i\delta + \frac{i}{\hbar} \int_0^B p_-(x') dx' - \frac{i}{\hbar} p_B (B-x) + \frac{i}{\hbar} \int_x^B \Delta p(x') dx'} \quad (\text{A.13}) \\ &= \left\{ e^{-\frac{i}{\hbar} p_B B} e^{i\delta + \frac{i}{\hbar} \int_0^B p_-(x') dx'} \right\} e^{\frac{i}{\hbar} p_B x} \left[e^{\frac{i}{\hbar} \int_x^B \Delta p(x') dx'} \right. \\ &\left. + e^{-i\delta - \frac{i}{\hbar} \int_0^B p_-(x') dx' + \frac{i}{\hbar} \int_0^B p_+(x') dx' - \frac{i}{\hbar} \int_x^B \Delta p(x') dx'} \right]. \end{aligned}$$

Comparing this with our expression for $\psi(x)$ derived above

$$\psi(x) \propto \left\{ \frac{1}{2} e^{-i \frac{\pi}{4}} \right\} e^{\frac{i}{\hbar} p_B(x)} \left[e^{\frac{i}{\hbar} \int_x^B \Delta p(x') dx'} + e^{i \frac{\pi}{2} - \frac{i}{\hbar} \int_x^B \Delta p(x') dx'} \right], \quad (\text{A.14})$$

we obtain

$$-i\delta - \frac{i}{\hbar} \int_0^B p_-(x') dx' + \frac{i}{\hbar} \int_0^B p_+(x') dx' = i \frac{\pi}{2} \quad (\text{A.15})$$

which leads to the obvious generalization of the usual WKB phase shift formula:

$$\delta = \frac{1}{\hbar} \left[\int_0^B p_+(x') dx' + \int_0^B p_-(x') dx' \right] - \frac{\pi}{2}. \quad (\text{A.16})$$

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FIGURE CAPTIONS

- Fig. 1: Singlet and octet states for the π^+ example.
- Fig. 2: Singlet and octet "potentials" for the π^+ example.
- Fig. 3: Orbits in phase space for one scenario of frozen color:
(a) relativistic kinetic energy and linear potential,
(b) non-relativistic kinetic energy and linear potential,
(c) states.
- Fig. 4: Orbits in phase space for another scenario of frozen color:
(a) relativistic kinetic energy and linear potential,
(b) non-relativistic kinetic energy and linear potential,
(c) states.
- Fig. 5: Possible orbits in phase space.
- Fig. 6: The turning point region.
- Fig. 7: Orbits in phase space for π^+ with broken color.
- Fig. 8: Quasi-degeneracy at high energy.
- Fig. 9: Orbit in phase space for general scalar Hamiltonian.

$$(\pi^+; \underline{8}) : \frac{(\pi^+; \pi^-)}{(\pi^+; K^-)} \quad \frac{(\pi^+; \eta) \quad (\pi^+; \pi^0)}{(\pi^+; K^0) \quad (\pi^+; \bar{K}^0)} \quad \frac{(\pi^+; \pi^+)}{(\pi^+; K^+)}$$

$$(\pi^+; \underline{1}) :$$

$$\underline{(\pi^+; \eta')}$$

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Fig. 1

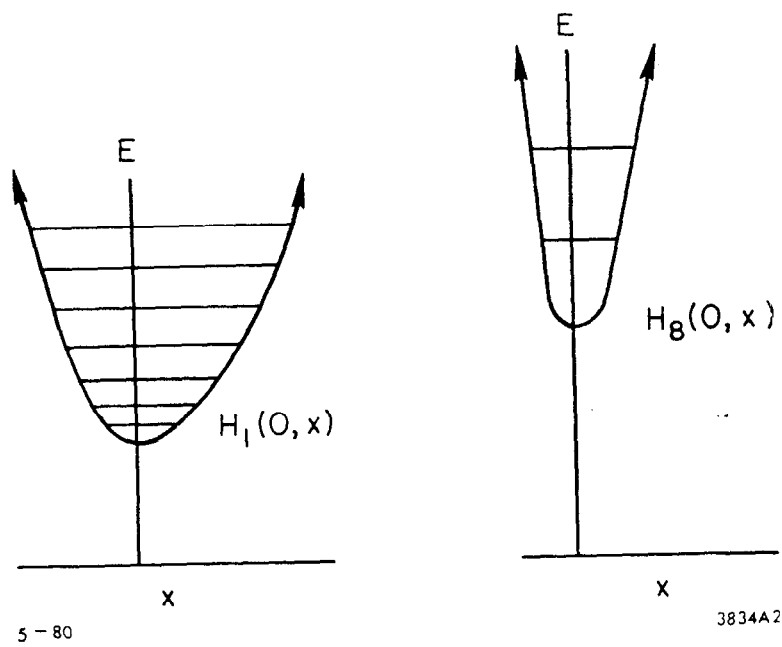


Fig. 2

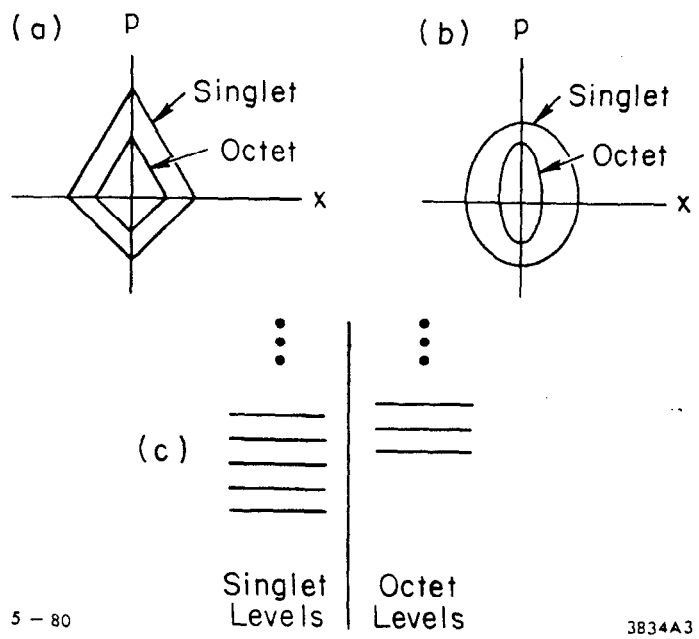


Fig. 3

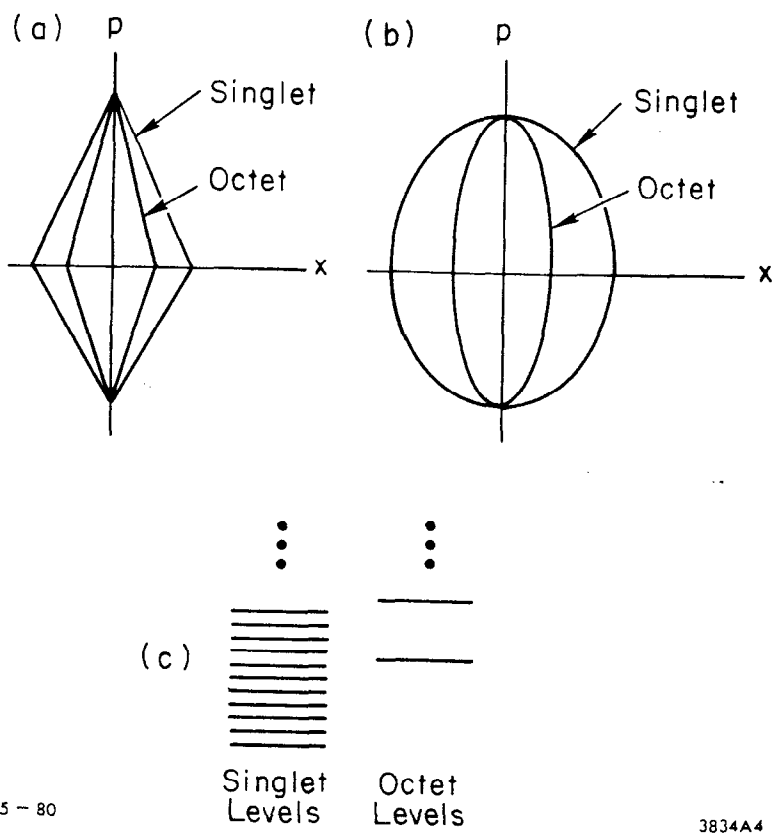


Fig. 4

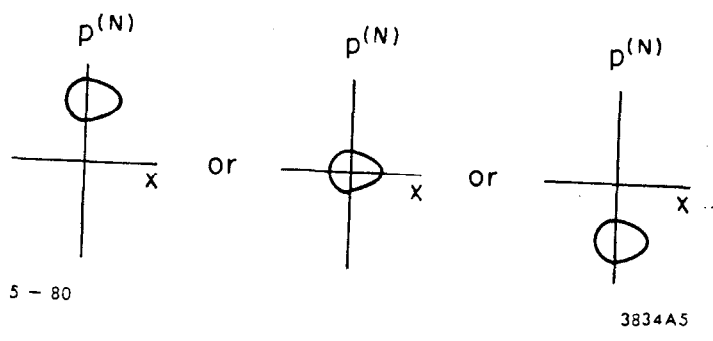
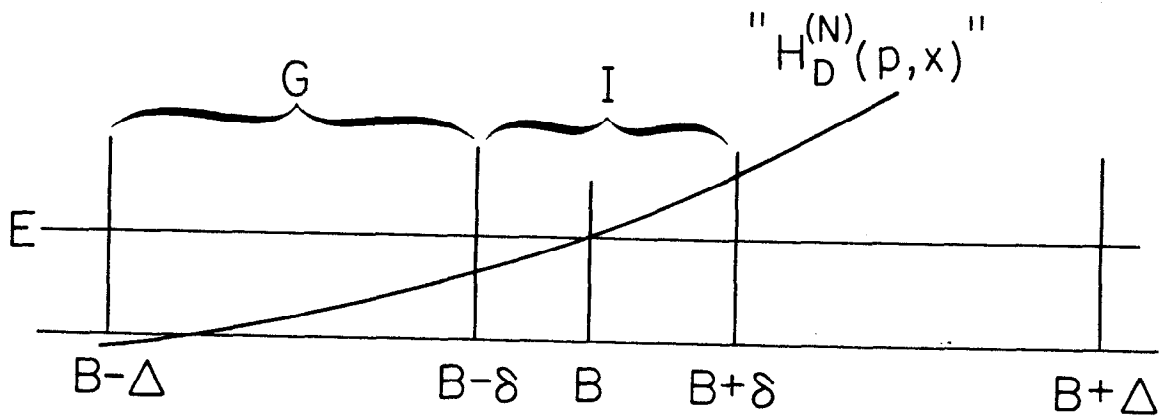


Fig. 5



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Fig. 6

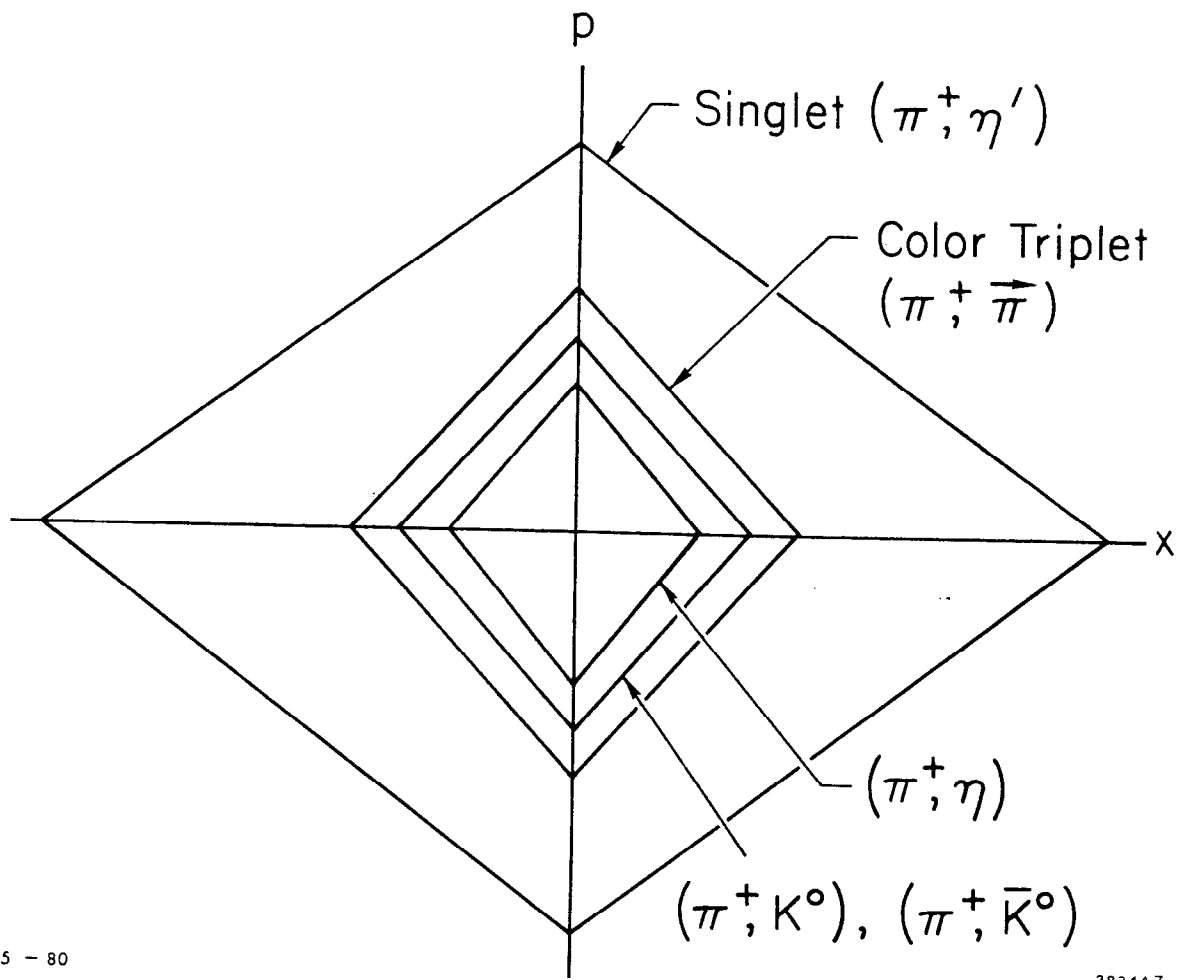


Fig. 7

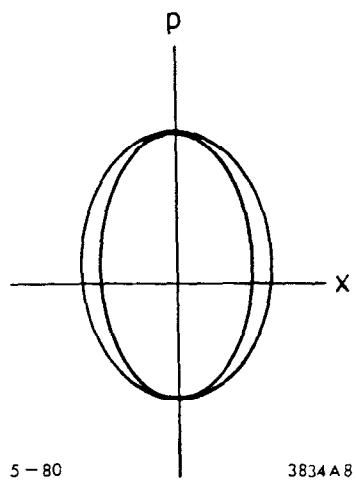


Fig. 8

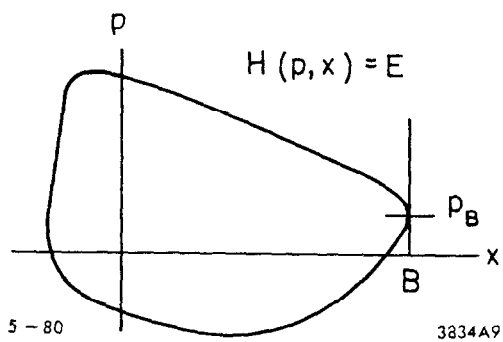


Fig. 9