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SYMMETRY BREAKING PATTERNS FOR UNITARY AND ORTHOGONAL GROUPS*

F. Buccella
Istituto di Fisica, Universita di Roma
INFN Sezione di Roma

H. Ruegg
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305
and
Universite de Geneve, Switzerland

C. A. Savoy
Universite de Geneve, Switzerland
and
Centre de Recherches Nucleaires, Universite Strasbourg I

ABSTRACT

We discuss the spontaneous symmetry breaking pattern for SU(n) and O(10). It is based on the exact treatment of the absolute minimum of the Higgs potential as a function of scalar fields belonging to the fundamental <u>and</u> adjoint representations of SU(n), the spinor <u>and</u> adjoint representations of O(10).

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I. Introduction

The unified model[1] of weak, electromagnetic and strong interactions based on the gauge group SU(5) has many attractive features. However, it still involves too many arbitrary parameters. One way to reduce this number is to imbed SU(5) into higher simple groups. Various schemes have been proposed by many authors,[2] but since no fully convincing solution has been found so far, it seems worthwhile to keep the discussion as general as possible.

In all schemes we have in mind, the Lagrangian is invariant under a given gauge group. The symmetry is spontaneously broken, that is if a scalar Higgs field transforms as a representation of the group, some of its components develop non-zero vacuum expectation values (VEV). This defines a privileged direction in representation space and determines the pattern of symmetry breaking. The subgroup which leaves these VEV invariant remains unbroken.

The scalar fields may be elementary. In this case, one wants the Higgs potential to be renormalizable, which limits it to a polynomial of degree four. A natural condition for the non-zero VEV is to require that they minimize the Higgs potential. This is the criterion used here. It is also possible to consider the scalar fields as boundstates of the fundamental fields.[3,4] In the absence of a satisfactory dynamical theory, it turns out that in this case also it may be necessary to minimize an effective Higgs potential.[4] Hence, a general discussion of the absolute minima of scalar potentials is useful also for dynamical symmetry breaking schemes.

A general group theoretic discussion of the Higgs potential has been given by L. F. Li,[5] who considered scalar fields in various irreducible representations. However, this is insufficient. For example, to break SU(5) down to SU(3) × SU(2) × U(1) and eventually to the exactly conserved SU(3) × U(1) one needs at least two irreducible components. Various particular examples have been discussed in some approximate schemes.[6,7,8]

Here we consider Higgs fields belonging to the adjoint plus the fundamental representation of SU(n) and the adjoint plus the spinor representation of O(10). This is still not sufficient, but our result has the merit of being exact, i.e., we do not require any parameter to be small. This may be important if one studies the transition between one symmetry breaking regime to another, the parameter in the Higgs potential varying as functions of energy (or temperature). For example, we shall find that in the SU(5) gauge model, a continuous change in a certain ratio of parameters (see Section IV) changes the conserved subgroup from SU(4) to SU(3) × U(1). This may be relevant to the discussion of monopoles.

To our knowledge, no exact treatment of the spinor Higgs fields has been given so far. This may be due to certain unfamiliar properties of these representations. Generalization to O(n) is hampered by the exponential growth of the number of spinor components.

Results presented in this talk have been discussed in more detail elsewhere.[9,10] Some new features will be shown here.

II. SU(n): Higgs Potential

Let the complex field H_i ($i = 1, \ldots, n$) transform as the fundamental, the Hermitean, traceless field ϕ_i^j as the adjoint representation of SU(n). The most general renormalizable potential of degree four, invariant (for simplicity) under the discrete operation $\phi_i^j \to -\phi_i^j$, is, using the notation of Ref. 6:

$$V(\phi,H) = -\frac{1}{2}\mu^2 \operatorname{tr}\phi^2 + \frac{a}{4}(\operatorname{tr}\phi^2)^2 - \frac{1}{2}\nu^2 H^+H + \frac{\lambda}{4}(H^+H)^2$$

$$+ \alpha H^+H \operatorname{tr}\phi^2 + \frac{b}{2}\operatorname{tr}\phi^4 + \beta H^+\phi^2 H \tag{2.1}$$

V has to be minimized with respect to all components of H and ϕ. However, we are interested in the symmetry breaking pattern, that is the unbroken subgroups. This does not depend on the norm of H and ϕ, but only on their direction in representation space. Hence, we will discuss the absolute minimum of

$$F = \frac{b}{2}\operatorname{tr}\phi^4 + \beta H^+\phi^2 H \tag{2.2}$$

keeping $\operatorname{tr}\phi^2$ and H^+H fixed. Diagonalizing ϕ_i^j

$$\phi_i^j = a_i\,\delta_i^j \tag{2.3}$$

$$\sum_{i=1}^{n} a_i = 0 \quad ; \qquad \sum_{i=1}^{n} a_i^2 = \operatorname{tr}\phi^2 \tag{2.4}$$

F becomes

$$F = \frac{b}{2}\sum_{i=1}^{n} a_i^4 + \beta \sum_{i=1}^{n} |H_i|^2 a_i^2 \tag{2.5}$$

Clearly, if b and β are both positive, each sum in (2.5) has to be minimum; if b and β are both negative, each sum has to be maximum. If the signs are opposite, more discussion is required.

Minimizing with respect to H_i gives:[7]

$$|H_i| = 0 \quad , \quad i = 1, 2, \ldots, n-1 \tag{2.6}$$

If we choose

$$\begin{aligned} a_n^2 &< a_i^2 (i = 1, 2, \ldots, n-1) \quad \text{when } \beta > 0 \\ a_n^2 &> a_i^2 (i = 1, 2, \ldots, n-1) \quad \text{when } \beta < 0 \end{aligned} \tag{2.7}$$

with (2.6), one gets for F:

$$F = \frac{b}{2} \sum_{i=1}^{n} a_i^4 + \beta H^+ H a_n^2 \tag{2.8}$$

III. SU(n): Symmetry Breaking Pattern

We first minimize with respect to $a_i (i = 1, \ldots, n-1)$. The following Lemma, proved in Ref. 9, is very useful:

Lemma:* the absolute minimum or the absolute maximum of

$$f = \sum_{i=1}^{n-1} a_i^4 \tag{3.1}$$

where a_i are real numbers subject to the constraints

$$\sum_{i=1}^{n-1} a_i = \sigma \quad ; \quad \sum_{i=1}^{n-1} a_i^2 = \rho^2 \tag{3.2}$$

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*It can be shown that the Lemma holds also for the function

$$g = b \sum_{i=1}^{n-1} a_i^4 + d \sum_{i=1}^{n-1} a_i^3 \quad .$$

for fixed σ and ρ^2 ($\sigma^2 \leq (n-1)\rho^2$), occurs only if at most two of the n-1 variables a_i are distinct. Furthermore, the absolute maximum is obtained if at least n-2 variables a_i are equal. This Lemma can be applied to the minimization of $\mathrm{tr}\,\phi^4$, where ϕ is a second rank term with real eigenvalues.

According to the Lemma, one has:

$$n_1 \text{ times } a_1 , \qquad n_2 \text{ times } a_2$$

$$n_1 + n_2 = n - 1 \tag{3.3}$$

The subgroup structure after symmetry breaking is then:

$$SU(n_1) \times SU(n_2) \times U(1) \qquad \text{if } n_1 n_2 \neq 0$$

$$SU(n-1) \qquad \text{if } n_1 n_2 = 0$$

In order to find n_1 and n_2, rewrite (2.4) as

$$n_1 a_1 + n_2 a_2 = -a_n \tag{3.4}$$

$$n_1 a_1^2 + n_2 a_2^2 = \mathrm{tr}\,\phi^2 - a_n^2 \tag{3.5}$$

Solve for a_1 and a_2 and define

$$x = \frac{n_1 - n_2}{\sqrt{n_1 n_2}} \quad ; \qquad \sin^2\theta = \frac{a_n^2}{\mathrm{tr}\,\phi^2} \cdot \frac{n}{n-1} \left(0 \leq \theta \leq \frac{\pi}{2}\right) \tag{3.6}$$

The function F to be minimized can now be written

$$F = \frac{b}{2}\,\mathrm{tr}\,\phi^2 [\sin^4\theta (n^2 - 3n + 3) + 6 \sin^2\theta\cos^2\theta + n(1 + x^2)\cos^4\theta$$

$$+ 4 \frac{b}{|b|} x\sqrt{n} \sin\theta\cos^3\theta + \frac{\beta}{b}\, 2 \frac{H^+H}{\mathrm{tr}\,\phi^2} (n-1)^2 \sin^2\theta] \frac{1}{n(n-1)} \tag{3.7}$$

The result of the minimization is given in Table I. For $b > 0$, $\beta < 0$, the solution depends on the ratio β/b.

IV. SU(n): Discussion

The characteristics of the subgroup after symmetry breaking are the following:

1) One loses one rank. This is due to H in the fundamental representation. With $\nu = \lambda = \beta = 0$, one would get instead $SU(n-1) \times U(1)$ or $SU(n/2) \times SU(n/2) \times U(1)$ (n even) or $SU\ (n-1)/2\ \times SU\ (n-1)/2\ \times U(1)$ (n odd).

2) The subgroup is the product of at most three factors, with at most one U(1). This follows from the Lemma.

3) For $b > 0$ and $\beta < 0$, as the ratio β/b varies, one finds $n/2$, respectively $(n-1)/2$ different subgroups for n even, respectively n odd.

4) $SU(n-1)$ is obtained for $b < 0$, $\beta < 0$ and for part of the quadrant $b > 0$, $\beta < 0$. $SU(n/2) \times SU\big((n/2)-1\big) \times U(1)$ (n even) is obtained for $b > 0$, $\beta > 0$ and for part of the quadrant $b > 0$, $\beta < 0$.

5) For SU(5), the solution is: SU(4) for $b < 0$, $\beta < 0$ and part of the quadrant $b > 0$, $\beta < 0$. $SU(3) \times U(1)$ for $b < 0$, $\beta > 0$ and part of the quadrant $b > 0$, $\beta < 0$. $SU(2) \times SU(2) \times U(1)$ for $b > 0$, $\beta > 0$.

V. O(10): Higgs Potential

We consider the antisymmetric Higgs scalar, $\phi_{ij} = -\phi_{ji}$ $(i,j = 1 \ldots 10)$ belonging to the adjoint representation 45, and the $16 + \overline{16}$ Higgs scalars χ in the spinor representation. We are interested in the most general renormalizable potential of degree four, excluding odd terms by requiring invariance under $\phi \to -\phi$.

Notice that:

1) The product $16\times 16\times 16\times 16$ vanishes.

2) For the symmetric product of two 16's one gets $(16\times 16)_s = 10+126$ so that one might have only two independent invariants $16\times 16\times \overline{16}\times \overline{16}$.

3) From $16\times\overline{16} = 1+45+210$ and $(45\times 45)_s = 1+54+210+770$ one sees that there are only two independent invariants $16\times\overline{16}\times 45\times 45$.

From this it follows that the most general Higgs potential is:

$$
\begin{aligned}
V(\phi,\chi) &= V_0 + V_s + V_a + V_i \\
V_0 &= a\chi^+\chi + b\,\mathrm{tr}\,\phi^2 + c(\chi^+\chi)^2 + d(\mathrm{tr}\,\phi^2)^2 + e\chi^+\chi\,\mathrm{tr}\,\phi^2 \\
V_s &= \kappa \sum_{i=1}^{10} (\chi^T c\gamma_i\chi)(\chi^T c\gamma_i\chi)^* \qquad (5.1) \\
V_a &= \lambda\,\mathrm{tr}\,\phi^4 \; ; \qquad V_i = \mu\chi^+\left(\sum_{i,j=1}^{10} \sigma_{ij}\,\phi_{ij}\right)^2 \chi
\end{aligned}
$$

where c,γ are defined in Ref. 11, 12, 13.

VI. O(10): Orbits of the Spinor Representation and Symmetry Breaking

A given spinor χ can be considered as a point in a 16 dimensional space. Acting on χ with all the group elements of O(10), one gets a set of points called the orbit of χ. According to Michel and Radicati,[14] all smooth, real, invariant functions are stationary on what they call critical orbits. A fortiori, an absolute minimum of the Higgs potential will occur on a critical orbit.

To characterize these orbits, define the basic states $\chi_A (A = 1 \ldots 16)$. They are eigenstates of five mutually commuting generators $H_\alpha (\alpha = 1 \ldots 5)$ belonging to the Cartan subalgebra. The eigenvalues are $\lambda_\alpha^A = \pm\frac{1}{2}$, their product over α being positive for representation 16, negative for $\overline{16}$. Any spinor χ can be written as

$$\chi(c_A) = \sum_{A=1}^{16} c_A \chi_A \tag{6.1}$$

For a given basic state χ_A, define a $\chi_{\overline{A}}$ by changing the sign of four eigenvalues λ_α^A. For example, if χ_A is given by $(+\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2})$ then a $\chi_{\overline{A}}$ is given by $(+\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$. There are five possibilities for $\chi_{\overline{A}}$. It can be shown that any $\chi(c_A)$ can be transformed by an O(10) rotation into

$$\chi(\theta) = \chi_A \cos\theta + \chi_{\overline{A}} \sin\theta \tag{6.2}$$

Especially, any basic state χ_A can be transformed into another basic state $\chi_{A'}$. The invariant V_s of Eq. (5.1) becomes

$$V_s = \kappa |\chi^+ \chi|^2 \sin^2\theta \cos^2\theta \tag{6.3}$$

For each value θ one gets a different orbit. The critical orbits correspond to extrema of V_s. For $\kappa > 0$, V_s is minimum for $\theta = 0$ and $\theta = \pi/2$. The corresponding little groups, i.e., the subgroups of O(10) leaving $\chi(0)$ or $\chi(\pi/2)$ invariant are SU(5) groups conjugate to each other. For $\kappa < 0$, V_s is minimum for $\theta = \pi/4$. The little group of $1/\sqrt{2}\left(\chi(0) + \chi(\pi/2)\right)$ is O(7).

The set of orbits for $\theta \neq 0, \pi/4, \pi/2$ (modulo $\pi/2$) is called generic and not critical. The little group is $O(6) \sim SU(4)$.

Hence, the symmetry breaking due to the spinor representation alone yields the subgroups SU(5) or O(7).

VII. O(10): Symmetry Breaking Patterns

It remains to minimize V_i and V_a in (5.1). Through an O(10) transformation, one can always rotate the Higgs scalar ϕ^{ij} in such a way that the only non-zero components are ($\alpha = 1 \ldots 5$)

$$\phi_{2\alpha-1,2\alpha} = -\phi_{2\alpha,2\alpha-1} = a_\alpha$$

$$\sum_{\alpha=1}^{5} a_\alpha^2 = \mathrm{tr}\,\phi^2 \tag{7.1}$$

Replacing (7.1) in (5.1), we get

$$V_a = \lambda \sum_{\alpha=1}^{5} a_\alpha^4 \tag{7.2}$$

Keeping $\mathrm{tr}\,\phi^2$ fixed, the minimum of V_a is obtained for $a_1 = a_2 = a_3 = a_4 = a_5$ if $\lambda > 0$ and $a_1 = a_2 = a_3 = a_4 = 0 \neq a_5$ if $\lambda < 0$, the unbroken subgroups being $SU(5) \times U(1)$ and $O(8) \times U(1)$ respectively.[5]

With (6.1) and (5.1) one obtains

$$V_i = \frac{\mu}{4} \sum_{A=1}^{16} |c_A|^2 \sigma_A^2 \quad ; \qquad \sigma_A = 2 \sum_{\alpha=1}^{5} \lambda_\alpha^A a_\alpha \tag{7.3}$$

At fixed values of the a_α's we define $\sigma^2 = (\mathrm{Min}\ \mu\sigma_A^2)/\mu$ so that the absolute minimum $V_i = (\mu/4)\,\chi^+\chi\,\sigma^2$ is obtained for any direction χ_{A_0} such that $\sigma_{A_0} = \sigma$.

Now one has to minimize

$$V_i + V_a = \lambda \sum_{\alpha=1}^{5} a_\alpha^4 + \frac{\mu}{4} \chi^+ \chi \, \sigma^2$$
$$\sum_{\alpha=1}^{5} a_\alpha^2 = \mathrm{tr}\,\phi^2 \tag{7.4}$$

with respect to a_α, where $\sigma = \sigma_{A_0} = \sum_\alpha 2\lambda_\alpha^{A_0} a_\alpha$. As noticed before, the basic state χ_{A_0}, associated to $\sigma_{A_0} = \sigma$, can be transformed into any other basic state. Correspondingly, only the signs at the a_α's will be affected, $a_\alpha \to \pm a_\alpha$, in Eq. (7.3). Therefore we take $\chi_{A_0} = \chi_1$, $\lambda_\alpha = \lambda_\alpha^1 = +\frac{1}{2} (\alpha = 1, \ldots, 5)$, $\sigma = \sum_\alpha a_\alpha$, without any loss of generality.

We can now again apply the Lemma of Section III to the minimum of $\lambda \sum_\alpha a_\alpha^4$ keeping $\sum_\alpha a_\alpha^2 = \mathrm{tr}\,\phi^2$ and $\sum a_\alpha = \sigma$ fixed.

It is then a matter of simple algebra[10] to find the absolute minimum of (7.4). In particular, one finds that $a_\alpha \neq 0$ and $a_\alpha \neq -a_\beta$ $(\alpha, \beta = 1, \ldots, 5)$. The result is given in Table II.

The absolute minimum of the Higgs potential V of Eq. (5.1) is as follows. For $\kappa > 0$, the choice of the spinor χ on the critical orbit with invariance group SU(5) is the proper one to minimize both V_s and $V_i + V_a$. Minimizing with respect to a_i yields the subgroups of SU(5) given in Table II.

If, instead, $\kappa < 0$ and $\lambda\mu \neq 0$, it can be shown that in order to get the absolute minimum of V, χ cannot stay on the orbit with O(7) invariance group, which minimizes V_s. In general, χ will belong to the orbit defined by $\chi_A \cos\theta + \chi_{\bar{A}} \sin\theta$, with θ depending on the parameters of the potential V. If $|\kappa|$ is small enough, $\theta = 0$ (or $\theta = \pi/2$), and the pattern of symmetry breaking will again be given by Table II. Otherwise, the residual symmetry group will be a subgroup of rank 3 of SU(4).

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Table I

Su(n) symmetry breaking pattern. The subgroups in the Table leave invariant the minimum of the potential $F = (b/2)\,\mathrm{tr}\,\phi^4 + \beta H^+ \phi^2 H$, $\phi \in$ adjoint, $H \in$ fundamental representation of SU(n).

A) n even	
$b > 0,\ \beta > 0$:	$SU(n/2) \times SU\big((n/2)-1\big) \times U(1)$
$b > 0,\ \beta < 0$:*	$SU(n/2) \times SU\big((n/2)-1\big) \times U(1)$ $SU\big((n/2)+1\big) \times SU\big((n/2)-2\big) \times U(1)$ - - - - - - - - - - - - - - - - $SU(n-1-m) \times SU(m) \times U(1)$ - - - - - - - - - - - - - - - - $SU(n-2) \times U(1)$ $SU(n-1)$
$b < 0,\ \beta < 0$:	$SU(n-1)$
$b < 0,\ \beta > 0$:	$SU(n-2) \times U(1)$
B) n odd	
$b > 0,\ \beta > 0$:	$SU\big((n-1)/2\big) \times SU\big((n-1)/2\big) \times U(1)$
$b > 0,\ \beta < 0$:*	$SU\big((n+1)/2\big) \times SU\big((n-3)/2\big) \times U(1)$ $SU\big((n+3)/2\big) \times SU\big((n-5)/2\big) \times U(1)$ - - - - - - - - - - - - - - - - $SU(n-1-m) \times SU(m) \times U(1)$ - - - - - - - - - - - - - - - - $SU(n-2) \times U(1)$ $SU(n-1)$
$b < 0,\ \beta < 0$:	$SU(n-1)$
$b < 0,\ \beta > 0$:	$SU(n-2) \times U(1)$

*Increasing the ratio β/b.

Table II*

O(10) symmetry breaking pattern. The subgroups in the Table leave invariant the minimum of the potential:

$$V = \kappa \sum_{i=1}^{10} \chi^T c\gamma_i \chi \left(\chi^T c\gamma_i \chi\right)^* + \lambda \mathrm{tr}\,\phi^4 + \mu\chi^+ \left(\sum_{i,j=1}^{10} \sigma ij \phi ij\right)^2 \chi ,$$

$\phi \in$ adjoint and $\chi \in$ spinor representation.

$\lambda > 0, \mu > 0$:	$SU(3) \times SU(2) \times U(1)$
$\lambda > 0, \mu < 0$:	$SU(5)$
$\lambda < 0, \mu > 0$:	$SU(4) \times U(1)$
$\lambda < 0, \mu > 0$:	$SU(4) \times U(1)$ or† $SU(5)$

*Table II gives the result for $\kappa > 0$. For $\kappa < 0$, see text.

†Depending on the ratio μ/λ.

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