

CHIRAL SYMMETRY BREAKDOWN IN LARGE-N CHROMODYNAMICS*

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ABSTRACT

Chromodynamics with n flavors of massless quarks is invariant under chiral $U(n) \times U(n)$. We show that in the limit of large number of colors, under reasonable assumptions, this symmetry group must spontaneously break down to diagonal $U(n)$.

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In nature, the gauge group of chromodynamics is $SU(3)$, and quarks are color triplets. Nevertheless, it is useful to consider generalizations in which the gauge group is $SU(N)$ and quarks are color N -tuplets. There are many observed properties of meson dynamics (e.g., Zweig's rule) that can be argued to be exact in the large- N limit; it is tempting to believe that this indicates that large- N chromodynamics is in some sense a good approximation to the real world.

In this note we study large- N chromodynamics with n massless quark N -tuplets. This theory is invariant² under the chiral symmetry group $U(n) \times U(n)$. This group contains many inequivalent subgroups; thus group theory allows many possible patterns of spontaneous symmetry breakdown. We shall argue here that in the large- N limit, under reasonable assumptions, the pattern of chiral symmetry breakdown is uniquely fixed: chiral $U(n) \times U(n)$ necessarily breaks down to diagonal $U(n)$. Hearteningly, this is the pattern observed in nature.

Our assumptions are as follows:

- 1) We assume that the large- N limit exists, that chromodynamics has an asymptotic expansion in powers of $1/N$.
- 2) We assume that chromodynamics confines for arbitrarily large N .
- 3) We assume that the breakdown of chiral symmetry is characterized by a non-zero value of some order parameter which is bilinear in the quark fields and which transforms according to the representation $(n, \bar{n}) \times (\bar{n}, n)$ of the chiral group.
- 4) We assume that the ground states of the theory are found by minimizing some effective potential, V , an invariant function

of the order parameter, constructed in the standard way by summing (an infinite number of) connected Feynman graphs.

- 5) We assume that in the large-N limit, the effective potential does not display accidental degeneracy, that any of its minima can be obtained from any other by the action of the chiral group.

Assumptions 1), 2), 4), and 5) are more or less standard. Assumption 3), though, requires comment, because it restricts the pattern of symmetry breakdown even before we invoke large-N dynamics. Let us label the order parameter by a (not necessarily Hermitian) $n \times n$ matrix, M . For example, the simplest candidate for M is

$$M_j^i = \langle \bar{\psi}^i (1 + \gamma_5) \psi_j \rangle, \quad (1)$$

where i and j are flavor indices, the brackets indicate vacuum expectation value, and the sum over (suppressed) color indices is implied. We stress that this is just an example; for our purposes some non-local or smeared-out version of this will do as well. All we need are the chiral transformation properties of M ,

$$(u, v) : M \rightarrow u M v^+, \quad u, v \in U(n). \quad (2)$$

It is easy to show that by a transformation of this form we can always make M real, diagonal, and non-negative. The squares of the diagonal entries are the eigenvalues of $M^+ M$ (or, equivalently, of MM^+). Thus V can depend only on these eigenvalues, and the pattern of chiral symmetry breakdown is determined by the pattern of eigenvalues at the minimum of

V. For example, if all the eigenvalues vanish at the minimum, there is no symmetry breakdown; if they are all equal but non-zero, the symmetry breaks down to diagonal $U(n)$; if they are all unequal and non-zero, it breaks down to $U(1)^n$, etc. Note that under our assumption, breakdown beyond $U(1)^n$ is impossible. If we had assumed two order parameters, M and M' , or if we had assumed different chiral transformation properties for the order parameter, further breakdown would have been allowed.

This concludes our introductory discussion. The remainder of this note is the proof of the announced result.

If we expand V in powers of M and M^+ , we will encounter terms like $\text{Tr}(MM^+)^r$, $\text{Tr}(MM^+)^r \text{Tr}(MM^+)^s$, etc. Because traces of quark operators arise in Feynman graphs from sums over quark loops, the terms of the first kind come from graphs with one quark loop, those of the second kind from graphs with two quark loops, etc. However, it is known¹ that in the large- N limit, connected graphs with L quark loops are $O(N^{2-L})$. Thus, the dominant graphs are those with only one quark loop, and, to leading order in $1/N$,

$$V = N \text{Tr} F(MM^+) \quad , \quad (3)$$

where F is some N -independent function. If we denote the eigenvalues of MM^+ by λ_i , $i=1\dots n$, then

$$V = \sum_i N F(\lambda_i) \quad . \quad (4)$$

Since the eigenvalues are independent variables, to minimize this sum is to minimize each term. Each eigenvalue must be at the minimum of F , and

thus the eigenvalues are either all zero (no symmetry breakdown) or all equal and non-zero (breakdown to $U(n)$).

We shall now eliminate the first alternative. We shall apply a method of analysis recently devised by 't Hooft,³ based on the Adler-Bell-Jackiw anomaly.⁴ The simplicity of the large- N theory makes the application particularly clean; there is no need of the supplementary assumptions required in the examples considered by 't Hooft.

Let us consider a chiral current

$$j_\mu = \bar{\psi} A (1 + \gamma_5) \gamma_\mu \psi, \quad (5)$$

where A is an $n \times n$ Hermitian matrix, and let us define the three-current Green's function by

$$\Gamma_{\mu\nu\lambda}(p, q, r) = \int d^4x d^4y e^{ip \cdot x} e^{iq \cdot y} T \langle j_\mu(x) j_\nu(y) j_\lambda(0) \rangle, \quad (6)$$

where r is $-(p+q)$. Γ is symmetric under simultaneous permutations of (p, q, r) and (μ, ν, λ) . The anomaly equation^{4,2} states that

$$r^\lambda \Gamma_{\mu\nu\lambda} = (N/\pi^2) (\text{Tr} A^3) \epsilon_{\mu\nu\lambda\sigma} p^\lambda q^\sigma. \quad (7)$$

We will choose A such that $\text{Tr} A^3$ is not zero.

Equation (7) implies that Γ cannot be analytic at $p=q=r=0$. Proof: If Γ is analytic, it has a Taylor expansion, and the right-hand side of Eq. (7) must come from a first-order term in this expansion. If we neglect the permutation symmetry of Γ , there are two independent first-order pseudotensors, $\epsilon_{\mu\nu\lambda\sigma} p^\sigma$ and $\epsilon_{\mu\nu\lambda\sigma} q^\sigma$. However, when we symmetrize these, each becomes $\epsilon_{\mu\nu\lambda\sigma} (p+q+r)^\sigma = 0$. Q.E.D.

It is known that in leading order in $1/N$, the only singularities in Green's functions made of strings of quark bilinears are poles.¹ For a three-bilinear Green's function, like Γ , these poles are at values of p^2 , q^2 , and/or r^2 equal to the masses of the particles made by applying the individual bilinears to the vacuum. Because Γ is not analytic at $p=q=r=0$, j_μ must create at least one massless particle when applied to the vacuum. If we were dealing with massive particles, a vector current could create either vector or scalar particles. For massless particles, though, Lorentz invariance forbids the creation of vector particles; only scalar particles are allowed.⁵ But for a conserved current, like j_μ , this is the Goldstone alternative: the current creates a massless scalar particle from the vacuum if and only if the associated symmetry suffers spontaneous breakdown.

Thus the first of our two alternatives, no symmetry breakdown at all, is excluded, and only the second, breakdown to diagonal $U(n)$, remains. This completes the argument.

Some comments: Our argument fell into two parts. For the first part (either no breakdown or breakdown to $U(n)$) we did not need assumption 2), the assumption of confinement. For the second part (exclusion of no breakdown), we did not need assumptions 3) to 5). In particular, we did not need to assume anything about the chiral transformation properties of the order parameter. Regrettably, no part of our argument gives any insight at all into the mechanism of symmetry breakdown.

NOTES

1. G. 't Hooft, Nucl. Phys. B72, 461 (1974), B75, 461 (1974); G. Rossi and G. Veneziano, Nucl. Phys. B123, 507 (1977); G. Chew and C. Rosenzweig, Phys. Reports 41C, 263 (1978); E. Witten, Nucl. Phys. B160, 57 (1979); S. Coleman, "1/N," lectures delivered at the 1979 School of Subnuclear Physics "Ettore Majorana" (to appear in the proceedings of the school, to be published by Plenum Publishers).
2. The chromodynamic anomaly in the U(1) axial current is irrelevant in the large-N limit; its effects first appear in next-to-leading order in 1/N. See E. Witten, Nucl. Phys. B156, 269 (1979); P. Di Vecchia, Phys. Lett. 85B, 357 (1979); G. Veneziano, Nucl. Phys. B159, 213 (1979). These papers all assume that the observed pattern of chiral symmetry breakdown persists in the large-N limit. The analysis given above replaces this assumption with a much weaker one.
3. G. 't Hooft, lectures at the 1979 Cargese School (to appear in the proceedings of the school).
4. S. L. Adler, Phys. Rev. 177, 2426 (1969); J. S. Bell and R. Jackiw, Nuovo Cimento 60A, 107 (1967); R. Jackiw, in Current Algebra and its Applications, ed. S. Treiman (Princeton University Press, Princeton, 1972).
5. The little group of a null vector, k , is isomorphic to the two-dimensional Euclidean group. Under this group, the single helicity state of a scalar particle transforms according to the trivial representation, while the two helicity states of a vector particle

each transform according to non-trivial one-dimensional representations. On the other hand, of the four components of the current, only the one aligned with k transforms according to a one-dimensional representation, the trivial representation.