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SU(3) WAVE SOLUTIONS*

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(Running Head: SU(3) Wave Solutions)

ABSTRACT

An ansatz yielding propagating wave solutions of pure SU(3) gauge theories is exhibited. The solutions are self-dual and have a superposition property like their SU(2) analogues. Possible generalizations of the ansatz which may be used to obtain additional irreducible SU(3) solutions are also suggested.

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I. INTRODUCTION

SU(2) Yang-Mills theories provide the simplest examples of theories with a non-Abelian gauge symmetry and hence have been the focus of most investigations. However, since the strong interactions may be mediated by an octet of colour SU(3) gauge fields, it is also important to consider classical solutions of SU(3) gauge theories. Apart from the possibility that such solutions will be relevant to the quantum theory, the study of SU(3) solutions is attractive for a number of other reasons. Firstly, results obtained for SU(2) theories generalize readily to higher rank gauge groups since it is always possible to embed known SU(2) solutions. Secondly, because SU(3) has an inherently more complex structure than SU(2), it is possible that some non-trivial generalizations of SU(2) solutions exist, apart from the straightforward embeddings. Finally, unlike the SU(2) case, it is possible to construct a stable solution to the pure SU(3) gauge field equations without the introduction of explicit scalar fields [1].

A large number of SU(3) solutions have now been discovered. These include generalizations of the 't Hooft-Polyakov monopole [1-6] and the Prasad-Sommerfield monopole [7], as well as SU(3) dyons [8]. Models of SU(3) monopoles coupled to fermions have also been considered [9].

SU(3) instantons with topological charges of ±1 and ±4, corresponding to the two inequivalent embeddings of the group SU(2) inside SU(3) [10,11] have been obtained. Irreducible SU(3) solutions have resulted from an O(3) symmetric ansatz [12,13] corresponding to an SU(3) generalization of Witten's cylindrically symmetric multi-instanton solution [14]. SU(3) versions of meron [15] and multi-meron [16] configurations also exist and investigation of complex SL(3,C) self-dual fields [17] has yielded a number of interesting non-trivial solutions.

The Corrigan-Fairlie-'t Hooft-Wilczek (CRtHW) ansatz [18], however, has not yet been generalized to SU(3) due to the difficulty of finding an analogue of the 't Hooft tensor, $\eta_{a\mu\nu}$. We have utilized a particularly simple version of this ansatz for the investigation of propagating wave solutions in SU(2) gauge theories [19]. In addition to the natural interest in SU(3) versions of these solutions, we might hope to gain some indication of possible generalizations of $\eta_{a\mu\nu}$ appropriate to SU(3).

In Section II we exhibit a generalization of the O(3) symmetric ansätze used by other authors [8,13] to find SU(3) solutions. This generalization is shown to yield the SU(3) version of the wave solutions mentioned above. The self-duality properties of the ansatz are considered in detail in Section III and possible generalizations which may be useful for finding the SU(3) analogue of the CFtHW ansatz are discussed in Section IV.

II. SU(3) WAVE SOLUTIONS

We begin by writing the gauge potential A_μ and field strength $F_{\mu\nu}$ as matrices in the space of infinitesimal group generators

$$A_\mu = \frac{A_\mu{}^a T^a}{2i} \quad , \tag{2.1a}$$

$$F_{\mu\nu} = \frac{F_{\mu\nu}{}^a T^a}{2i} \quad , \tag{2.1b}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad . \tag{2.1c}$$

The equation of motion may thus be written as

$$D^{\nu}F_{\mu\nu} = \partial^{\nu}F_{\mu\nu} + [A^{\nu}, F_{\mu\nu}] = 0 \quad . \tag{2.2}$$

For SU(2) gauge theories the matrices T^a are given by the 2×2 Pauli matrices, σ^a, $a = 1,\ldots,3$ whereas for SU(3) gauge theories they are chosen to be the usual 3×3 Gell-Mann matrices, λ^a, $a = 1,\ldots,8$.

As discussed in an earlier paper [19], a suitable ansatz for wavelike solutions of SU(2) gauge theories characterized by a propagation vector k_μ is given by

$$A_\mu = i\sigma_{\mu\nu}\, k^{\nu}\, f(k\cdot x) \quad , \tag{2.3}$$

where $k\cdot x = k^{\mu}x_{\mu}$. Equation (2.3) is just a special case of the CFtHW ansatz. The antisymmetric matrices $\sigma_{\mu\nu}$ satisfy the 0(4) commutation relations and are defined as usual by

$$\sigma_{ij} = \frac{1}{4i}\,[\sigma_i, \sigma_j] \quad ,$$

and (2.4)

$$\sigma_{i4} = \frac{1}{2}\,\sigma_i \quad ,$$

or

$$\sigma_{\mu\nu} = \frac{\eta_{a\mu\nu}\sigma^a}{2} \quad ,$$

with

$$\begin{aligned} \eta_{a\mu\nu} &= \varepsilon_{a\mu\nu} \qquad a,\mu\ \nu = 1,2,3 \quad , \\ &= \delta_{a\mu} \quad , \qquad \nu = 4 \quad . \end{aligned} \tag{2.5}$$

The CFtHW ansatz may be extended to Minkowski space[1] by defining

$$\sigma_{i0} = \frac{i\sigma_i}{2} \quad .$$

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[1]Our metric is $g^{\mu\nu} = \mathrm{diag}\ (+ - - -)$.

The set of $\sigma_{\mu\nu}$ matrices thus obtained satisfy $O(3,1)$ commutation relations and yield complex solutions in Minkowski space. In the particular case of ansatz (2.3) one class of self-dual solutions with a restricted superposition property is obtained provided that $k^2 = k_\mu k^\mu = 0$. Since the function $f(k\cdot x)$ remains completely arbitrary, these solutions may be regarded as non-abelian generalizations of electromagnetic plane waves. The Euclidean space version of these solutions with $k^2 = 0$ is of course trivial.

In order to obtain SU(3) versions of these SU(2) wave solutions, we begin by defining a generalization of the O(3) symmetric ansätze used by Horvath and Palla [8] and Bais and Weldon [13]. As in the SU(2) case it is possible to exhibit the ansatz in either Euclidean or Minkowski space.

A. Euclidean Space Version

We choose the SU(3) gauge potentials to be

$$\begin{aligned} A_i = i\varepsilon_{ijk}k_jL_kH(v) + i\varepsilon_{ijk}\frac{k_jk_p}{k}Q_{kp}G(v) + iL_ik_4D(v) \\ + iQ_{ij}\frac{k_jk_4}{k}E(v) + ik_jL_jk_iA(v) + iQ_{rs}\frac{k_rk_sk_i}{k}B(v) \quad . \end{aligned} \tag{2.6a}$$

$$A_4 = -iL_ak_a\,C(v) - iQ_{ab}\frac{k_ak_b}{k}F(v) \quad . \tag{2.6b}$$

where $v = k_\mu x_\mu = k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4$,

$k = |\vec{k}|$,

$(L_a)_{ij} = i\varepsilon_{iaj}$,

$(Q_{ab})_{ij} = \delta_{ai}\,\delta_{bj} + \delta_{aj}\,\delta_{bi} - \frac{2}{3}\delta_{ab}\,\delta_{ij}, \qquad i,j,a,b = 1,3$,

and H, G, D, E, A, B, C, F are unknown functions of v. It can be seen that ansatz (2.6) corresponds to the most general form of the potential which can be constructed from L_a, Q_{ab} and the vector k_μ, and reduces to the simple SU(2) embedding if

$$G = D = E = A = B = F \equiv 0 \quad .$$

L_a and Q_{ab} satisfy the commutation relations:

$$[L_a, L_b] = i\,\varepsilon_{abc}\,L_c \quad , \tag{2.7a}$$

$$[L_a, Q_{bc}] = i\,(\varepsilon_{cna}\,Q_{bn} + \varepsilon_{bna}\,Q_{cn}) \quad , \tag{2.7b}$$

$$[Q_{ab}, Q_{cd}] = i(\delta_{ad}\,\varepsilon_{bcs} + \delta_{bd}\,\varepsilon_{acs} + \delta_{bc}\,\varepsilon_{ads} + \delta_{ac}\,\varepsilon_{bds})\,L_s \quad . \tag{2.7c}$$

The Lorentz condition, $\partial_\mu A_\mu = 0$ is satisfied if

$$D' + \frac{\vec{k}^2\,A'}{k_4} = C' \quad , \tag{2.8a}$$

$$E' + \frac{\vec{k}^2\,B'}{k_4} = F' \quad , \tag{2.8b}$$

where the prime denotes differentiation with respect to $v = k_\mu x_\mu$.

It is straightforward to calculate the field strengths from Eq. (2.1c)

$$F_{ij} = i\{(\varepsilon_{jrs}k_i - \varepsilon_{irs}k_j)\;[k_r L_s (H' + \frac{{k_4}^2\,E^2}{k^2} + k_4 AD + 2k_4 BE)$$

$$+ \frac{Q_{ps}k_p k_r}{k}(G' - HG + k_4 AE + 2k_4 BD)]$$

$$- \varepsilon_{ijr}k_r k_s L_s (H^2 + G^2) - \varepsilon_{ijp}L_p {k_4}^2 (D^2 + E^2) - 2\varepsilon_{ijn}\,\frac{Q_{mn}k_m}{k}\,{k_4}^2\,DE$$

$$+ (k_i L_j - k_j L_i)\;[k_4 D' - k_4(HD + 2GE) - \vec{k}^2\,(AH + 2BG)]$$

$$- 2\,\varepsilon_{ijs}\,k_s\,\frac{Q_{np}\,k_n k_p}{k}\,HG$$

$$+ \left(\frac{Q_{jm}k_m k_i}{k} - \frac{Q_{im}k_m k_j}{k}\right) [k_4 E' - k_4(2HE + DG) - \vec{k}^2 (AG + 2BH)]$$

$$- (\varepsilon_{irm} Q_{mj} - \varepsilon_{jrm} Q_{mi}) \frac{k_r}{k} (\vec{k}^2 HG + k_4{}^2 DE)\} \quad , \qquad (2.9a)$$

$$F_{4i} = i \{\varepsilon_{ijk}k_j L_k (H' - CD - 2EF) k_4 + \varepsilon_{ijk}k_j k_p \frac{Q_{kp}}{k} k_4 (G' - CE - 2FD)$$

$$+ L_i[k_4{}^2 D' + \vec{k}^2(HC + 2FG)] + \frac{Q_{ij}k_j}{k} [k_4{}^2 E' + \vec{k}^2(2HF + CG)]$$

$$+ k_i[L_a k_a(C' + k_4 A' - HC - 2FG)$$

$$+ \frac{Q_{rs}k_r k_s}{k} (F' + k_4 B' - 2HF - CG)]\} \quad . \qquad (2.9b)$$

Equations (2.9) may now be inserted into the field Eqs. (2.2). In order to illustrate the simplifications required to obtain the field equations in a suitable form it is convenient to consider

$$\partial_i F_{4i} + [A_i, F_{4i}] = 0 \quad . \qquad (2.10)$$

From Eq. (2.9b) it follows immediately that

$$\partial_i F_{4i} = i \{L_i k_i(k_4{}^2 D'' + \vec{k}^2 C'' + \vec{k}^2 k_4 A'') + \frac{Q_{ab}k_a k_b}{k} (k_4{}^2 E''$$

$$+ \vec{k}^2 F'' + k_4 \vec{k}^2 B'')\} \quad . \qquad (2.11a)$$

The Lorentz condition (2.8) may be used to simplify this expression to

$$\partial_i F_{4i} = i \{L_i k_i \; k_\mu{}^2 \; C'' + \frac{Q_{ab}k_a k_b}{k} k_\mu{}^2 \; F'' \} \quad , \qquad (2.11b)$$

where $k_\mu{}^2 = k_4{}^2 + \vec{k}^2$.

Similarly, the inhomogeneous term in Eq. (2.10) becomes

$$[A_i, F_{4i}] = i\,\{k_j L_j [-k_4^{\,2}\ (2CD^2 + 2CE^2 + 8EFD) + 2DH' - 2HD'$$

$$-\vec{k}^2 (2CH^2 + 2CG^2 + 8HGF) + 2EG' - 2GE']$$

$$+ \frac{Q_{ij} k_i k_j}{k} [-\ k_4^{\,2}\ (6FD^2 + 6FE^2 + 6ECD) + 3DG' - 3GD'$$

$$-\ \vec{k}^2\ (6FH^2 + 6FG^2 + 6GHC) + 3EH' - 3HE'\ ]\} \quad . \qquad (2.12a)$$

Upon choosing $D = \{ {H \atop G} \}$ and $E = \{ {G \atop H} \}$, Eq. (2.12a) simplifies to

$$[A_i,\ F_{4i}] = i\ \{-k_j L_j k_\mu^{\,2}\ (2CH^2 + 2CG^2 + 8HGF)$$

$$-\ \frac{Q_{ij} k_i k_j}{k}\ k_\mu^{\,2}\ (6FH^2 + 6FG^2 + 6CHG)\} \quad . \qquad (2.12b)$$

The important point to note in Eqs. (2.11b) and (2.12b) is that they are now multiplied by a factor $k_\mu^{\,2}$. When these solutions are continued to Minkowski space, the class of solutions with $k_\mu^{\,2} = 0$ will automatically satisfy the equations of motion and represent the SU(3) analogues of the self-dual propagating SU(2) wave solutions obtained in earlier work.

The field equations for the field strengths F_{ij} may be simplified by the methods described above, and after much tedious algebra, the equations of motion finally become

$$i\ k_\mu^{\,2}\ \{k_j L_j (C'' - 2H^2 C - 2CG^2 - 8HGF)$$

$$+ \frac{Q_{ij} k_i k_j}{k}\ (F'' - 6FH^2 - 6FG^2 - 6CHG)\} = 0 \qquad (2.13a)$$

and

$$i\,k_\mu^{\,2}\,\{\left(\varepsilon_{jap}k_aL_p + \left\{\begin{matrix} k_4L_j \\ \dfrac{Q_{jn}k_nk_4}{k} \end{matrix}\right\}\right)[H'' - H^3 - 7HG^2 - 4GFC - 4HF^2 - HC^2]$$

$$+ \left(\varepsilon_{jan}\frac{k_ak_sQ_{sn}}{k} + \left\{\begin{matrix} \dfrac{Q_{jn}k_nk_4}{k} \\ k_4L_j \end{matrix}\right\}\right)[G'' - G^3 - 7H^2G - 4CHF - 4GF^2 - GC^2]$$

$$+ \frac{k_4}{k^2}k_jL_ak_a\left[C'' - \left\{\begin{matrix} H'' \\ G'' \end{matrix}\right\} - 2H^2C - 2G^2C - 8HFG\right.$$

$$\left. + \left\{\begin{matrix} H^3 + 7HG^2 + 4GFC + 4HF^2 + HC^2 \\ G^3 + 7H^2G + 4CHF + 4GF^2 + GC^2 \end{matrix}\right\}\right]$$

$$+ \frac{k_jQ_{im}k_ik_mk_4}{k^3}\left[F'' - \left\{\begin{matrix} H'' \\ G'' \end{matrix}\right\} - 6FH^2 - 6FG^2 - 6GHC\right.$$

$$\left. + \left\{\begin{matrix} H^3 + 7HG^2 + 4GFC + 4HF^2 + HC^2 \\ G^3 + 7H^2G + 4CHF + 4GF^2 + GC^2 \end{matrix}\right\}\right]\} = 0 \quad . \tag{2.13b}$$

where the terms in braces correspond to choosing $D = \{ {H \atop G} \}$, $E = \{ {G \atop H} \}$.

Hence solutions of Eq. (2.13) are obtained if

(i) $$C'' - (2CH^2 + 2CG^2 + 8HGF) = 0 \quad , \tag{2.14a}$$

(ii) $$F'' - 6(CHG + FH^2 + FG^2) = 0 \quad , \tag{2.14b}$$

(iii) $$H'' - (H^3 + 7HG^2 + HC^2 + 4GCF + 4HF^2) = 0 \quad , \tag{2.14c}$$

(iv) $$G'' - (G^3 + 7H^2G + GC^2 + 4HCF + 4GF^2) = 0 \quad . \tag{2.14d}$$

or $$k_\mu^{\,2} = 0 \tag{2.15}$$

The trivial SU(2) embedding is recovered by setting

$$H = C \quad ,$$

$$G = F = 0 \quad .$$

B. Minkowski Space Version

By analogy with the SU(2) case, the ansatz for the Minkowski space version of the above solutions may be written as

$$A_i = i\varepsilon_{ijk}k^jL_kH + i\varepsilon_{ijk}\frac{k^jk^PQ_{kp}G}{k} - L_ik^oD - \frac{Q_{ij}k^jk^oE}{k}$$

$$+ k^jL_jk_iA + \frac{Q_{rs}k^rk^sk_iB}{k} \quad , \tag{2.16a}$$

$$A_o = L_ak^aC + Q_{ab}\frac{k^ak^b}{k}F \quad . \tag{2.16b}$$

Inserting ansatz (2.16) into Eq. (2.1c) yields the field strengths:

$$F_{oi} = \{i\varepsilon_{ijk}k^jL_k(H' + CD + 2EF)k^o + i\varepsilon_{ijk}\frac{k^jk^PQ_{kp}k^o}{k}(G' + CE + 2FD)$$

$$+ L_i[-k_o{}^2D' - \vec{k}^2(HC + 2FG)] + Q_{ij}\frac{k^j}{k}[-k_o{}^2E' - \vec{k}^2(CG + 2HF)]$$

$$+ k_i[L_jk^j(k_oA' - C' - HC - 2FG)$$

$$+ \frac{Q_{ab}k^ak^b}{k}(k_oB' - F' - CG - 2HF)]\} \quad , \tag{2.17a}$$

and

$$F_{ij} = \{(i\varepsilon_{jrs}k_i - i\varepsilon_{irs}k_j)\ [L_sk^r(H' + \frac{k_o{}^2}{k^2}E^2 + k^oAD + 2k^oBE)$$

$$+ Q_{ps}\frac{k^Pk^r}{k}(G' + HG + k^oAE + 2k^oBD)] - i\varepsilon_{ijr}k^rL_sk^s(H^2 + G^2)$$

$$+ i\varepsilon_{ijp}L_pk_o{}^2(D^2 + E^2) + 2i\varepsilon_{ijn}\frac{Q^{mn}k_m}{k}k_o{}^2DE$$

$$-2i\varepsilon_{ijs}\ k^s\ Q_{np}\frac{k^nk^p}{k}\ HG$$

$$+ (k_iL_j - k_jL_i)[-k^oD' - k^o(HD + 2GE) - \vec{k}^2(AH + 2BG)]$$

$$+ (Q_{jm}\frac{k^mk_i}{k} - Q_{im}\frac{k^mk_j}{k})\ [-k^oE' - k^o(GD + 2HE) - \vec{k}^2(AG + 2BH)]$$

$$- (\varepsilon_{irm}Q_{mj} - \varepsilon_{jrm}Q_{mi})\frac{k^r}{k}(i\vec{k}^2HG - ik_o{}^2\ DE)\} \quad . \tag{2.17b}$$

Substituting these expressions into the field equations it is found that just as in the SU(2) case, solutions are obtained if the Eqs. (2.14) or (2.15) are satisfied, where now $k_\mu^{\ 2} = k_o^{\ 2} - \vec{k}^2$. Hence the fields (2.17), with the functions C, F, H and G remaining arbitrary and depending on a propagation vector k_μ such that $k_\mu^{\ 2} = 0$, describe the SU(3) analogues of SU(2) non-Abelian plane waves. Clearly for these solutions just as in the SU(2) case, it is possible to superpose gauge fields of the form (2.16) and still obtain a solution of the field equations.

As expected, the properties of SU(3) plane wave solutions do not differ significantly from those for SU(2) gauge theories.

The important result of this section is not the investigation of these properties, but rather the construction of the ansatz (2.6) or (2.16) in such a way as to ensure that the equations of motion reduce to expressions multiplied by an overall factor of $k_\mu^{\ 2}$.

III. SELF-DUALITY PROPERTIES OF THE SU(3) WAVE SOLUTIONS

It is interesting to examine the consequence of demanding that the field strengths (2.9) satisfy the self-duality condition. For convenience the Euclidean space solutions are considered; however, analagous results may be derived for the Minkowski space solutions.

To obtain the duals of the field strengths (2.9) it is much more convenient to express them in a covariant form. Accordingly, the gauge fields (2.6) are written as

$$A_\mu = i\,\tau_{\mu\nu} k_\nu H + i\,\xi_{4\mu\alpha\beta}\, k_\alpha\, k_\beta \frac{G}{k} + i\,\xi_{44\alpha\beta} \frac{k_\alpha k_\beta}{k\, k_4} K_\mu B$$

$$+ i\,\tau_{4\nu}\, k_\nu\, K_\mu\, \frac{A}{k_4} \quad , \tag{3.1}$$

where
$$K_\mu = \delta_{\mu 4}\, k_\gamma k_\gamma - k_\mu k_4 \quad , \tag{3.2a}$$

$$k = |\vec{k}| \quad , \tag{3.2b}$$

$$\tau_{\mu\nu} = \eta_{a\mu\nu}\, L_a \quad , \qquad a = 1,\ldots,3 \tag{3.2c}$$

$$\xi_{\mu\nu\alpha\beta} = -\eta_{a\mu\alpha}\, \eta_{b\nu\beta}\, Q_{ab} \quad , \qquad a,b = 1,\ldots,3 \quad , \tag{3.2d}$$

and $\eta_{a\mu\nu}$ is the 't Hooft tensor defined by Eq. (2.5).

The tensors $\tau_{\mu\nu}$ are seen to correspond to the embedding of the SU(2) tensors $\sigma_{\mu\nu}$ inside SU(3) which yields instantons with charges of $q = \pm 4$. The tensors $\xi_{\mu\nu\alpha\beta}$ have no analogue in SU(2). With the aid of the somewhat cumbersome commutation relations given in Appendix A, the field strengths may be evaluated:

$$F_{\mu\nu} = i\,\{(\tau_{\nu\rho} k_\mu - \tau_{\mu\rho} k_\nu)\, k_\rho (H' - H^2 - G^2) + \tau_{\nu\mu} k_\gamma k_\gamma (H^2 + G^2)$$

$$+ (\tau_{\nu\rho} K_\mu - \tau_{\mu\rho} K_\nu) \frac{K_\rho}{k_4} \left(AH + 2GB + \frac{G^2 k_4}{\vec{k}^2}\right)$$

$$+ \frac{1}{k_4} (K_\nu\, k_\mu - k_\nu\, K_\mu)\, \tau_{4\rho}\, k_\rho\, (A' - AH - 2GB - G^2 k_4/\vec{k}^2)$$

$$+ (\xi_{4\nu\alpha\beta}\, k_\mu - \xi_{4\mu\alpha\beta}\, k_\nu) \frac{k_\alpha k_\beta}{k} (G' - 3GH) - 3\xi_{4\mu\alpha\nu} \frac{k_\alpha}{k} GHk_\gamma k_\gamma$$

$$+ (K_\nu k_\mu - K_\mu k_\nu)\, \xi_{44\alpha\beta} \frac{k_\alpha k_\beta}{kk_4} (B' - GA - 2HB)$$

$$+ (\xi_{4\nu\alpha\beta}\, K_\mu - \xi_{4\mu\alpha\beta}\, K_\nu) \frac{K_\beta k_\alpha}{kk_4} (GA + 2HB)\} \quad . \tag{3.3}$$

It is trivial to check that Eq. (3.3) is identical to the field strengths (2.9) for D = H and E = G.

The dual of Eq. (3.3), ${}^*F_{\mu\nu} = \frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta}\,F_{\alpha\beta}$, may now be calculated using the formulae given in Appendix B:

$$\begin{aligned}
{}^*F_{\mu\nu} = i\,\{ &- (\tau_{\nu\alpha}\,k_\mu - \tau_{\mu\alpha}k_\nu)\,k_\alpha(H' - H^2 - G^2) + \tau_{\nu\mu}k_\gamma k_\gamma\,H' \\
&- (\tau_{\nu\alpha}\,K_\mu - \tau_{\mu\alpha}K_\nu)\,\frac{K_\alpha}{k_4}\,(AH + 2GB + \frac{k_4G^2}{\vec{k}^2}) \\
&- \tau_{\mu\nu}\,k_\gamma\,k_\gamma\,\frac{\vec{k}^2}{k_4}\,(AH + 2GB + \frac{G^2k_4}{\vec{k}^2}) \\
&+ \varepsilon_{\mu\nu\alpha 4}\,k_\alpha\,k_\gamma\,k_\gamma\,\frac{\tau_{4\rho}k_\rho}{k_4}\,(A' - AH - 2GB - \frac{G^2k_4}{\vec{k}^2}) \\
&- (\xi_{4\nu\gamma\delta}\,k_\mu - \xi_{4\mu\gamma\delta}\,k_\nu)\,\frac{k_\gamma k_\delta}{k}\,(G' - 3GH) - \xi_{4\mu\gamma\nu}\,\frac{k_\gamma G'}{k} \\
&- \xi_{4\mu\gamma\nu}\,\frac{k_\gamma\vec{k}^2}{kk_4}\,(GA + 2HB)\,k_\alpha k_\alpha \\
&- (\xi_{4\nu\gamma\delta}\,K_\mu - \xi_{4\mu\gamma\delta}\,K_\nu)\,\frac{K_\delta k_\gamma}{kk_4}\,(GA + 2HB) \\
&+ \varepsilon_{\mu\nu\alpha 4}\,k_\alpha k_\gamma k_\gamma\,\xi_{44\alpha\beta}\,\frac{k_\alpha k_\beta}{kk_4}\,(B' - GA - 2HB)\} \quad .
\end{aligned} \tag{3.4}$$

Comparing Eqs. (3.3) and (3.4) it is easy to see that the anti-self-duality condition, ${}^*F_{\mu\nu} = -F_{\mu\nu}$, is satisfied if

$$k_\gamma k_\gamma = 0 \quad , \tag{3.5}$$

or

$$\text{(i)} \qquad H' + H^2 + 2G^2 + \frac{\vec{k}^2}{k_4}(AH + 2GB) = 0 \quad ; \tag{3.6a}$$

$$\text{(ii)} \qquad A' - (AH + 2GB + G^2 k_4/\vec{k}^2) = 0 \quad ; \tag{3.6b}$$

$$\text{(iii)} \qquad G' + 3GH + \frac{\vec{k}^2}{k_4}(GA + 2HB) = 0 \quad ; \tag{3.6c}$$

$$\text{(iv)} \qquad B' - GA - 2HB = 0 \quad . \tag{3.6d}$$

It is not difficult to check that the self-duality Eqs. (3.6) imply the equations of motion (2.14), by remembering the Lorentz condition for the ansatz (3.1) is satisfied if

$$H + \frac{\vec{k}^2 A}{k_4} = C \quad ,$$

$$G + \frac{\vec{k}^2 B}{k_4} = F \quad .$$

Hence, as expected the $k_\mu{}^2 = 0$ solutions for SU(3) gauge theories are anti-self-dual, just like their SU(2) analogues. The system of Eqs. (3.6) gives a set of first order equations for wavelike SU(3) solutions, which although simpler than the corresponding equations of motion, are still not trivial to solve.

IV. REMARKS

(i) The form of the ansatz (3.1) is very suggestive. With a suitable choice of the functions G, B and A it reduces to an embedding of the CFtHW ansatz inside SU(3). From a generalization of Eq. (3.1), a possible candidate for an SU(3) version of the CFtHW ansatz may be written as

$$A_\mu = i\,\tau_{\mu\nu}\,\partial_\nu \ln h + i\,\xi_{\mu\rho\alpha\beta}\,[(\partial_\rho \ln C)(\partial_{\alpha\beta} \ln f)$$

$$- (\partial_\rho \ln f)(\partial_{\alpha\beta} \ln C)] \quad , \qquad (4.1)$$

where h, f and C are some superpotentials. It is not difficult to show the Eq. (4.1) satisfies the Lorentz condition.

The field strengths and their duals resulting from ansatz (4.1) have been evaluated. The algebra is rather involved and unfortunately applying the self-duality condition does not lead to the nice simplification which occurs for SU(2). However, it is still possible that (4.1) results in some simplification of the equations of motion, so further investigation of this ansatz is indicated.

(ii) We have recently obtained the most general self-dual SU(2) plane wave solutions [20] by the use of Yang's R-gauge equations [21]. Yang's formulation has also been extended to the gauge group SU(3) [22]. Just as in the SU(2) case, it is not difficult to see that the most general self-dual SU(3) plane wave solutions may be obtained by simply requiring that the functions used in the R-gauge ansatz be dependent on the Lorentz scalar k•x.

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APPENDIX A

The commutation relations for the tensors $\tau_{\mu\nu}$ and $\xi_{\mu\nu\rho\sigma}$ as defined by Eqs. (3.2c)-(3.2d) are given by

$$[\tau_{\mu\nu}, \tau_{\rho\delta}] = i\,(\tau_{\mu\rho}\,\delta_{\nu\delta} - \tau_{\mu\delta}\,\delta_{\nu\rho} + \tau_{\nu\delta}\,\delta_{\mu\rho} - \tau_{\nu\rho}\,\delta_{\mu\delta}) \quad , \tag{A1}$$

$$\begin{aligned}[\xi_{\mu\nu\alpha\beta}, \tau_{\rho\sigma}] = i\,(&\xi_{\mu\nu\alpha\rho}\,\delta_{\beta\sigma} - \xi_{\mu\nu\alpha\sigma}\,\delta_{\beta\rho} + \xi_{\mu\beta\alpha\sigma}\,\delta_{\nu\rho} - \xi_{\mu\beta\alpha\rho}\,\delta_{\nu\sigma} \\ &+ \xi_{\nu\mu\beta\rho}\,\delta_{\alpha\sigma} - \xi_{\nu\mu\beta\sigma}\,\delta_{\alpha\rho} + \xi_{\nu\alpha\beta\sigma}\,\delta_{\mu\rho} - \xi_{\nu\alpha\beta\rho}\,\delta_{\mu\sigma}) \quad ,\end{aligned} \tag{A2}$$

$$\begin{aligned}[\xi_{\mu\nu\alpha\beta}\,\xi_{\rho\sigma\gamma\delta}] = &(\delta_{\mu\sigma}\,\delta_{\alpha\delta} - \delta_{\mu\delta}\,\delta_{\alpha\sigma})[\tau_{\nu\beta}, \tau_{\rho\gamma}] \\ &+ (\delta_{\nu\sigma}\,\delta_{\beta\delta} - \delta_{\nu\delta}\,\delta_{\beta\sigma})[\tau_{\mu\alpha}, \tau_{\rho\gamma}] \\ &+ (\delta_{\nu\rho}\,\delta_{\beta\gamma} - \delta_{\nu\gamma}\,\delta_{\beta\rho})[\tau_{\mu\alpha}, \tau_{\sigma\delta}] \\ &+ (\delta_{\mu\rho}\,\delta_{\alpha\gamma} - \delta_{\mu\gamma}\,\delta_{\alpha\rho})[\tau_{\nu\beta}, \tau_{\sigma\delta}] \quad .\end{aligned} \tag{A3}$$

APPENDIX B

Using the well-known identity

$$\varepsilon_{\nu k\sigma\delta}\ \eta_{a\mu\delta} = \eta_{a\nu k}\ \delta_{\mu\sigma} + \eta_{a\sigma\nu}\ \delta_{\mu k} + \eta_{ak\sigma}\ \delta_{\mu\nu} \quad , \tag{B1}$$

the duals of the tensors $\tau_{\mu\nu}$ and $\xi_{\mu\nu\alpha\beta}$ may be evaluated:

$$\frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta}\ \tau_{\beta\rho} = \tau_{\nu\mu}\ \delta_{\alpha\rho} + \tau_{\alpha\nu}\ \delta_{\rho\mu} + \tau_{\mu\alpha}\ \delta_{\rho\nu} \quad , \tag{B2}$$

$$\frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta}\ \xi_{\beta\rho\gamma\delta} = -\frac{1}{2}\,(\xi_{\mu\rho\nu\delta}\ \delta_{\gamma\alpha} + \xi_{\alpha\rho\mu\delta}\ \delta_{\gamma\nu} + \xi_{\nu\rho\alpha\delta}\ \delta_{\gamma\mu}) \quad , \tag{B3}$$

$$\frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta}\ \xi_{\beta\rho\alpha\delta} = -\xi_{\mu\rho\nu\delta} \quad , \tag{B4}$$

$$\frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta}\ \xi_{\alpha\sigma\delta\beta} = -\frac{1}{2}\ \varepsilon_{\mu\nu\alpha\beta}\ \xi_{\beta\sigma\delta\alpha} = -\frac{1}{4}\ \varepsilon_{\mu\nu\alpha\beta}\ \xi_{\sigma\beta\delta\alpha} = \frac{1}{2}\ \xi_{\mu\sigma\nu\delta} \quad . \tag{B5}$$

The following formulae are also useful

$$\frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta}\ (\tau_{\beta\rho}\ k_{\alpha} - \tau_{\alpha\rho}\ k_{\beta})\ k_{\rho} = -(\tau_{\nu\alpha}\ k_{\mu} - \tau_{\mu\alpha}\ k_{\nu})\ k_{\alpha} + \tau_{\nu\mu}\ k_{\gamma}\ k_{\gamma} \quad , \tag{B6}$$

$$\begin{aligned} &\frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta}\ (\xi_{4\beta\gamma\delta}\ k_{\alpha} - \xi_{4\alpha\gamma\delta}\ k_{\beta})\ k_{\gamma}\ k_{\delta} \\ &\quad = -(\xi_{4\nu\gamma\delta}\ k_{\mu} - \xi_{4\mu\gamma\delta}\ k_{\nu})\ k_{\gamma}\ k_{\delta} - \xi_{4\mu\gamma\nu}\ k_{\gamma}\ k_{\delta}\ k_{\delta} \quad . \end{aligned} \tag{B7}$$

REFERENCES

1. Marciano, W., Pagels, H.: Phys. Rev. D12, 1093-1095 (1975)
2. Wu, A.C.T., Wu, T. T.: J. Math. Phys. 15, 53-54 (1974)
3. Chakrabarti, A.: Ann. Inst. H. Poincaré, 23, 235-249 (1975)
4. Corrigan, E., Olive, D. I., Fairlie, D. B., Nuyts, J.: Nucl. Phys. B106, 475-492 (1976)
5. Sinha, A.: Phys. Rev. D14, 2016-2022 (1976)
6. Horvath, Z., Palla, L.: Nucl. Phys. B116, 500-524 (1976)
7. Kerner, R.: Phys. Lett. 81B, 21-24 (1979); Wilkinson, D., Alexander Bais, F.: Phys. Rev. D19, 2410-2415 (1979); Czechowski, A.: CERN-TH-2282, 1977
8. Horvath, Z., Palla, L.: Phys. Rev. D14, 1711-1714 (1976)
9. Chakrabarti, A.: Nucl. Phys. B101, 159-172 (1975); Jacobs, L.: Phys. Rev. D14, 1739-1746 (1976)
10. Wilczek, F.: Phys. Lett. 65B, 160-162 (1976); Bernard, C. W., Christ, N. H., Guth, A. H., Weinberg, E. J.: Phys. Rev. D16, 2967-2977 (1977)
11. Marciano, W., Pagels, H., Parsa, Z.: Phys. Rev. D15, 1044-1049 (1977)
12. Yates, R. G.: Nucl. Phys. B123, 333-338 (1977)
13. Alexander Bais, F., Arthur Weldon, H.: Phys. Rev. D18, 561-566 (1978)
14. Witten, E.: Phys. Rev. Lett. 38, 121-124 (1977)
15. Cervero, J., Horvath, Z., Palla, L.: University of Salamanca preprint 54/79
16. Imbrie, J. Z.: Lett. Math. Phys. 2, 483-492 (1978)
17. Elizade, E.: Phys. Rev. D19, 1798-1801 (1979)

18. Corrigan, E., Fairlie, D. B.: Phys. Lett. 67B, 69-71 (1977); 't Hooft, G.: unpublished; Wilczek, F.: In: Quark Confinement and Field Theory (eds. D. Stump, D. Weingarten). New York:John Wiley 1979

19. Kovacs, E., Lo, S. Y.: Phys. Rev. D19, 3649-3652 (1979)

20. Lo, S. Y., Desmond, P., Kovacs, E.: Phys. Lett. 90B, 419-424 (1980)

21. Yang, C. N.: Phys. Rev. Lett. 38, 1377-1379 (1977)

22. Prasad, Phys. Rev. D17, 3243-3246 (1978)