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# ANSATZE FOR SOLUTIONS WITH HIGHER TOPOLOGICAL CHARGE*

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## ABSTRACT

The correspondence between topologically stable kinks of scalar $\phi^4$ theory, vortices of the Abelian Higgs model, monopoles of the 't Hooft-Polyakov model and instantons of SU(2) Euclidean gauge theories is exhibited. The use of multi-instanton ansätze for finding multi-monopole and multivortex configurations is then investigated. A partial solution of the N-vortex configuration for the Abelian Higgs model is obtained.

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## 1. Introduction

Solutions of Yang-Mills theories with topologically non-trivial properties have been known for sometime. The investigation of models with an SU(2) gauge theory coupled to a Higgs triplet led to the discovery of the 't Hooft-Polyakov monopole,[1] a static, finite energy, spherically symmetric extended object in three spatial dimensions with the important property of a topologically conserved magnetic charge. Subsequently, the Prasad-Sommerfield (PS) monopole,[2] representing an exact solution of the field equations in the special case of vanishing Higgs potential was discovered. Further work also revealed the existence of dyons.[3]

As a result of these successes many investigations concerned with the topological properties of Yang-Mills theories in general were initiated. It was realized that the behavior of the Higgs field on a sphere at infinity, $S^2_\infty$, essentially determines the magnetic charge of the monopole by defining a mapping from $S^2_\infty$ onto the unit sphere in group space.[4] Such a mapping can be characterized by an integer N, the winding number, corresponding to the number of coverings of the sphere in group space. All mappings with the same value of N are continuously deformable or homotopic to each other and the value of N for a particular mapping is a gauge invariant quantity.

Hence, the mappings fall into a group of disjoint classes (the homotopy group) labeled by the winding number which thus provides a convenient way of classifying monopole solutions. In fact, it is easily shown that the magnetic charge of a monopole is precisely 1/e times the winding number N, where e is the gauge coupling constant. The 't Hooft-

Polyakov monopole corresponds to a mapping with winding number equal to one and is thus the lowest finite energy, topologically stable solution possible. This result is not surprising in view of its high degree of symmetry.

Topological considerations have not ruled out the existence of solutions with higher magnetic charges since the winding number can assume any integral value. However, despite much effort, no such solutions have been found to date. In fact, as a result of these unsuccessful searches it has emerged[5] that there are no finite energy spherically symmetric field configurations with magnetic charge greater than one. So, if higher charged solutions exist they will have a more complex structure.

A similar state of affairs exists in two spatial dimensions for the Abelian Higgs model, or equivalently Landau-Ginzburg theory. For certain values of the coupling constants, topological arguments indicate the possible existence of stable vortices with quantized magnetic flux.[6,7] The behavior of the Higgs field is, as before, the factor determining the number of units of magnetic flux. In this case the relevant mappings are from a circle in real space $S^1_\infty$, onto the group U(1) which may also be parameterized by a circle. The mappings from $S^1_\infty$ to $S^1$ thus fall into equivalence classes labeled by the winding number N, which may again take on any integral value. The mangetic flux is then equal to $2\pi/e$ times the winding number, where e is the gauge coupling constant of the theory.

De Vega and Schaposnik[8] have found an exact series solution for this model provided that e, and the quartic coupling for the scalar potential $\lambda$, satisfy the relationship $2\lambda = e^2$. In Landau-Ginzburg theory this corresponds to the transition between type I and type II superconductors. They have explicitly exhibited the solution for the case $N = 1$, and in principle the exact solution for a multicharged vortex is also calculable. To date, no explicit multivortex solutions, that is stable configurations of separated vortices, have been found.

In one spatial dimension the kink[7] (or antikink) of scalar $\phi^4$ theory is an exact topologically stable solution. The topological charge may only take on values of $\pm 1$ corresponding to the trivial mapping of the vacuum field values $\phi = \pm F$ onto the points $x = \pm\infty$, so in this case there are no stable solutions with higher topological charge.

Attempts to solve Yang-Mills theories in four Euclidean dimensions have met with considerably more success. Just like their one, two and three dimensional counterparts, these solutions exhibit topological stability. In this case however, it is the behavior of the gauge field at infinity that plays the crucial role in determining the topological quantum number, q. This quantity again can be shown to assume integral values. The first solution discovered, the Belavin-Polyakov-Schwarz-Tyupkin (BPST) instanton,[9] is spherically symmetric (in four dimensions) and has $q = 1$. Subsequently, Witten[10] found multi-instanton solutions consisting of N instantons arranged along the time axis. Multi-instanton solutions with instantons located at arbitrary spacetime points have also been found using the Corrigan-Fairlie-t' Hooft-Wilczek (CFtHW)[11-13] ansatz.

Despite the apparent diversity of the theories considered above, the topologically stable solutions which they admit show remarkable similarities. In all cases the solutions satisfy a set of first order equations which imply the equations of motion.[14] They exhibit bounded action (or energy), the bound being proportional to the topological charge, and all appear not to interact with each other although in cases where no explicit higher charged solutions have been found this has not been verified directly.[15] The similarities between the properties of these solutions points to a relationship between them.

In Section 2, we consider in turn, each of the models discussed above and show how they are related.[16,17] Whilst interesting in its own right, this relationship motivates the use of successful techniques developed for finding multi-instanton solutions in the search[18] for multimonopoles and multivortices.

In Section 3, the most general cylindrically symmetric ansatz for the gauge potential[19] motivated from Witten's solution is used to obtain a set of first order non-linear equations describing a cylindrically symmetric monopole in the PS limit of the 't Hooft-Polyakov model. The use of the CFtHW ansatz in the search for cylindrically symmetric monopoles is also discussed.

In Section 4, Witten's technique for finding multi-instanton solutions is also shown to give a partial solution of the N-vortex configuration of the Abelian Higgs model when $2\lambda = e^2$. The paper concludes with a summary of the work in Section 5.

## 2. The Correspondence between Instantons, Monopoles, Vortices and Kinks

Table I summarizes the properties of topologically stable solutions in one, two, three and four dimensions as outlined in Section 1.

To make the transition from four dimensional Euclidean Yang-Mills theory to the PS limit of the 't Hooft-Polyakov model we begin with the usual SU(2) gauge potentials, $A_\mu^a$ and define the field strengths by

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + e\varepsilon^{abc} A_\mu^b A_\nu^c \quad . \tag{2.1}$$

The Yang-Mills Lagrangian density given in Table I may then be written as

$$\mathscr{L}_{YM} = \frac{1}{4} F_{ij}^a F_{ij}^a + \frac{1}{2} F_{oi}^a F_{oi}^a \quad , \tag{2.2a}$$

$$= \frac{1}{4}\left(F_{ij}^a \mp \varepsilon_{ijk} F_{0k}^a\right)^2 \pm E_i^a B_i^a \quad , \tag{2.2b}$$

where $E_i^a$ and $B_i^a$ are the electric and magnetic fields, respectively. When the term in brackets in Eq. (2.2b) is set to zero, the familiar self-duality condition is obtained. The second term in Eq. (2.2b) then gives a lower bound for the action and is of course proportional to the topological charge density as given in Table I.

Now, choosing time independent fields and replacing $A_0^a$ by $\phi^a$,[18] the Higgs field, remembering that $F_{k0}^a$ becomes $D_k\phi^a$ with this replacement, we see that $\mathscr{L}_{YM}$ becomes precisely the expression for the Hamiltonian density of the PS model as obtained by Bogomol'nyi.[14] That is

$$\mathscr{L}_{YM} \to \mathscr{H} = \frac{1}{4}\left(F_{ij}^a \pm \varepsilon_{ijk} D_k\phi^a\right)^2 \mp B_i^a D_i\phi^a \quad . \tag{2.3}$$

The first term in brackets, when set to zero, is the static limit of the self-duality condition in four Euclidean dimensions[17,18] and is identical to the first-order Bogomol'nyi equations for the PS monopole. The second term in Eq. (2.3) is proportional to the static limit of the four dimensional topological charge density and is easily shown to be proportional to the magnetic charge of the PS monopole.

Hence PS monopoles are mathematically equivalent to static Euclidean instantons. The properties of quantized topological charge, bounded action and absence of instanton interactions in four dimensions are seen to have a precise correspondence with quantized magnetic charge, bounded mass and absence of PS monopole interactions in three dimensions.

The fact that the static limit of Euclidean Yang-Mills theory is equivalent to the PS limit of the 't Hooft-Polyakov model has been known for some time and exploited by a number of authors. In particular, Ju[16] has shown that choosing the most general static, spherically symmetric ansatz for the gauge potentials yields the PS monopole as the only regular, finite energy solution. This result thus provides a nice demonstration of the uniqueness of this solution. Manton[18] has used the CFtHW ansatz to search for static self-dual field configurations and obtained the PS monopole in the spherically symmetric case. He has also shown that the PS monopole may be recovered by searching for time independent solutions of Witten's multi-instanton equations. This result is not surprising in view of the fact that the static limit of Witten's ansatz is spherically symmetric in three spatial dimensions.

Furthermore, recent investigations[19] of the symmetry properties of multi-instanton solutions have revealed that even the two instanton version of Witten's solution, with both instantons located at the origin is not, contrary to expectation, spherically symmetric. It is therefore tempting to conjecture that the result[5] concerning the absence of higher charged spherically symmetric monopoles is also true for self-dual solutions in four dimensions.

To obtain the Hamiltonian density for the Abelian Higgs model, it is first necessary to renormalize the energy scale by adding a constant term to Eq. (2.3) giving

$$\mathcal{H} = \frac{1}{4} F^a_{ij} F^a_{ij} + \frac{1}{2} D_i\phi^a D_i\phi^a + e^2 \frac{F^4}{8} \quad , \qquad (2.4a)$$

$$= \frac{1}{4}\left(F^a_{ij} \pm \varepsilon_{ijk} D_k\phi^a\right)^2 \mp \frac{1}{2} \varepsilon_{ijk} F^a_{ij} D_k\phi^a + \frac{e^2F^4}{8} \quad , \qquad (2.4b)$$

where F is the magnitude of the vacuum expectation value for the Higgs field. Making the substitutions

$$A^a_m = V_m \delta^{a3} \quad , \qquad m = 1,2 \quad , \qquad (2.5a)$$

$$A^p_3 = \frac{\varepsilon^{pq}\Phi^q}{\sqrt{2}} \quad , \qquad p,q = 1,2 \quad , \qquad (2.5b)$$

$$A^3_3 = 0 \quad , \qquad (2.5c)$$

$$\phi^p = \Phi^p/\sqrt{2} \quad , \qquad (2.5d)$$

$$\phi^3 = i F^2 \quad , \qquad (2.5e)$$

we shall see that $V_m$ and $\Phi^p$ may be identified as the usual electromagnetic gauge field and two component scalar field respectively. Now, assuming all fields in Eqs. (2.5) to be independent of z, Eq. (2.4a) becomes

$$\mathcal{H} = \frac{1}{4} F_{mp}F_{mp} + \frac{1}{2} D_m\Phi^p D_m\Phi^p + \frac{e^2}{8}\left(\Phi^m\Phi^m - F^2\right)^2 \quad , \qquad (2.6)$$

where

$$F_{mp} = \partial_m V_p - \partial_p V_m \quad ,$$

$$D_m\Phi^p = \partial_m\Phi^p + e\varepsilon^{pb} V_m \Phi^b \quad .$$

Equation (2.6) is just the Hamiltonian density for the Abelian Higgs model considered by Bogomol'nyi for the special case when the coupling constants satisfy $2\lambda = e^2$. Note that to obtain Eq. (2.6), $\phi^3$ must be chosen with a non-zero, constant vlaue.

Now, making the substitutions (2.5) in Eq. (2.4b) and going to Bogomol'nyi's dimensionless variables gives

$$\begin{aligned}\mathscr{H} = \; & \frac{e^2F^4}{4}\left[\frac{1}{4}\left(f_{mn}^2 \pm 2f_{mn}\varepsilon_{mn}(-Q^aQ^a) + (1-Q^aQ^a)^2\right)\right.\\ & \left. + \frac{1}{2}\left(\varepsilon_{pq}D_iQ^q \mp \varepsilon_{im}D_mQ^p\right)^2\right]\\ & \mp \frac{e^2F^4}{4}\left[\frac{1}{2}f_{mn}\varepsilon_{mn}(-Q^aQ^a) - \varepsilon_{mn}\varepsilon_{pq}D_mQ^qD_nQ^p\right] \quad . \qquad (2.7)\end{aligned}$$

The first and second terms in square brackets correspond to the t and z independent limits of the Euclidean self-duality condition and topological charge density respectively. In contrast to the situation in three dimensions however, to extract the Bogomol'nyi equations and the lower bound on the mass of the vortex from Eq. (2.7) it is necessary to add zero cleverly disguised as

$$\frac{e^2F^4}{2}\left[\pm\frac{1}{2}f_{mn}\varepsilon_{mn} \mp \frac{1}{2}f_{mn}\varepsilon_{mn}\right] \quad . \qquad (2.8)$$

When Eq. (2.8) is absorbed into Eq. (2.7), Bogomol'nyi's expression for the Hamiltonian density of the Abelian Higgs model with $2\lambda = e^2$ follows immediately.

So, in this case although the two dimensional limits of the topological charge density and the self-duality condition require some manipulation to produce the correct form, the instanton-like properties of vortices can still be seen to have a direct relationship with their four dimensional counterparts. Interestingly, unlike the instanton case, the series solutions of De Vega and Schaposnik indicate that a vortex of charge N is radially symmetric. It appears that the Abelian Higgs model in two dimensions has sufficient linearity to admit multi-charged solutions with the maximal symmetry whereas non-Abelian theories in higher dimensions do not.

Continuing down to dependence on one spatial dimension only, the energy density for the kink can be recovered. Setting

$$F_{mn} = 0 \quad , \tag{2.9a}$$

$$\Phi^1 = \phi \quad , \quad \Phi^2 = 0 \quad , \tag{2.9b}$$

$$\frac{e^2}{4} = \lambda \quad , \quad F^2 = m^2/2\lambda \quad , \tag{2.9c}$$

where $m^2$ and $\lambda$ are now the usual quadratic and quartic couplings respectively, Eq. (2.6) becomes

$$\mathcal{H} = \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + \frac{\lambda}{2}\left((\phi)^2 - \frac{m^2}{2\lambda}\right)^2 \quad . \tag{2.10}$$

Making the substitutions (2.9) in Eq. (2.7) reduces the first term in square brackets to Eq. (2.10) and causes the second term to vanish. Hence to obtain the Bogomol'nyi expression for the kink energy density it is necessary to add the identically zero term

$$\pm\sqrt{\lambda}\left((\phi)^2 - \frac{m^2}{2\lambda}\right) \mp \sqrt{\lambda}\left((\phi)^2 - \frac{m^2}{2\lambda}\right) \quad . \tag{2.11}$$

So, even one dimensional $\phi^4$ theory can be regarded as a special limit of self-dual Euclidean Yang-Mills theories. That is, the kink is a "one dimensional instanton."

The foregoing discussion has illustrated that the notion of self-duality, which has proved so useful in finding solutions of Yang-Mills theories in four dimensions could also be employed to find solutions in lower dimensions. Such solutions would be stable since for a particular value of the topological charge they saturate the lower bound on the energy. The idea then, is to take the appropriate limits of the ansätze used to obtain multi-instanton solutions in the hope of finding multi-monopole and multivortex field configurations.

## 3. Ansätze for Cylindrically Symmetric Monopoles

Setting

$$A_\mu = \frac{eA_\mu^a \sigma^a}{2i} \quad ,$$

where $\sigma^a$ are the Pauli matrices, the most general ansatz for a gauge potential with spatial cylindrical symmetry may be written as[19]

$$A_\mu = \frac{e\vec{\sigma}}{2i} \cdot \Big\{ \left(\hat{t}_\mu f_1 + \hat{z}_\mu f_2 + \hat{\rho}_\mu f_3 + \hat{\phi}_\mu f_4\right)\hat{z} + \left(\hat{t}_\mu f_5 + \hat{z}_\mu f_6 + \hat{\rho}_\mu f_7 + \hat{\phi}_\mu f_8\right)\hat{\rho}$$

$$+ \left(\hat{z}_\mu f_9 + \hat{\rho}_\mu f_{10} + \hat{\phi}_\mu f_{11}\right)\hat{\phi} \Big\} \quad , \tag{3.1a}$$

where

$$\hat{t}_\mu = (1\, , 0\, , 0\, , 0) \quad , \tag{3.1b}$$

$$\hat{z}_\mu = (0, 0\, , 0\, , 1) \quad , \tag{3.1c}$$

$$\hat{\rho}_\mu = \frac{1}{\rho}\,(0\, , x\, , y\, , 0) \quad , \tag{3.1d}$$

$$\hat{\phi}_\mu = \frac{1}{\rho}\,(0\, , -y\, , x\, , 0) \quad , \tag{3.1e}$$

$$\rho^2 = x^2 + y^2 \quad , \tag{3.1f}$$

and $\hat{z}$, $\hat{\rho}$, and $\hat{\phi}$ are similar unit vectors in group space. The functions $f_1, f_2, \ldots, f_{11}$ are in general functions of $\rho$, z and t. From the discussion in Section 2, it is known that ansatz (3.1a) may also be used to search for monopole solutions by simply requiring that the functions $f_i$, $i = 1, \ldots, 11$ be independent of time and identifying $A_0^a$ with $\phi^a$, the Higgs field. So, with this restriction on the $f_i$'s, the components of $A_\mu$ may be written explicitly as

$$eA_j^a = \left\{ \left( \frac{\rho_j}{\rho} f_3 + \frac{\varepsilon^{jst} z^s \rho^t}{\rho} f_4 \right) z^a + \left( \frac{\rho_j}{\rho} f_7 + \frac{\varepsilon^{jst} z^s \rho^t}{\rho} f_8 \right) \frac{\rho^a}{\rho} + \left( \frac{\rho_j}{\rho} f_{10} + \frac{\varepsilon^{jst} z^s \rho^t}{\rho} f_{11} \right) \frac{\varepsilon^{abc} z^b \rho^c}{\rho} \right\} , \quad j = 1,2 \qquad (3.2a)$$

$$eA_3^a = z^a f_2 + \frac{\rho^a}{\rho} f_6 + \frac{\varepsilon^{abc} z^b \rho^c}{\rho} f_9 , \qquad (3.2b)$$

$$e\phi^a = eA_0^a = z^a f_1 + \frac{\rho^a}{\rho} f_5 , \qquad (3.2c)$$

where now

$$z^a = (0, 0, 1) ,$$

$$\rho^a = (x, y, 0) ,$$

$$\varepsilon^{abc} z^b \rho^c = (-y, x, 0) ,$$

and

$$a,b,c,s,t = 1,2,3 .$$

The ansatz (3.2) is precisely the form obtained when Witten's ansatz for multi-instanton solutions is rewritten in a manifestly cylindrically symmetric form dependent on $\rho$ and z only.

The field strengths may be calculated from Eq. (2.1) in a straightforward manner and are given by

$$F^a_{12} = \frac{1}{e}\left\{\frac{z^a}{\rho}\left(\partial_\rho g_4 + g_{11}f_7 - g_8 f_{10}\right) + \frac{\rho^a}{\rho^2}\left(\partial_\rho g_8 - g_{11}f_3 + f_{10}g_4\right) + \frac{\varepsilon^{abc}z^b\rho^c}{\rho^2}\left(\partial_\rho g_{11} + g_8 f_3 - f_7 g_4\right)\right\} , \qquad (3.3a)$$

$$\begin{aligned} F^a_{i3} = \frac{1}{e}\Bigg\{ & \frac{\varepsilon^{abc}z^b\rho^c}{\rho^2}\rho_i\left(\partial_\rho f_9 - \partial_z f_{10} + f_3 f_6 - f_2 f_7\right) \\ & + \left(\partial_\rho f_6 - \partial_z f_7 + f_2 f_{10} - f_3 f_9\right)\frac{\rho_i\rho^a}{\rho^2} + \frac{\rho_i z^a}{\rho}\left(\partial_\rho f_2 - \partial_z f_3 + f_9 f_7 - f_6 f_{10}\right) \\ & + \frac{\varepsilon^{ist}z^s\rho^t z^a}{\rho^2}\left(-\partial_z g_4 - f_6 g_{11} + f_9 g_8\right) + \frac{\varepsilon^{ist}z^s\rho^t\rho^a}{\rho^3}\left(-\partial_z g_8 - g_4 f_9 + f_2 g_{11}\right) \\ & + \frac{\left(\delta^{ai}\rho^2 - \rho^a\rho^i\right)}{\rho^3}\left(-\partial_z g_{11} + g_4 f_6 - g_8 f_2\right)\Bigg\} , \qquad i = 1,2, \qquad (3.3b) \end{aligned}$$

$$F^a_{30} = D_3\phi^a = \frac{1}{e}\left\{z^a\left(\partial_z f_1 - f_9 f_5\right) + \frac{\rho^a}{\rho}\left(\partial_z f_5 + f_9 f_1\right) + \frac{\varepsilon^{abc}z^b\rho^c}{\rho}\left(f_2 f_5 - f_6 f_1\right)\right\} , \qquad (3.3c)$$

$$\begin{aligned} F^a_{i0} = D_i\phi^a = \frac{1}{e}\Bigg\{ & \frac{\rho^a\rho^i}{\rho^2}\left(\partial_\rho f_5 + f_{10}f_1\right) + \frac{z^a\rho^i}{\rho}\left(\partial_\rho f_1 - f_{10}f_5\right) \\ & + \frac{\varepsilon^{abc}z^b\rho^c\rho^i}{\rho^2}\left(f_5 f_3 - f_1 f_7\right) + \frac{\left(\delta^{ai}\rho^2 - \rho^a\rho^i\right)}{\rho^3}\left(f_5 g_4 - f_1 g_8\right) \\ & - \frac{\varepsilon^{ist}z^s\rho^t z^a}{\rho^2}\left(g_{11}f_5\right) + \frac{\varepsilon^{ist}z^s\rho^t\rho^a}{\rho^3}\left(g_{11}f_1\right)\Bigg\} , \quad i = 1,2, \qquad (3.3d) \end{aligned}$$

where it is convenient to define

$$f_8 = g_8/\rho \quad , \tag{3.3e}$$

$$f_4 = (g_4 - 1)/\rho \quad , \tag{3.3f}$$

$$f_{11} = g_{11}/\rho \quad , \tag{3.3g}$$

and $\partial_\rho$ and $\partial_z$ are derivatives with respect to $\rho$ and $z$ respectively.

Equations (3.3) may be substituted into the static self-duality condition giving

$$\partial_z f_1 - f_9 f_5 = \mp \frac{1}{\rho}\left(\partial_\rho g_4 - g_8 f_{10} + g_{11} f_7\right) \quad , \tag{3.4a}$$

$$-f_6 f_1 + f_2 f_5 = \mp \frac{1}{\rho}\left(\partial_\rho g_{11} - g_4 f_7 + g_8 f_3\right) \quad , \tag{3.4b}$$

$$\partial_z f_5 + f_9 f_1 = \mp \frac{1}{\rho}\left(\partial_\rho g_8 - g_{11} f_3 + g_4 f_{10}\right) \quad , \tag{3.4c}$$

$$-f_1 f_7 + f_5 f_3 = \pm \frac{1}{\rho}\left(\partial_z g_{11} - g_4 f_6 + g_8 f_2\right) \quad , \tag{3.4d}$$

$$\partial_\rho f_5 + f_{10} f_1 = \pm \frac{1}{\rho}\left(\partial_z g_8 + g_4 f_9 - g_{11} f_2\right) \quad , \tag{3.4e}$$

$$\partial_\rho f_1 - f_{10} f_5 = \pm \frac{1}{\rho}\left(\partial_z g_4 - g_8 f_9 + g_{11} f_6\right) \quad , \tag{3.4f}$$

$$\partial_\rho f_6 - \partial_z f_7 - f_3 f_9 + f_2 f_{10} = \pm f_1 g_{11}/\rho \quad , \tag{3.4g}$$

$$\partial_\rho f_2 - \partial_z f_3 - f_6 f_{10} + f_9 f_7 = \mp f_5 g_{11}/\rho \quad , \tag{3.4h}$$

$$\partial_\rho f_9 - \partial_z f_{10} - f_2 f_7 + f_3 f_6 = \pm \frac{1}{\rho}\left(f_5 g_4 - f_1 g_8\right) \quad . \tag{3.4i}$$

Equations (3.4) are the most general equations describing a cylindrically symmetric PS monopole. In fact, since a choice of gauge such as $\partial_\mu A_\mu = 0$ yields in general another three equations for the unknown functions, the above system is overconstrained.

Manton[18] has also searched for cylindrically symmetric monopoles using an ansatz which may be obtained from Eqs. (3.2) by setting

$$f_7 = f_3 = f_6 = f_2 = g_{11} = 0 \quad . \tag{3.5}$$

Remembering that Manton used a value for the gauge coupling constant corresponding to $e = -1$, it is found that the self-duality Eqs. (3.4) reduce to those obtained in the special case when Eq. (3.5) is true.

As a further check on Eqs. (3.4), it can be ensured that the PS monopole is recovered when the functions $f_i$ are chosen so that ansatz (3.2) reduces to spherically symmetric form. This is accomplished by choosing

$$f_1 = -z\phi_3(r)/r \quad , \tag{3.6a}$$

$$f_5 = \rho f_1/z \quad , \tag{3.6b}$$

$$g_{11} = \rho\phi_2(r)/r \quad , \tag{3.6c}$$

$$f_2 = \rho g_{11}/r^2 \quad , \tag{3.6d}$$

$$f_3 = f_6 = -zg_{11}/r^2 \quad , \tag{3.6e}$$

$$f_7 = zf_3/\rho \quad , \tag{3.6f}$$

$$f_{10} = -z(1+\phi_1(r))/r^2 \quad , \tag{3.6g}$$

$$g_8 = -\rho f_{10} \quad , \tag{3.6h}$$

$$f_9 = \rho(1+\phi_1(r))/r^2 \quad , \tag{3.6i}$$

$$g_4 = 1 - \rho f_9 \quad , \tag{3.6j}$$

where

$$r^2 = \rho^2 + z^2 \quad .$$

The reduced form of the self-duality Eqs. (3.4) thus obtained have only one regular solution corresponding to the PS monopole.[16]

Equations (3.4) give a somewhat intractable set of coupled non-linear equations, which in general will not be easy to solve. Therefore it is useful to consider possible simplifications.

It is well known that the CFtHW ansatz[20]

$$A_\mu = i\sigma_{\mu\nu}\partial^\nu \ln \Phi \quad , \tag{3.7}$$

reduces the self-duality condition to

$$\Box\Phi = 0 \quad , \tag{3.8}$$

and the Yang-Mills equations of motion to

$$\Box\Phi + \lambda\Phi^3 = 0 \quad , \tag{3.9}$$

where $\lambda$ is an arbitrary constant.

From the discussion in Section 2, a static version of the CFtHW ansatz may be used to search for cylindrically symmetric monopoles. In this case, to ensure static gauge fields, the derivatives of the superpotential $\Phi$ must be time independent, although $\Phi$ itself need not be. With these restrictions it has been shown that $\Phi$ must be of the form[18]

$$\Phi(\vec{x},t) = \sigma(\vec{x}) \exp \alpha t \quad , \tag{3.10}$$

where $\alpha$ is a constant. $\sigma(\vec{x})$ is a function of $\rho$ and $z$ only for cylindrical symmetry. The self-duality Eq. (3.8) thus becomes

$$\partial_\rho^2\sigma + \frac{\partial_\rho\sigma}{\rho} + \partial_z^2\sigma + \alpha^2\sigma = 0 \quad . \tag{3.11}$$

Since Witten's ansatz and the CFtHW ansatz yield gauge equivalent solutions,[21] it should be possible to recover Eq. (3.11) from the more general Eqs. (3.4). Choosing

$$
\begin{aligned}
f_1 &= \pm\partial_z q \quad , \\
f_5 &= \pm\partial_\rho q \quad , \\
f_6 &= f_3 = 0 \quad , \\
f_7 &= f_2 = g_{11}/\rho = \mp\partial_0 q \quad , \\
f_{10} &= -g_8/\rho = \partial_z q \quad , \\
f_9 &= -\partial_\rho q \quad , \\
g_4 &= 1-\rho f_9 = 1+\rho\partial_\rho q \quad ,
\end{aligned}
$$

where $q = \ln\Phi$, casts the ansatz (3.2) into the cylindrically symmetric CFtHW form and reduces Eqs. (3.4) as asserted.

A general solution of Eq. (3.11) may be obtained by separation of variables and does not lead to finite energy field configurations.

The CFtHW ansatz can also be used to search for solutions which are not self-dual. However, since the gauge potential must be static, the function $\Phi$ must be of the form given in Eq. (3.10). In this case, the equation of motion (3.9) reduces to

$$\nabla^2\sigma + \alpha^2\sigma + \beta\sigma^3 = 0 \quad , \tag{3.12}$$

where $\beta = \lambda e^{2\alpha t}$ is independent of $\rho$ and z. Equation (3.12) looks promising and is currently under investigation.

Finally, it is trivial to include dyon solutions by using the Prasad-Sommerfield procedure of defining a new gauge potential $B_\mu^a$ and Higgs field $\phi'^a$ given by

$$B_i^a = A_i^a \quad ,$$

$$B_0^a = \phi^a \sinh\gamma \quad ,$$

$$\phi'^a = \phi^a \cosh\gamma \quad ,$$

where $\gamma$ is an arbitrary constant.

## 4. Ansätze for Multivortices

For the case of the Abelian Higgs model, the two dimensional limit of the self-duality condition yields a system of equations very similar to those obtained by Witten for the multi-instanton solution. From Table I this condition when written out explicitly in Bogomol'nyi's dimensionless variables becomes

$$\partial_2 Q_2 - v_2 Q_1 = \mp\left(\partial_1 Q_1 + v_1 Q_2\right) \quad , \tag{4.1a}$$

$$\partial_2 Q_1 + v_2 Q_2 = \pm\left(\partial_1 Q_2 - v_1 Q_1\right) \quad , \tag{4.1b}$$

$$\partial_1 v_2 + \partial_2 v_1 = \mp\left(1 - Q_1^2 - Q_2^2\right) \quad , \tag{4.1c}$$

the upper and lower sign corresponding to the self-dual and anti-self-dual case respectively.

Following Witten's method of solution by setting

$$v_{a-} = \mp\varepsilon_{ab}\,\partial_b\,\psi \quad , \qquad a,b = 1,2 \tag{4.2a}$$

$$Q_1 = e^{\psi}\chi_1 \quad , \tag{4.2b}$$

$$Q_2 = e^{\psi}\chi_2 \quad , \tag{4.2c}$$

Equations (4.1a)-(4.1b) reduce to the Cauchy-Riemann equations for $f = \chi_1 \mp i\chi_2$, an analytic function of $z = x_1 + ix_2$. Choosing

$$\psi = -\frac{1}{2} \ln (f^* f) + \rho \quad , \tag{4.3a}$$

reduces Eq. (4.1c) to

$$\nabla^2 \rho = e^{2\rho} - 1 \quad . \tag{4.3b}$$

Equation (4.3b) has also been obtained by Lohe[17] and has no known analytic solutions.

The interesting result is that the equations for the Higgs fields are precisely those obtained by Witten. Since the calculation of the magnetic flux depends only on the Higgs fields it is seen from Witten's results that the number of zeroes of f determines N, the topological charge of the vortex, and the 2N parameters required to describe a multivortex solution are just those parameters specifying the positions of the zeroes of f. After the completion of this work, a preprint[22] was received which has reached the same conclusion.

## 5. Summary

An examination of the properties of topologically stable solutions in one, two, three and four spatial dimensions reveals a number of similarities between them. It is possible to exhibit explicitly the connection between the various models, so that in some sense solutions in lower dimensions can be regarded as special cases of instantons. Indeed, PS monopoles are mathematically equivalent to static Euclidean instantons.

The absence of explicit higher charged solutions in two and three spatial dimensions suggests the use of the appropriate limits of multi-instanton ansätze in the search for new solutions.

For the PS limit of the 't Hooft-Polyakov model equations for the most general cylindrically symmetric monopole have been obtained. Some simplification of these equations may be achieved by using the CFtHW ansatz. This ansatz is also found to simplify the equations of motions for monopoles which do not saturate the Bogomol'nyi bound.

For the Abelian Higgs model, the use of the Bogomol'nyi equations and Witten's methods show that the number of parameters required to specify on N-vortex configuration is just the number of parameters describing the zeroes of the Higgs fields.

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20. We use the usual notation $\sigma_{ij} = \frac{1}{2}\varepsilon_{ijk}\sigma^k$ and $\sigma_{i0} = \frac{1}{2}\sigma_i$. $\Phi$ is some scalar superpotential.

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TABLE I

Summarizes the main features of the topologically stable solutions which exist in one, two, three and four dimensions.

| Model | Euclidean Yang-Mills Theory | PS Limit of t' Hooft-Polyakov Model | Abelian Higgs Model, $2\lambda = e^2$ | Scalar $\phi^4$ Theory |
|---|---|---|---|---|
| Dimension n | 4 | 3 | 2 | 1 |
| Topologically Stable Solution | Instanton | Monopole | Vortex | Kink |
| Lagrangian Density | $\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}$ | $-\frac{1}{4}(F^a_{ij})^2$ $-\frac{1}{2}(D_i\phi^a)^2$ | $-\frac{1}{4}(F_{mn})^2 - \frac{1}{2}(D_m\phi^a)^2$ $-\frac{e^2}{8}(\phi^a\phi^a - F^2)^2$ | $-\frac{1}{2}(\partial_x\phi)^2$ $-\frac{\lambda}{2}\left(\phi^2 - \frac{m^2}{2\lambda}\right)^2$ |
| Self-Duality Condition | $F^a_{\mu\nu} = \pm \overset{*}{F}{}^a_{\mu\nu}$ | $F^a_{ij} = \mp\epsilon_{ijk} D_k\phi^a$ | $f_{mn} = \mp\epsilon_{mn}(1 - Q^aQ^a)$ $\epsilon_{mn} D_n Q^a = \pm\epsilon_{ab} D_m Q^b$ | $\frac{d\phi}{dx} = \pm\sqrt{\lambda}\left(\frac{m^2}{2\lambda} - \phi^2\right)$ |
| Topological Charge Density $\tilde{q}$ | $\frac{e^2}{8\pi^2} E^a_i B^a_i$ | $\frac{B^a_i D_i \phi^a}{F}$ | $\frac{1}{2e}\epsilon_{mn} f_{mn}$ | $\sqrt{\frac{\lambda}{2m^2}}\,\frac{d\phi}{dx}$ |
| Topological Charge $\int d^n x \tilde{q}$ | $q = \pm N$ Topological Charge | $4\pi M = \pm\frac{4\pi N}{e}$ Magnetic flux | $\Phi = \pm\frac{2\pi N}{e}$ Magnetic flux | $K = \pm 1$ Kink Number |
| Lower Bound on Action or Energy | $\frac{8\pi^2}{e^2}\lvert q\rvert$ | $4\pi MF$ | $\frac{eF^2}{2}\lvert\Phi\rvert$ | $\frac{\sqrt{2}m^3}{3\lambda}\lvert K\rvert$ |
| Interaction | Non-Interacting Multi-Instanton Configurations Exist | PS Monopoles are Non-Interacting | Vortices for $2\lambda = e^2$ are Non-Interacting | No Higher Charged Solutions |
| Symmetry Properties | BPST Instanton only Spherically Symmetric Solution known | No Spherically Symmetric Solutions with $N > 1$ | De Vega and Schaposnik Series Solution Radially Symmetric | --- |