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THE DIMENSION OF A QUANTUM—MECHANICAL PATH *

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ABSTRACT

We show that the observed path of a particle in quantum mechanics is a fractal curve with Hausdorff dimension two.

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I. Introduction

In quantum mechanics, the distance which a particle travels in a fixed period of time depends on the resolution of the detecting apparatus which is used to track the particle. Similar correlations between distance and resolution are found in curves which are given by everywhere continuous but nowhere differentiable functions. Mathematicians have developed various concepts and techniques for dealing with such curves[1] and Mandelbrot,[2] in particular, has studied their application to natural phenomena.[3] In this article, we review some of these concepts and apply them to quantum-mechanical motion.

To introduce the mathematical ideas we will be using let us begin by considering a well-known example of an everywhere continuous but nowhere differentiable curve, the Koch curve. Its construction is shown in Figure 1. The Koch curve is the final product of an infinite sequence of steps like those in Figure 1. When each step in the construction is performed the length of the curve increases by a factor of 4/3, so the final curve being the result of an infinite number of steps is infinitely long. The similarities between this curve and the path of a quantum-mechanical particle[4] become apparent when we consider viewing the Koch curve with a finite spatial resolution. In this case, the infinitely many wiggles in the curve which are smaller than some minimum length Δx, cannot be detected and the measured length of the curve will be finite. However, this length will depend on Δx and will increase without limit as $\Delta x \to 0$. For example, suppose that we examine the Koch curve resolving distances greater than some scale Δx and measure its length to be ℓ. Then, if we improve our resolution so that $\Delta x' = (1/3)\,\Delta x$, the next level of wiggles in the curve will become visible and we will measure a new length $\ell' = (4/3)\ell$. Since the conventional definition of length, when applied to curves like the Koch curve, gives a quantity which depends on the resolution with which the curve is examined (even for very small Δx), it is not very useful. Hausdorff proposed a modified definition of length to be used in these cases. The Hausdorff length, L, is given by

$$L = \ell(\Delta x)^{D-1} \quad , \tag{1.1}$$

where ℓ is the usual length measured when the resolution is Δx. D is a number chosen so that L will be independent of Δx, at least in the limit $\Delta x \to 0$. (Hausdorff gave a precise definition of "resolution" by covering the curve with $(\ell/\Delta x)$ spheres of diameter Δx). Note that when $D = 1$, the Hausdorff definition just reduces to the usual concept of length. For the Koch curve, we determine D by requiring that

$$L' = \ell'(\Delta x')^{D-1} = \left(\frac{4}{3}\,\ell\right)\left(\frac{1}{3}\,\Delta x\right)^{D-1} = L = \ell(\Delta x)^{D-1} \; . \tag{1.2}$$

This implies that

$$D = \ln 4 / \ln 3 \quad . \tag{1.3}$$

Clearly the Hausdorff definition of length is more practical than the conventional one because it eliminates the resolution or scale dependence of the measured value of the length of a curve. In addition, the Hausdorff dimension D for a particular curve is a useful parameter for describing its properties. The fact that $D \neq 1$ for the Koch curve identifies it, in the language of Mandelbrot, as a <u>fractal curve</u>.

To define the quantum-mechanical path of a particle we imagine measuring the position of a free particle with a spatial resolution Δx

at times separated by an interval Δt. The path is then defined as the curve determined by drawing straight lines between the points where the particle was located at sequential times. To be more precise one should draw straight lines between the centers of regions within which the particle is known to lie, since we are working with an uncertainty Δx. At the classical level, this path will just be a straight line with dimension $D = 1$ (or in the case of a particle at rest, a point of zero dimension). However, at the quantum level, the localization of a particle within a region of "size" Δx results, according to the Heisenberg uncertainty principle, in an uncertainty in the momentum of order $\hbar/\Delta x$. Thus, as the particle is more and more precisely located in space, its path will become increasingly erratic. Of course, in quantum mechanics we can only speak of a particle's path in the statistical sense and we must work with average values (denoted by $\langle\ \rangle$). If we measure the position of a particle at times t_0, $t_1 = t_0 + \Delta t$, ... , $t_N = t_0 + N\Delta t$, with $T = t_N - t_0 = N\Delta t$, the length of the particle's path will be

$$\langle \ell \rangle = N\langle \Delta\ell \rangle \tag{1.4}$$

where $\langle \Delta\ell \rangle$ is the average distance which the particle travels in a time Δt. Let us first consider the case where the average momentum of the particle is zero (i.e., in the classical limit the particle is at rest). Then, as the uncertainty principle would suggest, and as we will show in Section II,

$$\langle \Delta\ell \rangle \propto \hbar\Delta t/m\Delta x \quad . \tag{1.5}$$

Thus

$$\langle \ell \rangle \propto \hbar T/m\Delta x \tag{1.6}$$

and, as in the case of the Koch curve, the length of a particle's path in quantum mechanics depends on the detection resolution Δx, and diverges in the limit $\Delta x \to 0$. Following Hausdorff, we can introduce a modified definition of length,

$$\langle L \rangle = \langle \ell \rangle (\Delta x)^{D-1} \quad . \tag{1.7}$$

Requiring that $\langle L \rangle$ be independent of Δx gives, from Equation (1.6), that $D = 2$. The path of a particle in quantum mechanics is therefore a fractal of dimension two.[5]

The Koch curve has another interesting property, which under certain circumstances, is shared by the quantum-mechanical particle path. This is self-similarity. If we view a Koch curve with a resolution $\Delta x' = (1/3)\ \Delta x$ then the curve we see is, up to repetitions and translations, just a scaled down version of the curve we saw when the distances being resolved were of size Δx. The path of a quantum-mechanical particle will be self-similar if

$$\langle \Delta \ell \rangle \propto \Delta x \quad . \tag{1.8}$$

Comparing Equations (1.8) and (1.5), we see that to get a self-similar path we must scale the time between position measurements of the particle in proportion to the square of Δx. That is, if

$$\Delta t \propto \frac{m(\Delta x)^2}{\hbar} \tag{1.9}$$

then the resulting path is self-similar. In Section II, Equation (1.9) will arise naturally in our derivation of Equation (1.5) because it is related through the uncertainty principle to the energy-momentum relation $E = p^2/2m$. Thus, just as the fractal nature of the quantum-mechanical path reflects the Heisenberg uncertainty principle, the condition for self-similarity, Equation (1.9), is a reflection of the underlying dynamics $E = p^2/2m$.

Finally, it is interesting to consider the case when the particle has some nonzero average momentum (i.e., in the classical limit the particle is moving) since then the transition from the classical result $D = 1$ to the quantum result $D = 2$ can be seen. This is done in Section III.

II. Derivation of the Hausdorff Dimension for a Quantum-Mechanical Path

Recall that we define the path of a quantum-mechanical particle by measuring the position of the particle with a resolution Δx at times separated by an interval Δt. The fact that a position measurement only localizes the particle within a region of "size" Δx is taken into account by assuming that the wave function of a particle, measured to be at the origin, is

$$\psi_{\Delta x}(\vec{x}) = \frac{(\Delta x)^{3/2}}{\hbar^3} \int_{\mathbb{R}^3} \frac{d^3 p}{(2\pi)^{3/2}} \, f\left(\frac{|\vec{p}|\,\Delta x}{\hbar}\right) e^{i\vec{p}\cdot\vec{x}/\hbar} \qquad (2.1)$$

Defining a dimensionless vector $\vec{k} = \vec{p}(\Delta x)/\hbar$, the normalization condition on the wavefunction implies that

$$\int_{\mathbb{R}^3} d^3 k \, \left| f(|\vec{k}|) \right|^2 = 1 \quad . \qquad (2.2)$$

The actual form chosen for the function f will not be important in our discussion, we just assume it is a form sufficient to localize the particle to a region of "size" Δx. The expression (2.1) follows from simple dimensional arguments for a detection procedure which is isotropic and can be characterized by a single dimensional parameter Δx.

An interesting realization of (2.1) is

$$f\left(|\vec{k}|\right) = \sqrt{\frac{3}{4\pi}}\ \theta\left(1-|\vec{k}|\right) \quad . \tag{2.3}$$

This corresponds to using a momentum cut off $\Lambda = \hbar/\Delta x$ in all calculations (i.e., putting the particle in a momentum box of radius Λ centered about the origin of momentum space) or equivalently to using a spatial lattice with spacing Δx.

The crucial quantity in the discussions of Section I was the average distance the particle travels in a time Δt. When $\langle \Delta\ell \rangle$ is large compared with Δx it is given by [6]

$$\langle \Delta\ell \rangle = \int_{\mathbb{R}^3} d^3x\, |\vec{x}|\, \left| \psi_{\Delta x}(\vec{x},\Delta t) \right|^2 \quad . \tag{2.4}$$

From Equation (2.1)

$$\psi_{\Delta x}(\vec{x},\Delta t) = \frac{(\Delta x)^{3/2}}{\hbar^3} \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^{3/2}} f\left(\frac{|\vec{p}|\Delta x}{\hbar}\right) e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar} - \frac{i|\vec{p}|^2\Delta t}{2m\hbar}} \quad . \tag{2.5}$$

Using the dimensionless vectors $\vec{k} = \vec{p}\Delta x/\hbar$ and $\vec{y} = \vec{x}/\Delta x$, Equations (2.4) and (2.5) can be rewritten in the form

$$\langle \Delta\ell \rangle = \Delta x \int_{\mathbb{R}^3} d^3y\, |\vec{y}|\, \left| F\left(\vec{y}, \hbar\Delta t/2m(\Delta x)^2\right) \right|^2 \quad , \tag{2.6}$$

where

$$F\left(\vec{y},\hbar\Delta t/2m(\Delta x)^2\right) = \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^{3/2}}\, f\left(|\vec{k}|\right)\, e^{i\vec{k}\cdot\vec{y} - \frac{i|\vec{k}|^2\hbar\Delta t}{2m(\Delta x)^2}} \quad . \tag{2.7}$$

From Equation (2.6) it is evident that if the time interval Δt is scaled so that

$$\frac{\hbar \Delta t}{2m(\Delta x)^2} = \text{constant} \quad , \tag{2.8}$$

then

$$\langle \Delta \ell \rangle \propto \Delta x \propto \frac{\hbar \Delta t}{m \Delta x} \quad , \tag{2.9}$$

which is precisely the result used in the introduction to derive the condition for self-similarity and fractal dimension of the quantum path.

It should be noted that condition (2.8) is only required if we wish the path to be self-similar. To determine the Hausdorff dimension we can contemplate keeping Δt fixed while varying Δx. In this case as $\Delta x \to 0$, $\hbar \Delta t/2m(\Delta x)^2 \to \infty$ and $f(|\vec{k}|)$ in Equation (2.7) can be approximated by a Gaussian (this is essentially a stationary phase approximation).[7] The net result is then,

$$\langle \Delta \ell \rangle \propto \frac{\hbar \Delta t}{m \Delta x} \sqrt{1 + \left(\frac{2m(\Delta x)^2}{\hbar \Delta t}\right)^2} \quad . \tag{2.10}$$

This is more complicated than Equation (2.9) but reduces to Equation (1.5) when $\Delta x \ll \sqrt{(\hbar \Delta t/2m)}$. Thus when Δt is held fixed we can define a Hausdorff dimension in the limit $\Delta x \to 0$. As in the self-similar case the result is $D = 2$

III. The Transition from D = 1 to D = 2

In order to exhibit the transition from the classical result D = 1 to the quantum-mechanical result D = 2, we consider the case where the particle has some non-zero average momentum $\vec{p}_{av}$. This is done by replacing Equation (2.1) by

$$\psi_{\Delta x}(\vec{x}) = \frac{(\Delta x)^{3/2}}{\hbar^3} \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^{3/2}} f\left(\frac{|\vec{p}|\Delta x}{\hbar}\right) e^{\frac{i(\vec{p}+\vec{p}_{av})\cdot\vec{x}}{\hbar}}$$

$$= \frac{(\Delta x)^{3/2}}{\hbar^3} \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^{3/2}} f\left(\frac{|\vec{p}-\vec{p}_{av}|\Delta x}{\hbar}\right) e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar}} \tag{3.1}$$

In this case Equation (2.6) is replaced by

$$\langle\Delta\ell\rangle = (\Delta x) \int_{\mathbb{R}^3} d^3y \left| \vec{y} + \frac{\vec{p}_{av}\Delta t}{m\Delta x} \right| \left| F\left(\vec{y}, \frac{\hbar\Delta t}{2m(\Delta x)^2}\right) \right|^2 \tag{3.2}$$

where F is given by Equation (2.7). For simplicity assume that Δt is scaled so that

$$\frac{\hbar\Delta t}{2m(\Delta x)^2} = b \tag{3.3}$$

where b is a constant of order unity. Then

$$\langle\Delta\ell\rangle = \frac{|\vec{p}_{av}|\Delta t}{m} \int_{\mathbb{R}^3} d^3y \left| \frac{\vec{p}_{av}}{|\vec{p}_{av}|} + \frac{\hbar\vec{y}}{2|\vec{p}_{av}|(\Delta x)b} \right| \left|F(\vec{y},b)\right|^2 . \tag{3.4}$$

Finally,

$$\langle L \rangle = N \langle \Delta\ell \rangle (\Delta x)^{D-1}$$

$$= \frac{|\vec{p}_{av}|T}{m} \int_{\mathbb{R}^3} d^3y \left| \frac{\vec{p}_{av}}{|\vec{p}_{av}|} + \frac{\hbar \vec{y}}{2|\vec{p}_{av}|(\Delta x)b} \right| |F(\vec{y},b)|^2 (\Delta x)^{D-1}. \tag{3.5}$$

Requiring that the Hausdorff length $\langle L \rangle$ be independent of Δx gives[8] $D = 1$ when the distances being resolved are much larger than the particle's wavelength (i.e., $\Delta x \gg \hbar / |\vec{p}_{av}|$) and $D = 2$ when the distances being resolved are much smaller than the particle's wavelength (i.e., $\Delta x \ll \hbar / |\vec{p}_{av}|$). These are respectively the classical and quantum-mechanical limits. In the region between these limits the Hausdorff dimension D is not well defined since it is rapidly varying with Δx.

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References

1. W. Hurewicz and H. Wallman, Dimension Theory, Princeton University Press, Princeton, New Jersey (1941).

2. B. B. Mandelbrot, Fractals: Form, Chance and Dimension, W. H. Freeman and Company, San Francisco, California (1977).

3. In another application, A. B. Kraemmer, H. B. Nielson and H. C. Tze, Nuc. Phys. B81, 145 (1974), consider the dimension of a particle in quantum field theory.

4. In the path integral approach, Feynmann and Hibbs {R. P. Feynman and A. R. Hibbs, Quantum Mechanics and Path Integrals, McGraw-Hill, New York, New York (1965)} have noted that the typical path of a quantum-mechanical particle is continuous and non-differentiable.

5. The path of a particle undergoing Brownian motion is also fractal with $D = 2$. The similarities between Brownian motion and quantum-mechanical motion have been stressed frequently. See, for example, E. Nelson, Phys. Rev. 150, 1079 (1966) and references therein.

6. The precise definition of the length, $\langle \Delta \ell \rangle$, is somewhat arbitrary. For example one could use

$$\langle \Delta \ell \rangle = \left\{ \int_{\mathbb{R}^3} d^3x \, |\vec{x}|^2 \left| \psi_{\Delta x}(\vec{x}, \Delta t) \right|^2 \right\}^{1/2}$$

instead of Equation (2.4). This, however, would not change our results.

7. When $f(|\vec{k}|)$ is the Gaussian

$$f(|\vec{k}|) = \left(\frac{2}{\pi}\right)^{3/4} e^{-|\vec{k}|^2} ,$$

the integrals in Equations (2.6) and (2.7) can be done exactly. In particular Equation (2.7) becomes

$$F\left(\vec{y}, \frac{\hbar\Delta t}{2m(\Delta x)^2}\right) = \frac{(2\pi)^{-3/4}}{\left(1 + \frac{i\hbar\Delta t}{2m(\Delta x)^2}\right)^{3/2}} \exp\left[\frac{-|\vec{y}|^2}{4\left(1 + \frac{i\hbar\Delta t}{2m(\Delta x)^2}\right)}\right] .$$

8. To consider the case where the path is not self-similar, and b is much greater than unity, we let f be the Gaussian

$$f(|\vec{k}-\vec{k}_{av}|) = \left(\frac{2}{\pi}\right)^{3/4} e^{-\left|\vec{k} - \vec{k}_{av}\right|^2} ,$$

where $\vec{k}_{av} = \vec{p}_{av}\Delta x/\hbar$. Then

$$\langle L\rangle \propto \frac{|\vec{p}_{av}|T}{m} \int_{\mathbb{R}^3} d^3z \left| \frac{\hbar\vec{z}}{(\Delta x)|\vec{p}_{av}|}\sqrt{1 + b^{-2}} + \sqrt{2}\,\frac{\vec{p}_{av}}{|\vec{p}_{av}|}\right| e^{-z^2} (\Delta x)^{D-1} .$$

Thus, even for $b >> 1$, $D = 1$ when $\Delta x >> \hbar/|\vec{p}_{av}|$ and $D = 2$ when $\Delta x << \hbar/|\vec{p}_{av}|$.

Figure Caption

1. Construction of the Koch curve.

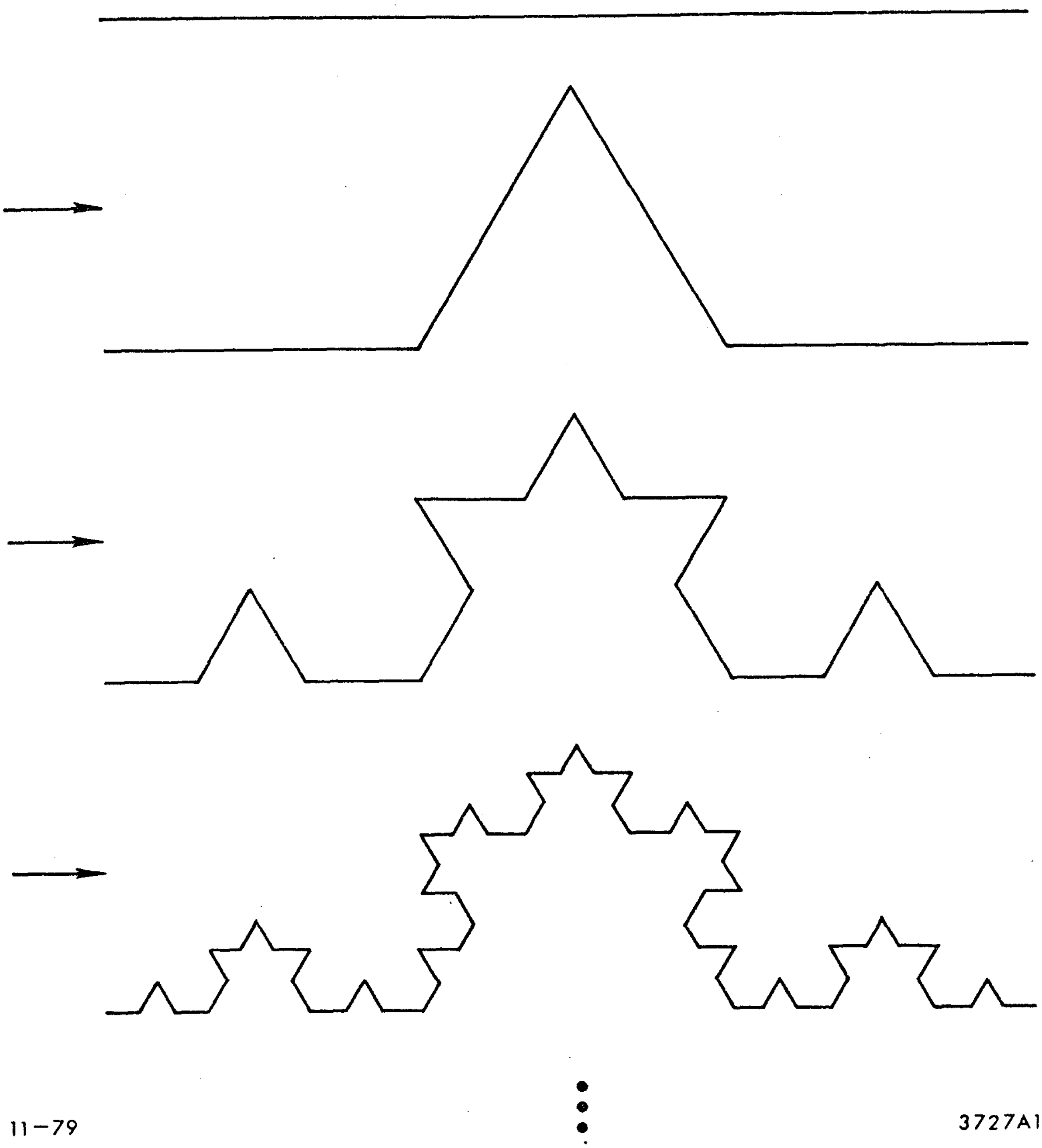

Fig. 1