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RADIATION SPECTRA AND ANGULAR DISTRIBUTION OF EMITTED QUANTA FOR PLANAR CHANNELLED PARTICLES: DEPENDENCE ON PARTICLE ENERGY. *

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ABSTRACT

Using a method of calculating the radiation characteristics for the motion in an arbitrary one-dimensional potential, developed in the previous paper,[7] we look here for the maximum in the number of emitted photons as a function of particle energy for different amplitudes of oscillations, divergence angles in a plane parallel to the trapping crystal planes and harmonic numbers.

The problem is treated in the classical approximation. A numerical example is given for positron channelling in the (1,1,0) direction of a Silicon crystal. The influence of the refraction index is discussed briefly.

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I. Introduction

Radiation of channelled particles trapped between planes of a crystal is a powerful tunable source of polarized radiation in an interesting range of frequencies. It can also be used to study the properties and characteristics of the crystal itself. For both applications of the phenomena, one needs knowledge of the radiation spectra and angular distribution of emitted quanta. All the known theoretical results [1-5] for these radiation characteristics are obtained in the linear approximation, where only the first (quadratic in coordinate) term of the power series expansion of the potential in which the particle moves is retained and considered. In the recent work of Pantell and Swent, [6] the experimental radiation spectrum is fit with a spectrum calculated quantum mechanically for a particularly chosen empirical potential of channelling motion.

In our own work [7] a method was developed which permits one to calculate radiation characteristics for the motion in an arbitrary one-dimensional potential. By using this method one can calculate all the characteristics of the channelling radiation without any assumption on the value of anharmonicity. In particular, expressions for radiation frequencies, polarization angles, and the number of emitted photons as functions of quanta angles, particle energy, amplitude of oscillations, and divergence in a plane parallel to the trapping crystal planes for any given harmonic number are found.

In this work we are looking for the dependence of the spectra on particle energy. Taking as our example the number of quanta emitted in the forward direction, we find that this quantity has a maximum at a certain particle energy for a given crystal and incident particle direction.

In Section II, we present formulae for the number of emitted photons per interval of solid angle and for the emission frequency, taken from work[7] to which we direct the reader for more detailed discussion. Numerical results for positrons can be found in Section III. In Section IV, we discuss the influence of the refraction index.

II. Some Useful Formulae

Let a relativistic particle with energy E (and Lorentz factor γ) be trapped between the planes of a crystal. We choose a coordinate frame in which the crystal planes are parallel to the yz plane. The particle is assumed to have only one relativistic velocity component β_z ($\beta = v/c$) along the z axis, the other two components being small:

$$\beta_x / \beta_z \ll 1 \quad , \tag{1}$$

$$\beta_y / \beta_z \ll 1 \quad . \tag{2}$$

The length of one "oscillation" of the relativistic channelled particle is much longer than any lattice periods. Consequently, the force exerted on the channelled particle can be derived from a time independent one-dimensional potential V(x) (for the "oscillations" occurring in the x - z plane) which is the planar average (over y and z) of the true electrostatic potential V(x,y,z) within the lattice. We choose further for convenience, $V(0) = 0$.

The first integral of the equation of motion gives

$$\beta_x = \sqrt{2\,[\,V(x_m) - V(x)\,]\,/\,E} \quad , \tag{3}$$

where x_m is the value of x at the point $\beta_x = 0$. x_m is the maximum excursion of the particle from the plane $x = 0$ and we call this quantity the "amplitude" of (nonlinear) oscillations. In (3) and for the rest of the paper we use the following abbreviations:

$$V_u = V(u)\,, \qquad V_m = V(x_m) \quad . \tag{3}$$

The "frequency", of oscillations is $\Omega = 2\pi / \oint du \,/\, \sqrt{2c^2(V_m - V_u)/E}$, where the sign $\oint$ means integration over the full period of oscillation. For the case of potential symmetric in $x\,(V_{-x} = V_x)$ we get:

$$\Omega = \pi c \Big/ 2 \int_0^{x_m} dx \Big/ \sqrt{2(V_m - V_x)\,/\,E} \quad . \tag{5}$$

Calling θ the azimuthal angle of the radiation direction with the z axis and φ the polar angle between its projection on the $x - y$ plane and the x axis, the wave vector $\vec{K}$ has the following components:

$$K_{x,y,z} = K\, n_{x,y,z} = \frac{\omega}{c}\,(\sin\theta\cos\varphi, \quad \sin\theta\sin\varphi, \quad \cos\theta) \tag{6}$$

If the number of oscillations over the length L (the crystal thickness) is a large number, the radiation occurs in the form of a line spectrum. The center of the k^{th} line is positioned at the frequency:

$$\omega_k = \frac{\pi c \gamma k B}{x_m T_1 (1 + B^2 + D^2 + \Theta^2 - 2\Theta D \sin\varphi)} , \qquad (7)$$

where we introduce the following useful variables:

$$B = \sqrt{2 V_m \gamma / m c^2} , \qquad (8)$$

$$D = \beta_y \gamma , \qquad (9)$$

$$\Theta = \theta\gamma . \qquad (10)$$

The definition of T_1 is shown below; see equation 16. In these variables the number of photons emitted on the k^{th} harmonic from the crystal length L equals:

$$\frac{dN_k}{\Theta d\Theta d\varphi} = \frac{k L B F_k}{137 \gamma x_m T_1^3 (1 + B^2 + D^2 + \Theta^2 - 2\Theta D \sin\varphi)^2} \qquad (11)$$

In this expression the following notations are used:

$$F_k = B^2 \Phi_{kx}^2 + (\Theta^2 + D^2 - 2\Theta D \sin\varphi)\, \Phi_{ky}^2 - 2 B \Theta \Phi_{kx} \Phi_{ky} \cos\varphi , \qquad (12)$$

where

$$\Phi_{kx} = \int_0^1 dw \left\{ \begin{matrix} \cos \\ \sin \end{matrix} \left(\frac{k\pi T_w}{2T_1} \right) \times \begin{matrix} \cos \\ \sin \end{matrix} \left(\frac{k\pi q w}{2T_1} \right) \right\} \begin{matrix} \text{if } k = 2p+1 \\ \text{if } k = 2p \end{matrix} , \qquad (13)$$

$$\Phi_{ky} = \int_0^1 \frac{dw}{\sqrt{1 - V_{wx_m}/V_m}} \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \left(\frac{k\pi T_w}{2T_1} \right) \times \begin{matrix} \sin \\ \cos \end{matrix} \left(\frac{k\pi q w}{2T_1} \right) \right\} \begin{matrix} \text{if } k = 2p+1 \\ \text{if } k = 2p \end{matrix} , \qquad (14)$$

$$q = 2B\Theta\cos\varphi \Big/ (1 + B^2 + D^2 + \Theta^2 - 2\Theta D \sin\varphi) \tag{15}$$

and

$$T_w = \int_0^w du \Big/ \sqrt{1 - V_{ux_m}/V_m} \quad . \tag{16}$$

In the forward direction ($\Theta = 0$) $q = 0$. Hence, in this case $\Phi_{2px} = 0$, and a particle moving in the x - z plane ($D = 0$) radiates only odd harmonics.

III. Dependence on Particle Energy

We are interested now in the dependence of the radiation on the incident particle energy. This information can be very useful for an optimal choice of experimental arrangement for given conditions. As an example, we choose channelling of positrons in the (1,1,0) direction of a Silicon (Si) crystal.

Summing up the contributions of adjacent crystal planes, we get the following expressions for the continuum Lindhard[8] potential at the distance x from the crystal plane:

$$V(u) = V_0 \sum_{p=1}^{P} \left\{ \sqrt{(2p-1-u)^2 + b^2} + \sqrt{(2p-1+u)^2 + b^2} - 2\sqrt{(2p-1)^2 + b^2} \right\}, \quad P \to \infty , \tag{17}$$

where $u = 2x/d$, $V_0 = \pi Ze^2 n\, d^2$, $b = 2\sqrt{3}\, a/d$, n is the number of atoms with atomic number Z per unit volume, d is the distance between crystal planes and a is the screening length of the electron-atom interaction for the Thomas-Fermi atom model. Expression (17) is valid in the region $0 < u < 1$.

For other values of u, one can use the relations $V(-u) = V(u)$ and $V(u+2p) = V(u)$, $p = 0, \pm1, \pm2, \ldots$. For the (1,1,0) direction in Si these constants have the following values: $V_0 = 117$ eV ($Z = 14$, $d = 1.920$ Å, $a = 0.194$ Å, $n = 4.994 \times 10^{22}$ cm^{-3}, $b = 0.350$). Practically speaking, only the first few terms of the sum in expression (17) contribute for any given value of u.

In order to present results in a form independent of a particular crystal's characteristics, let us rewrite (7) and (11) in the following way:

$$\omega_k = C_\omega \Omega_g \quad , \tag{18}$$

where

$$C_\omega = \frac{2\pi c\gamma}{d} k \quad , \tag{19}$$

$$\Omega_g = \frac{B}{uT_1(1 + B^2 + D^2 + \Theta^2 - 2\Theta D \sin\varphi)} \quad , \tag{20}$$

and

$$\frac{dN}{\theta d\theta d\varphi} = C_N N_g \quad , \tag{21}$$

where

$$C_N = \frac{2L\gamma}{137d} \quad , \tag{22}$$

$$N_g = \frac{k B F_k}{nT_1^3(1 + B^2 + D^2 + \Theta^2 - 2\Theta D \sin\varphi)^2} \quad . \tag{23}$$

The characteristics of a given crystal enter into the functions Ω_g and N_g only through parameters B and T_1. The dependence on particle energy (or Lorentz factor γ) can be tracked through parameters C_ω, C_N, B, D and Θ.

Figures 1 - 3 give the dependence of the first three odd harmonics ($k = 1,3,5$) of the radiation spectrum in the forward direction ($\Theta = 0$) on particle energy for $D = \beta_y \gamma = 0$ (9) and different amplitudes u. Plotted is the quantity N_g as a function of parameter B (8).

Figures 4 - 6 give the same dependence but for $D = 1.0$.

Figures 7 and 8 give the radiation frequency (the quantity Ω_g) of the first harmonic for $D = 0.0$ and $D = 1.0$, respectively.

Figures 9 - 11 give the angular dependence of several first harmonics of radiation spectra for different values of divergence angle $D = \beta_y \gamma$ (9) of the positron and polar angle φ of the radiation.

IV. Comment on the Influence of the Refraction Index

The formulae for the number of emitted quanta and their frequencies were obtained with the assumption that the refraction index ε of the crystal is equal to one. Indeed, at the frequencies of interest for an ultrarelativistic particle, ε is very close to 1. In the limit of $\omega \gg \omega_i$ (ω_i are the proper frequencies of equivalent oscillators for the atomic electrons)

$$\varepsilon(\omega) = 1 - 4\pi Z n e^2/m\omega^2 \equiv 1 - \omega_0^2/\omega^2; \qquad \omega_0 \ll \omega \quad . \tag{24}$$

For this case it is easy to show that all the results stay the same if in the brackets of expressions (7), (11) and (15) one adds the term $(\gamma\omega_0/\omega_k)^2$. For example

$$\omega_k = \frac{\pi c \gamma k B}{x_m T_1 (1 + B^2 + D^2 + \Theta^2 - 2\Theta D \sin\varphi + \gamma^2 \omega_0^2/\omega_k^2)} \quad . \tag{25}$$

Let us denote by $\bar{\omega}$ the value of ω_k at the limit $\omega_0 = 0$.

Then for $\varepsilon - 1 \neq 0$ the k^{th} line splits into two with frequencies

$$\omega_{\pm} = \frac{\bar{\omega}}{2} \pm \sqrt{\frac{\bar{\omega}^2}{4} - (\gamma\omega_0)^2} \quad , \qquad (16$$

if $\gamma\omega_0 < \bar{\omega}/2$ (the case $\gamma\omega_0 > \bar{\omega}/2$ is, from a practical point of view, very unusual and needs a special investigation). Usually not only $\omega_0 < \bar{\omega}$, but even $\gamma\omega_0 < \bar{\omega}/2$. If $\gamma\omega_0 << \bar{\omega}$ then

$$\omega_+ = \bar{\omega} - (\gamma\omega_0)^2/\bar{\omega} \quad , \qquad (27)$$

$$\omega_- = (\gamma\omega_0)^2/\bar{\omega} \quad . \qquad (28)$$

In some cases the influence of the refraction index can explain the unequal distances between the spectral lines. We shall not discuss this effect any further here.

Conclusion

The method[7] of calculating the frequency and angular spectra of channelling radiation gives us the possibility of finding the optimal particle energy for any single particle and quanta parameters. These results can further be used to obtain spectra averaged over the particle distribution in transverse phase space of a beam for any given geometry of experiment.

Acknowledgment

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Figure Captions

1. The dependence of the number of radiated quanta per solid angle for the first harmonic in the forward direction on parameter B (8) for $D=0$ (9) and different particle amplitudes:

   (1) $u=0.9$,

   (2) $u=0.7$,

   (3) $u=0.5$,

   (4) $u=0.3$,

   (5) $u=0.1$.

   Plotted is the quantity $N_g = \left(\frac{dN}{\theta d\theta d\varphi}\right) \Big/ \left(\frac{2L\gamma}{137d}\right)$ (see (23)).

2. The same as on Fig. 1, but for the third harmonic.

3. The same as on Fig. 1, but for the fifth harmonic.

4. The same as on Fig. 1, but for $D=1.0$.

5. The same as on Fig. 2, but for $D=1.0$.

6. The same as on Fig. 3, but for $D=1.0$.

7. The dependence of the radiation frequency of the first harmonic on parameter B (8) for $D=0$ (9) and different particle amplitudes:

   (1) $u=0.9$,

   (2) $u=0.7$,

   (3) $u=0.5$,

   (4) $u=0.3$,

   (5) $u=0.1$.

   Plotted is the quantity $\Omega_g = \omega_1 \Big/ \left(\frac{2\pi c\gamma}{d}\right)$ (see (20)).

8. The same as on Fig. 7, but for $D = 1.0$.

9. Angular ($\Theta = \theta\gamma$) dependence of several first harmonics of the radiation of a positron, $u = 0.9$, $\varphi = 0$, $D = 0$:

   (1) $k = 1$,

   (2) $k = 2$,

   (3) $k = 3$,

   (4) $k = 4$,

   (5) $k = 5$.

10. The same as on Fig. 9, but for $\varphi = \pi/2$.

11. The same as on Fig. 9, but for $D = 1.0$.

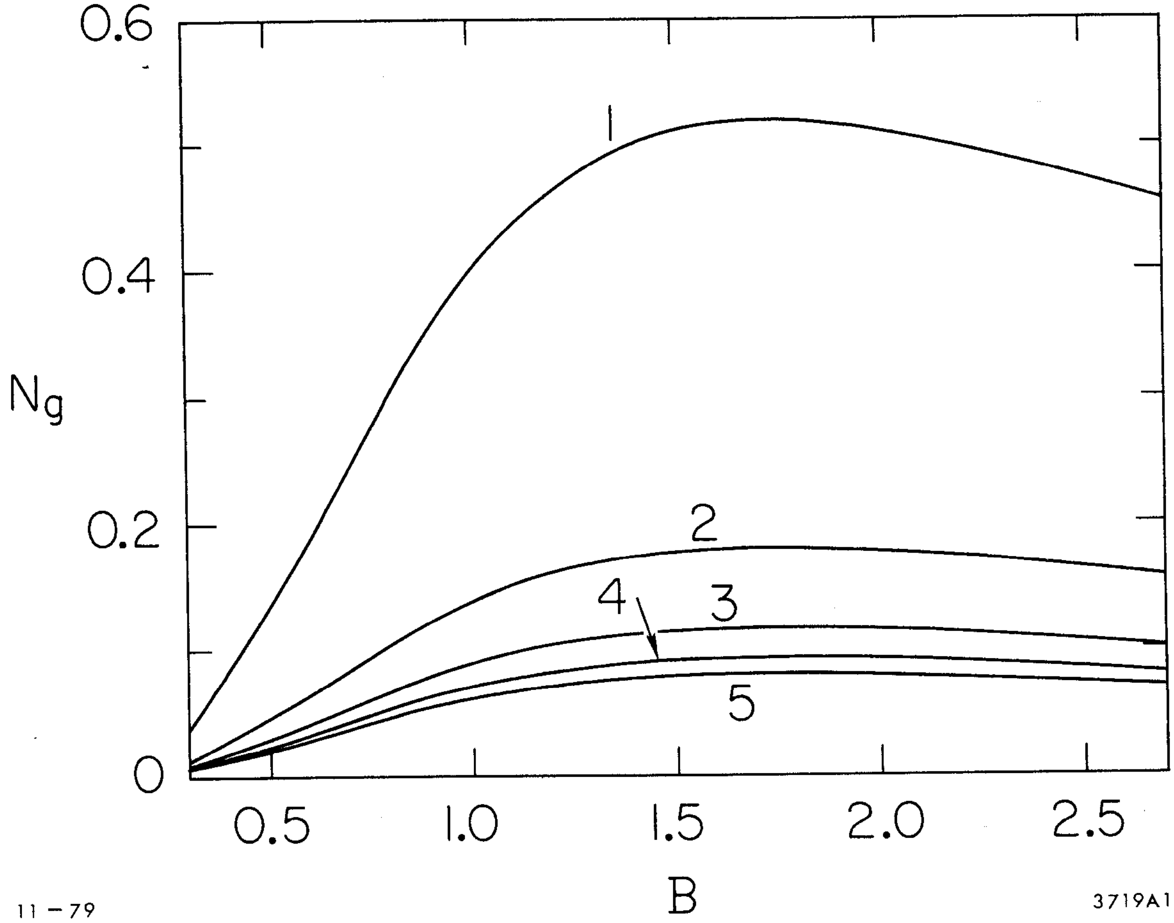


Fig. 1

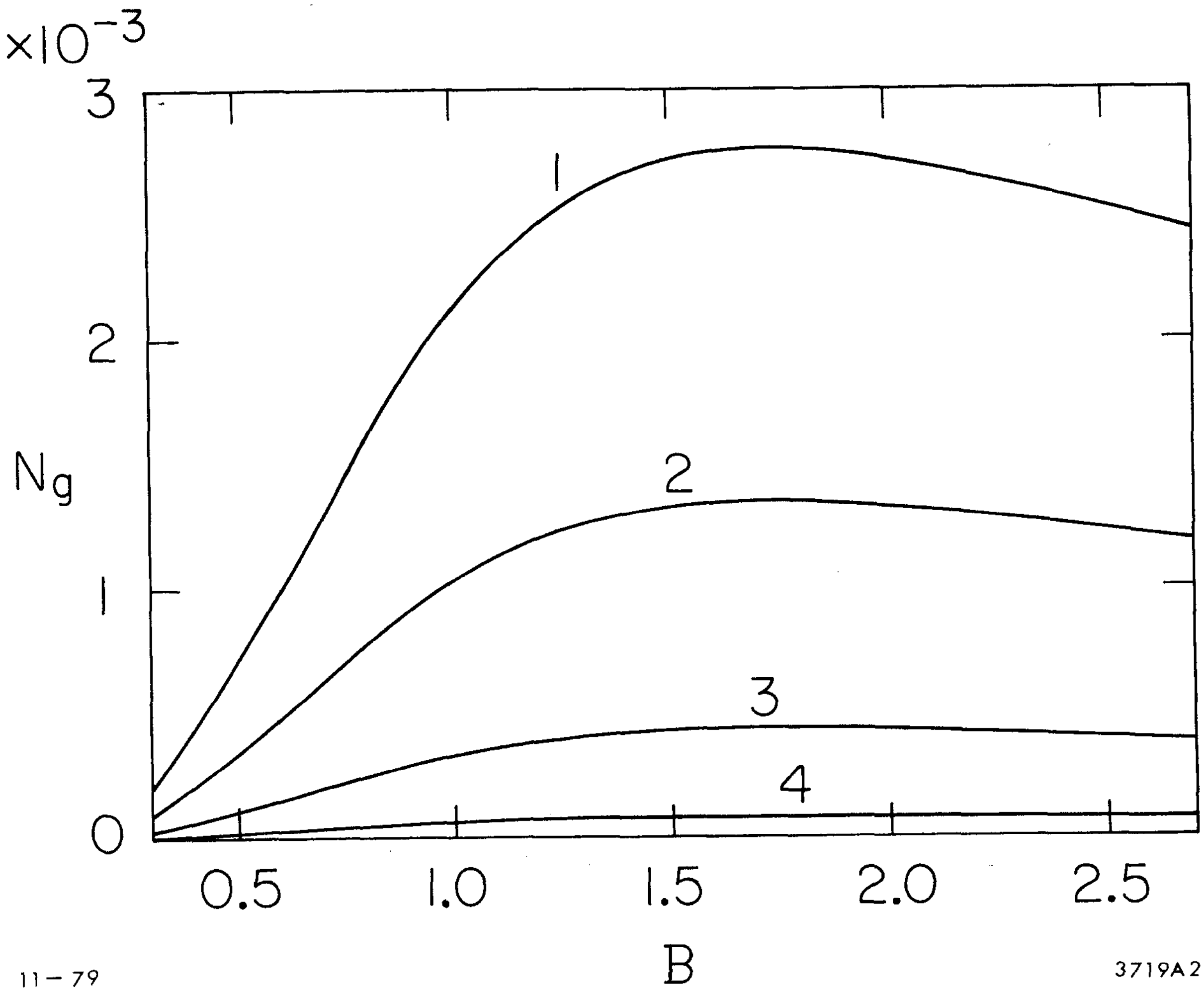


Fig. 2

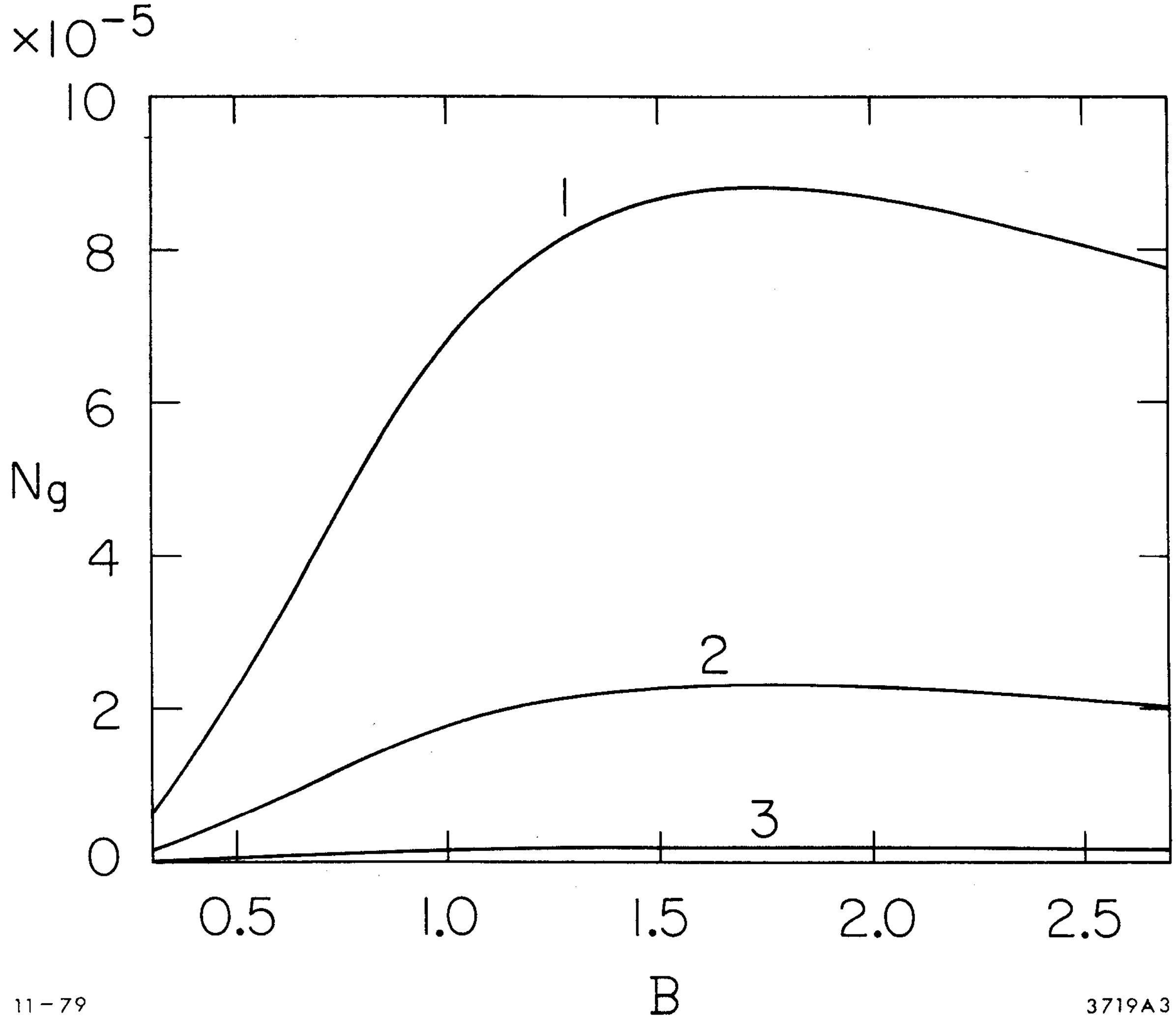
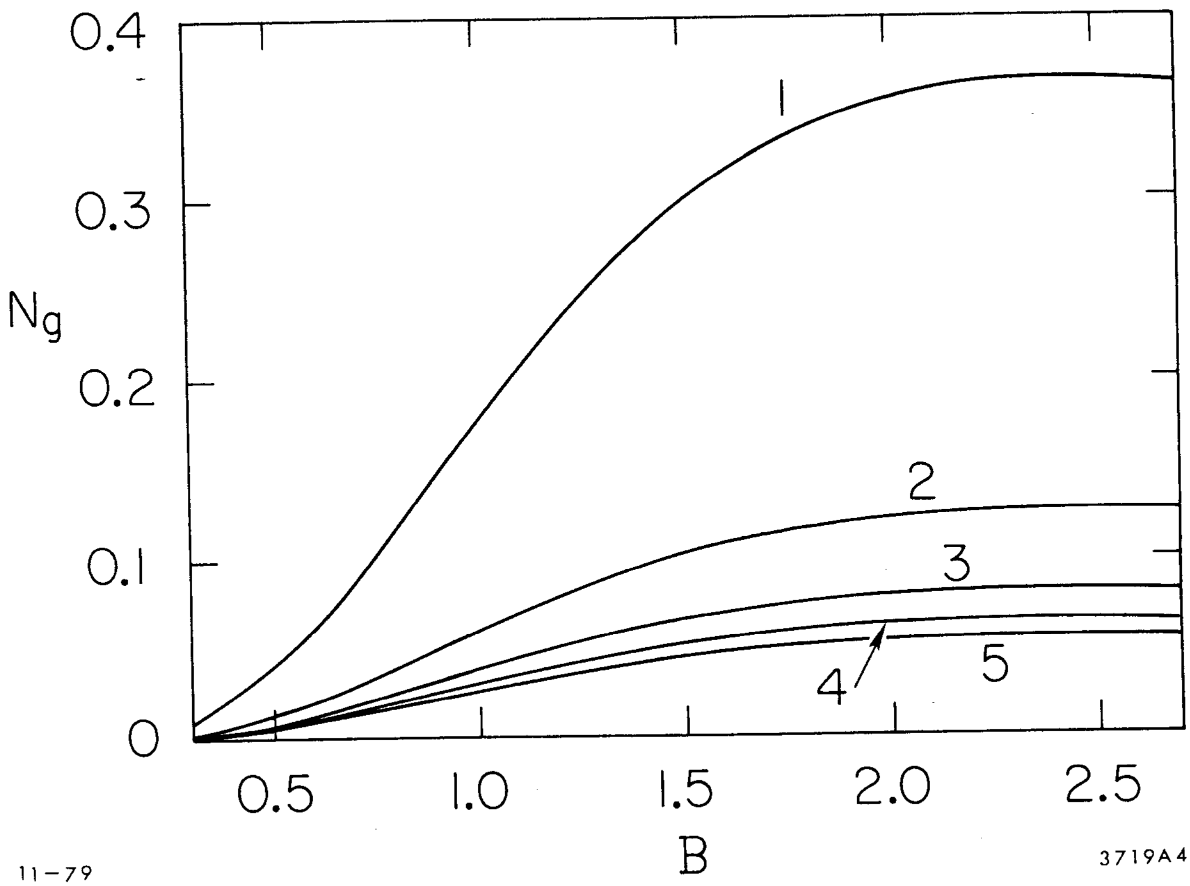


Fig. 3



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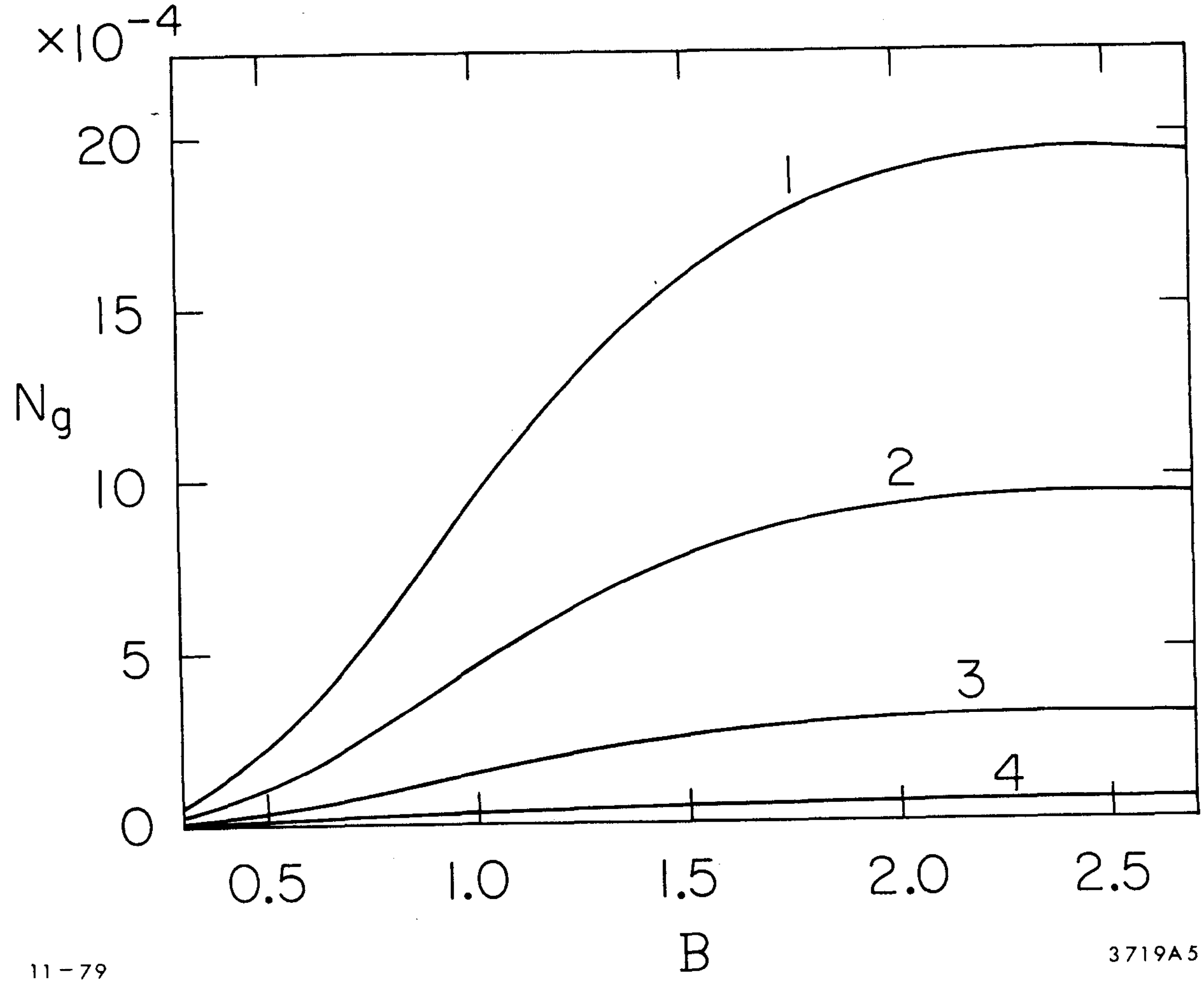

Fig. 5

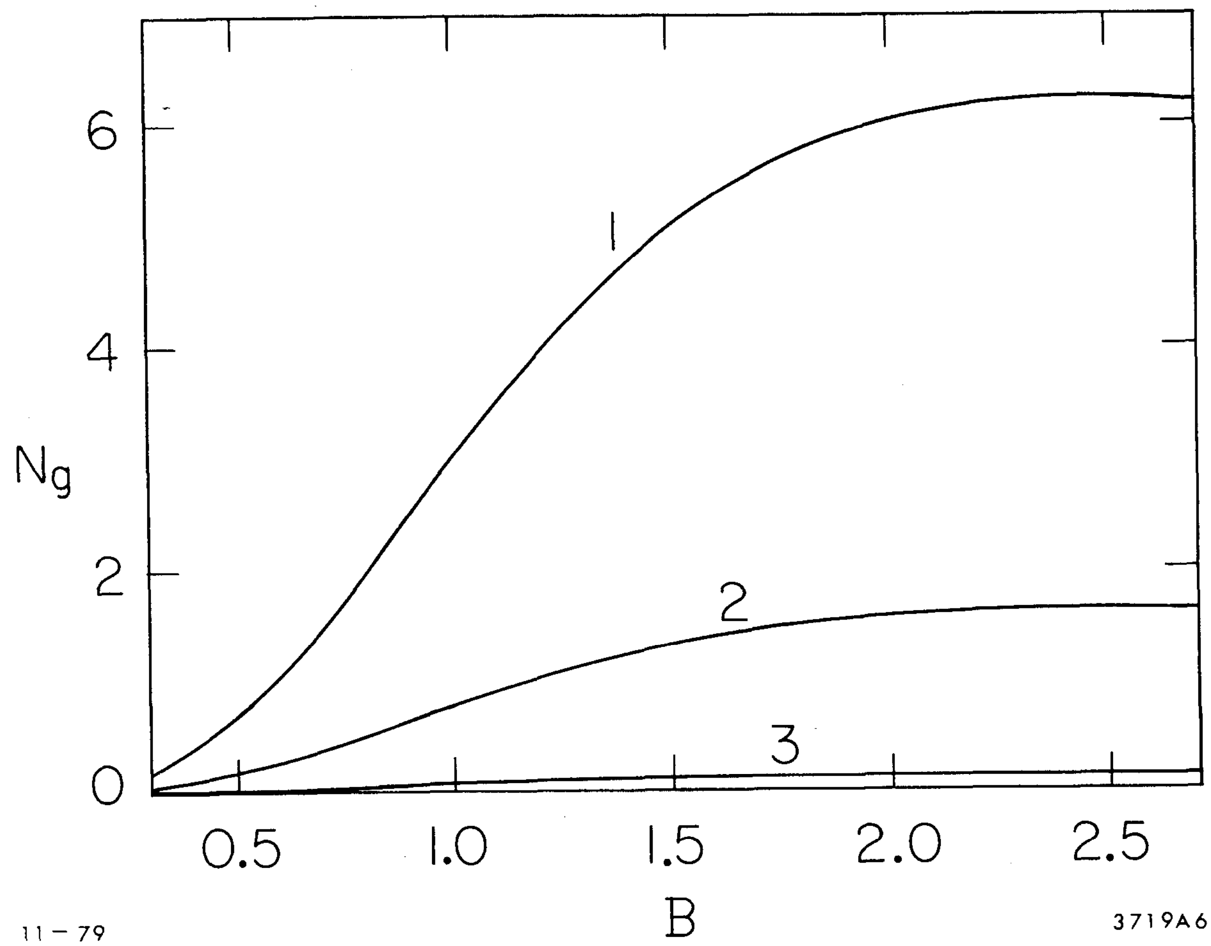

Fig. 6

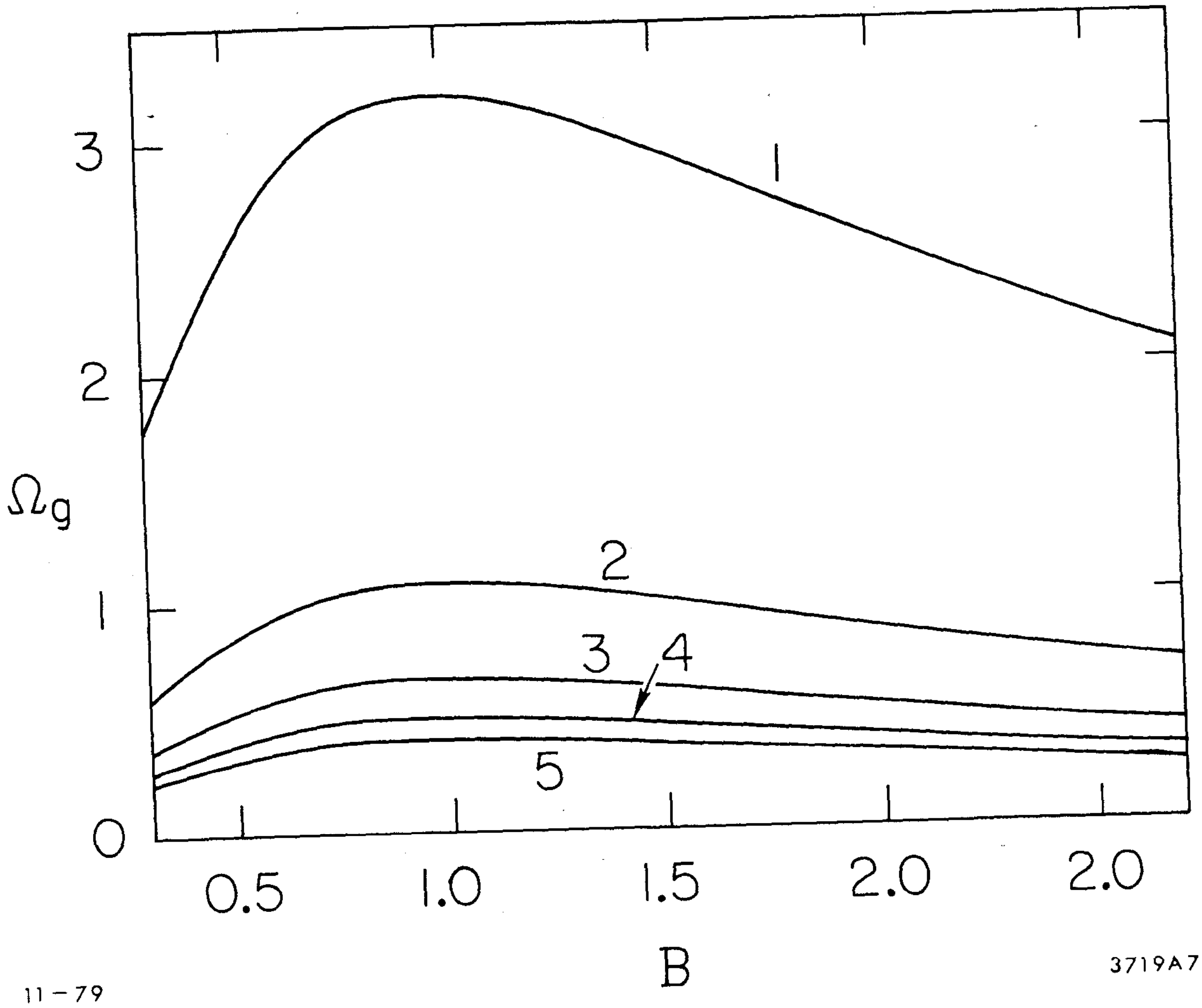

Fig. 7

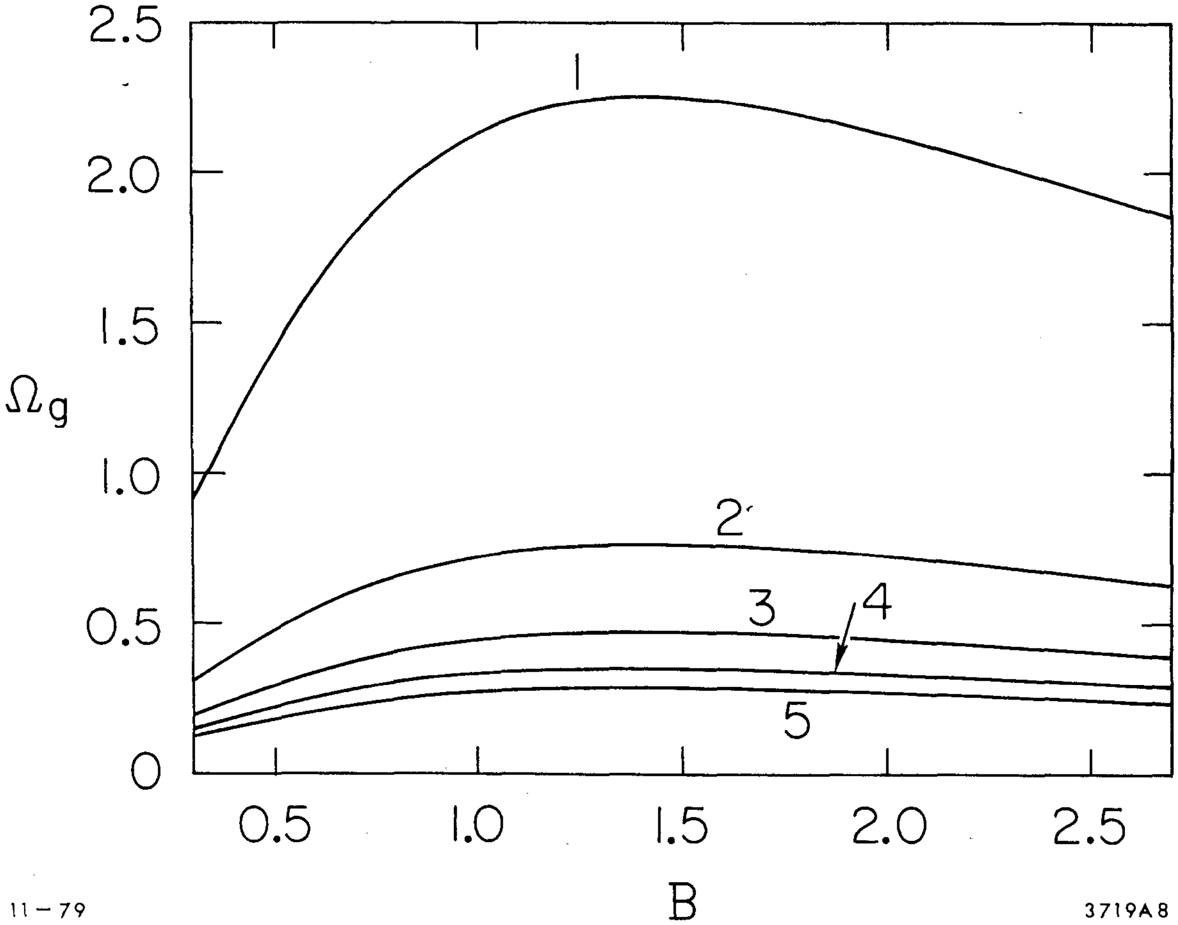

Fig. 8

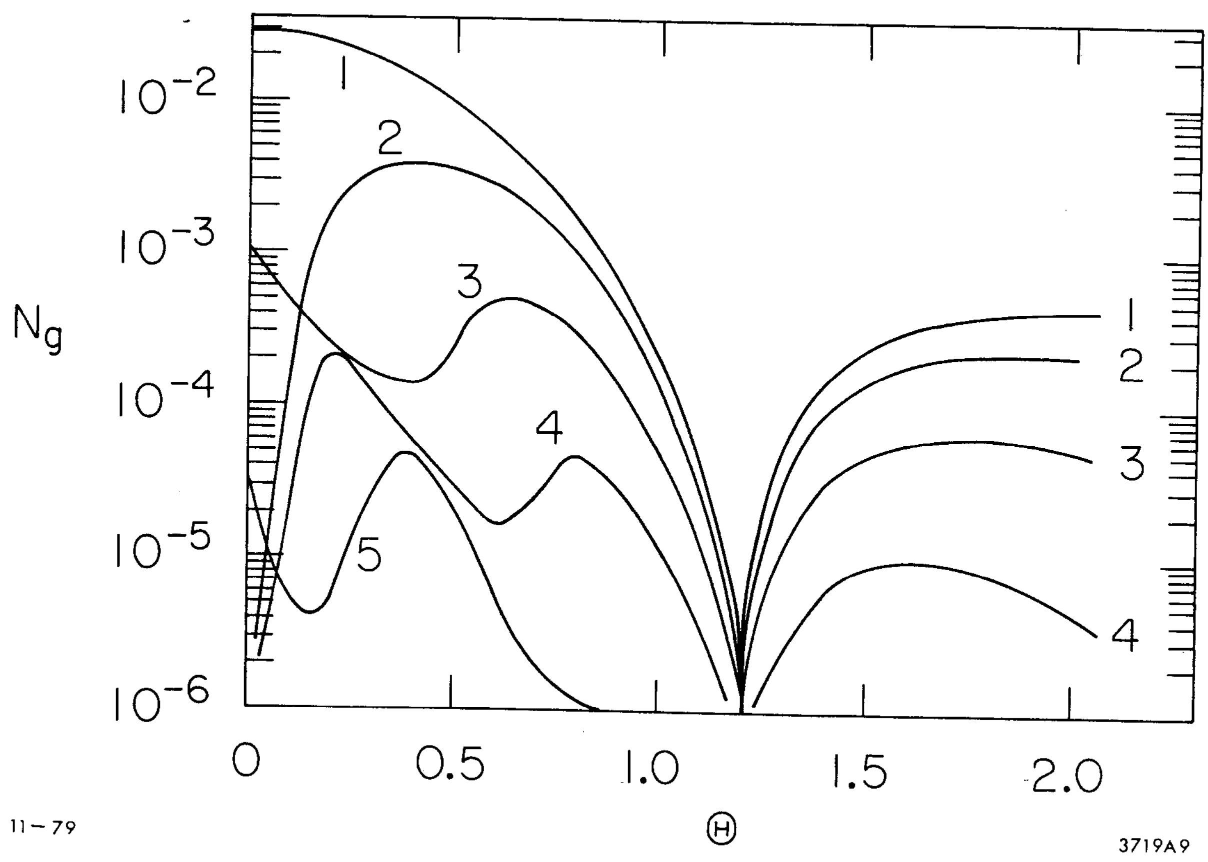

Fig. 9

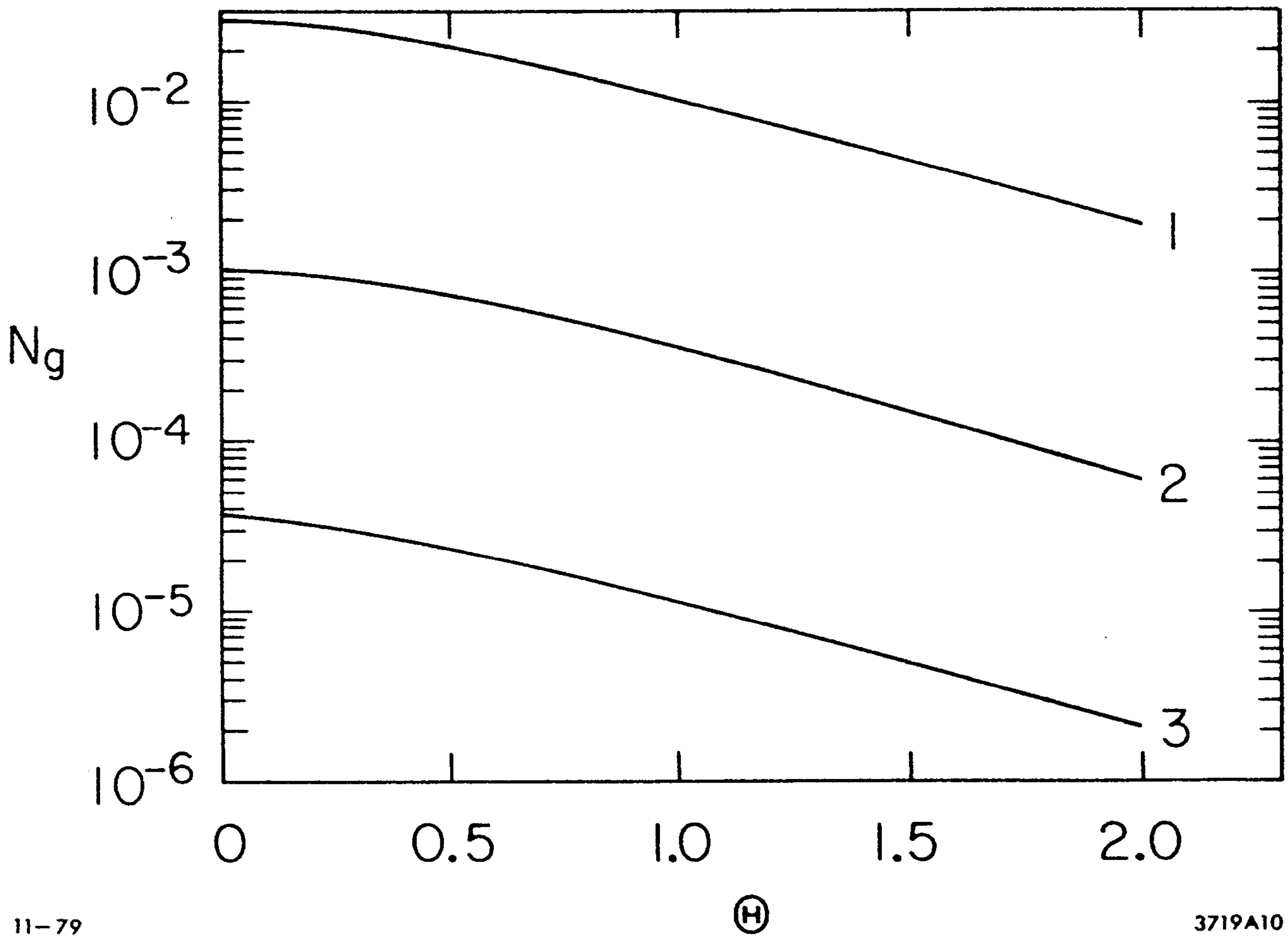

Fig. 10

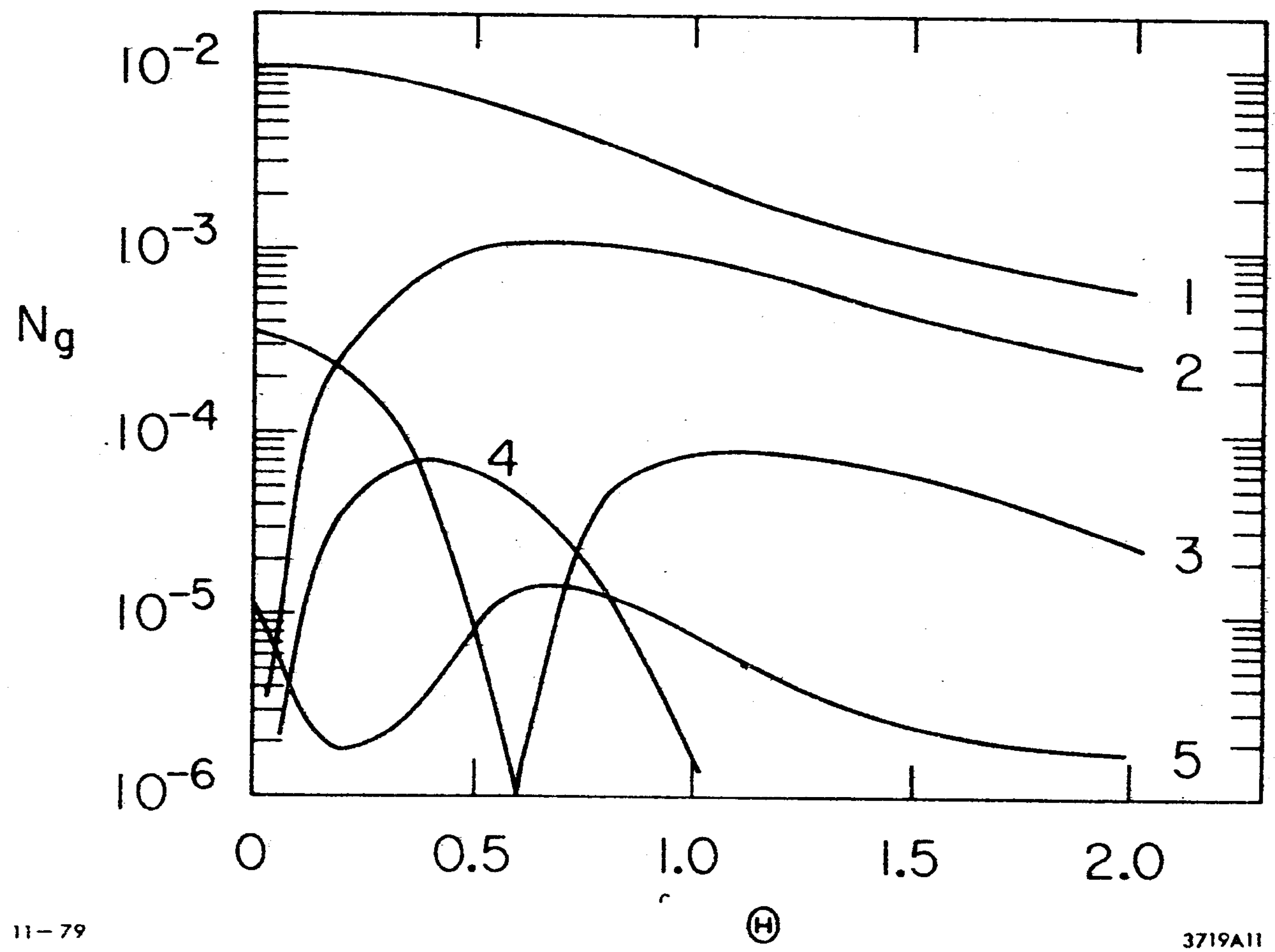

Fig. 11