SLAC-PUB-2407
September 1979
(T/E)

CONNECTION BETWEEN NEUTRINO AND GAUGE INVARIANCE: A TWO-DIMENSIONAL MODEL*

R. Ferrari**

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

ABSTRACT

We discuss a two-dimensional model of interaction between electrons and neutrinos. The absence of a neutrino mass allows for a local conservation law, similar to gauge invariance in quantum electrodynamics. We discuss the peculiar fact that the tree approximation gives wrong results for the conservation equation of the current associated with gauge transformations. Moreover, we show that the anomaly of the axial current is responsible for this property.

(Submitted to Nuovo Cimento Letters.)

---

*Work supported in part by the Department of Energy under contract number DE-AC03-76SF00515.

**Supported by a NATO Fellowship.
On leave of absence from Istituto di Fisica, Università di Pisa, Italy and CERN, Geneva, Switzerland.

The existence in nature of massless fermions rouses some puzzling questions, particularly from the theoretical point of view. The fact that at least one neutrino is massless is just the minimal requirement to render observed parity violating processes compatible with the Poincaré invariance of the physical laws.[†] However, to our knowledge, no other important symmetry or field theoretical property is associated with the fact that the neutrino is massless.

We would like to associate some local conservation law to the absence of the mass of the neutrino, similar to what happens to the connection between group invariance and photon. We shall investigate this problem in a two-dimensional model of interaction between an electron and a neutrino. The peculiarity of the two-dimensional world will allow a state of neutrino-antineutrino to play the role of gauge-particle.

In the tree approximation the theory is given by the Feynman diagram rules derived from the Lagrangian density[‡]

$$\mathscr{L} = \bar{\phi}(i\not{\partial} - m)\phi + \bar{\psi}(i\not{\partial})\psi + g\, j^{\mu} J_{\mu} \tag{1}$$

where

$$j^{\mu} = \bar{\psi}\gamma^{\mu}\psi \qquad J^{\mu} = \bar{\phi}\gamma^{\mu}\phi \tag{2}$$

We use the following conventions

$$g^{00} = 1 \qquad \epsilon^{01} = 1 \qquad \gamma^{0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^{1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$\gamma^{5} = \gamma^{0}\gamma^{1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad d^{2}x = dx^{0}dx^{1} \qquad \delta^{(2)}(x) = \delta(x^{0})\,\delta(x^{1}) \tag{3}$$

The regularization and the renormalization procedure will be elected in such a way to exert local gauge invariance

$$(4) \qquad \begin{aligned} \psi(x) &\rightarrow \exp i\left(\lambda(x) - \gamma^5\tilde{\lambda}(x)\right)\psi(x) \\ \phi(x) &\rightarrow \exp\left(-i\frac{g}{\pi}\lambda(x)\right)\phi(x) \end{aligned}$$

where $\partial_\mu\lambda(x) = \varepsilon_{\mu\nu}\partial^\nu\tilde{\lambda}(x)$ and consequently $\Box\lambda(x) = 0$. If we define J in a gauge-invariant manner, then the neutrino current should transform like

$$(5) \qquad j_\mu(x) \rightarrow j_\mu(x) - \frac{1}{\pi}\partial_\mu\lambda(x)$$

in order to get a gauge-invariant Lagrangian.

The product of the fields in the definition of J in (2) can be made finite and gauge-invariant by using any gauge-invariant subtraction procedure, e.g., dimensional regularization ([1]). On the other side, the split-point definition ([2]) seems to be the better way to enforce transformation (5) for j:

$$(6) \qquad j_\mu(x) \equiv \lim_{\substack{\xi\to0 \\ \xi^2\neq0}} Z_3(\xi^2)\left[T\left(\bar{\psi}(x+\xi)\gamma_\mu\psi(x)\right) - \langle 0|T\left(\bar{\psi}(\xi)\gamma_\mu\psi(0)\right)|0\rangle\right]$$

where $Z_3(\xi^2)$ is some renormalization factor. Let us suppose in fact that for small ξ the most singular part of $T\left(\bar{\psi}(x+\xi)\gamma_\mu\psi(x)\right)$ is proportional to the free propagator, i.e.,

$$\frac{1}{2\pi i}\,\frac{\mathrm{Tr}\left[\gamma_\mu\,\xi\cdot\gamma\right]}{\xi^2 - i\varepsilon}$$

then

$$(7) \qquad j'_\mu(x) - j_\mu(x) \alpha \lim_{\substack{\xi\to0 \\ \xi^2\neq0}} \frac{1}{2\pi i}\,\mathrm{Tr}\left[-i\xi^\nu\partial_\nu\left(\lambda(x) + \gamma^5\tilde{\lambda}(x)\right)\frac{\gamma_\mu\xi\cdot\gamma}{\xi^2 - i\varepsilon}\right] =$$

$$= -\frac{1}{\pi} \lim_{\substack{\xi\to 0 \\ \xi^2\neq 0}} \frac{\xi^2 \partial_\mu \lambda(x)}{\xi^2 - i\varepsilon} = \frac{1}{\pi} \partial_\mu \lambda(x)$$

We shall evaluate later the proportionality constant in (7).

The j-two-point function can be easily evaluated in the one-loop (OL) approximation. By employing the definition in (6), we get

$$(8) \quad \langle 0|T\left(j_\mu(x) j_\nu(0)\right)|0\rangle\Big|_{OL} = \frac{1}{(2\pi i)^2} \lim_{\substack{\xi\to 0 \\ \xi^2\neq 0}} \mathrm{Tr}\left[\gamma_\mu \frac{x\cdot\gamma}{x^2 - i\varepsilon} \gamma_\nu \frac{(x+\xi)\cdot\gamma}{(x+\xi)^2 - i\varepsilon}\right]$$

Consider light-cone coordinates (i.e., μ, $\nu = +, -$). By using the relations

$$(9) \quad \begin{array}{ll} \gamma^\pm \equiv \gamma^0 \pm \gamma^1 , & \left(\gamma^\pm\right)^2 = 0 , \\ \gamma^+\gamma^- = 2\left(1-\gamma^5\right) , & \gamma^-\gamma^+ = 2\left(1+\gamma^5\right) \end{array}$$

we arrive at

$$\langle 0|T\left(j^+(x) j^+(0)\right)|0\rangle\Big|_{OL} = \frac{1}{(2\pi i)^2} \lim_{\substack{\xi\to 0 \\ \xi^2\neq 0}} 4 \frac{x^+}{x^2 - i\varepsilon} \frac{(x+\xi)^+}{(x+\xi)^2 - i\varepsilon}$$

$$= \pi^{-2} \partial^+ \partial^+ \ell n\left(-x^2 + i\varepsilon\right)$$

and

$$\langle 0|T\left(j^+(x) j^-(0)\right)|0\rangle\Big|_{OL} = 0$$

i.e.,

$$(10) \quad \langle 0|T\left(j^\mu(x) j^\nu(0)\right)|0\rangle\Big|_{OL} = -\frac{1}{8\pi^2} \left(\Box g^{\mu\nu} - 2\partial^\mu\partial^\nu\right) \ell n(-x^2 + i\varepsilon)$$

$$(11) \quad \partial^\mu \langle 0|T\left(j_\mu(x) j^\nu(0)\right)|0\rangle\Big|_{OL} = \frac{i}{2\pi} \partial^\nu \delta^{(2)}(x) .$$

In the next section we shall prove that the result in (11) is exact to all orders of the perturbative series. We need a simple lemma for the unrenormalized theory.

Lemma: The one-loop connected part $\mathcal{T}(1\ldots n)$ of $\langle 0|T(j^{\mu_1}(x_1)\ldots j^{\mu_n}(x_n))|0\rangle$ vanishes identically for $n > 2$.

The proof of this result has been given by Y. Frishman ([3]). Here we provide some improvements. The Fourier transform for non-exceptional momenta (i.e., none of the internal lines are on-shell) is given by

$$\tilde{\mathcal{T}}(1\ldots n) \equiv \int dx_1\ldots dx_n \exp(ix_ip_i)\, \mathcal{T}(1\ldots n)$$

$$= i^n\, \delta^{(2)}(p_1 + \ldots + p_n) \lim_{\substack{\xi\to 0\\ \xi^2\neq 0}} \sum_{\text{Perm.}} \int d^2k \exp(iq_i\xi_{i+1})$$

$$\text{(12)}\qquad \mathrm{Tr}\left[\gamma^{\mu_1} \frac{q_1\cdot\gamma}{q_1^2 + i\varepsilon} \cdots \gamma^{\mu_n} \frac{q_n\cdot\gamma}{q_n^2 + i\varepsilon}\right] \quad ,$$

where $q_i = \sum_{j=1}^{i} p_j + k$ and the permutations are preformed on the indices of μ and p. Due to the relations in (9), $\tilde{\mathcal{T}}$ can be different from zero only if all the $\mu_i = +$ or $\mu_i = -$. In particular we have

$$\tilde{\mathcal{T}}^{+\ldots+}(1\ldots n) = (2i)^n\, \delta^{(2)}(p_1 + \ldots + p_n) \lim_{\substack{\xi\to 0\\ \xi^2\neq 0}} \sum_{\text{Perm.}} \frac{1}{2}\int dk^+\, dk^-$$

$$\text{(13)}\qquad \exp\left[iq_i\xi_{i+1}\right] \frac{q_1^+}{q_1^2 + i\varepsilon} \cdots \frac{q_n^+}{q_n^2 + i\varepsilon} \quad .$$

We cannot take the limit inside the integral, otherwise the integration is ambiguous. However, if we integrate first in k^-, the integral is zero for large positive or negative k^+ since the poles are then all located in the same imaginary half plane of k^-. Then we obtain $(n > 2)$:

$$\tilde{\mathscr{T}}^{+\cdots+}(1\ldots n) = (2i)^n \delta^{(2)}(p_1+\ldots+p_n) \sum_{\text{Perm.}} \frac{1}{2}\int dk^+ \int dk^-$$

$$\times \frac{q_1^+}{q_1^2 + i\varepsilon} \cdots \frac{q_n^+}{q_n^2 + i\varepsilon} \quad . \tag{14}$$

By using (14) one can easily show that

$$p_{1\mu_1} \tilde{\mathscr{T}}^{\mu_1\ldots\mu_n}(1\ldots n) = 0 \quad . \tag{15}$$

From (15) and from

$$g_{\mu_1\mu_2} \tilde{\mathscr{T}}^{\mu_1\mu_2\ldots\mu_n}(1\ldots n) = 0 \tag{16}$$

one obtains (3) that for all momenta

$$\tilde{\mathscr{T}}(1\ldots n) = 0 \tag{17}$$

From the lemma just proved, it follows that the j-two-point Green function is given only by the chain-graphs, where the one-loop j-two-point function (10) connects the one-particle-irreducible (including the propagator in (10)) blobs of $\langle 0|T(J_\mu J_\nu)|0\rangle$. Since J is built in a gauge-invariant way, the electron does not contribute to the divergence of the j-two point function. Thus (11) is valid for any number of loops.

A further result can be derived from the lemma. First notice that a simple use of the Ward identity shows that the renormalization constant Z_3 in (6) is also the wave function renormalization constant for the neutrino. By power counting arguments, it should have a logarithmic-type dependence on the cut-off. We are now in position to show that the proportionality constant in (7) is just one. In fact for any product B of fields, the most singular part of the limit $\xi = 0$ of

$$\langle 0|T[(\bar{\psi}(x+\xi)\ \gamma_\mu \psi(x))B]|0\rangle$$

is the disconnected part

(18) $$\langle 0|T(\bar{\psi}(\xi)\ \gamma_\mu \psi(0))|0\rangle\langle 0|T(B)|0\rangle \quad .$$

From the lemma it follows that any loop-insertion to the closing neutrino line vanishes in the limit, i.e.,

(19) $$Z_3\langle 0|T[\bar{\psi}(\xi)\ \gamma_\mu \psi(0)]|0\rangle \simeq \frac{1}{2\pi i}\ \frac{\mathrm{Tr}[\gamma_\mu\ \xi\cdot\gamma]}{\xi^2 - i\varepsilon}$$

Hence the transformation property in (5) is implemented by our renormalization procedure.

These facts suggest that the Lagrangian in (1) will be renormalized like in QED in four dimensions, i.e., with $Z_1 = Z_2$:

(20) $$\mathscr{L} = Z_1[\bar{\phi}(i\not{\partial} + g\gamma_\mu\ j_\mu)\ \phi] - m\ Z_m\ \bar{\phi}\ \phi + Z_3\ \bar{\psi}(i\not{\partial})\ \psi$$

A more general argument will follow from the analysis of the invariance of the perturbative series. Notice that, due to (10), the S-matrix in the electron sector is the same as in the massive Thirring model.

The currents associated with the symmetry transformation (4) have unusual and interesting properties. In particular the current obtained from the Noether theorem for the transformation in (4) is not conserved. This is due to our renormalization prescription. Moreover the axial currents present anomalies. These anomalies allow us to find a new local current which is conserved and generates the transformation (4). This is a quite new situation in field theory: the invariance properties of the theory appear only after the quantum corrections to the tree approximation have been performed.

The Noether current associated with transformation (4)

$$\mathcal{J}^{\mu}(x) = -\frac{g}{\pi}\lambda(x)J^{\mu}(x) + \{\lambda(x)g^{\mu\nu} + \tilde{\lambda}(x)\varepsilon^{\mu\nu}\}j_{\nu}(x) ,$$

where $j^{\mu}(x) = Z_1\bar{\phi}\gamma_{\mu}\phi(x)$, is not conserved. In order to find the right current, we shall analyze closer the currents J and j. The Feynman graphs, contributing to an amplitude involving j and any number of external neutrinos (dotted lines) and electrons (dashed lines) can be divided into two classes as indicated in figure 1. The contribution of the first class of graphs to the divergence of j is easily given by using (11)

$$\partial^{\mu}\langle 0|T(j_{\mu}(x)B)|0\rangle\Big|_{1^{st}} = -\frac{g}{2\pi}\partial^{\mu}\langle 0|T(J_{\mu}(x)B)|0\rangle \tag{21}$$

where B stands for the product of electron- and neutrino-fields. The second class of graphs gives the contribution

$$\partial^{\mu}\langle 0|T(j_{\mu}(x)\bar{\psi}(x_1)\ldots\bar{\psi}(x_n)\psi(y_1)\ldots\psi(y_n)\bar{\phi}(w_1)\ldots\bar{\phi}(w_m)\phi(z_1)\ldots\phi(z_m))|0\rangle\Big|_{2^{nd}}$$
$$= \sum_{i=1}^{n}[\delta^{(2)}(x-x_1)-\delta^{(2)}(x-y_i)]\langle 0|T(\bar{\psi}(x_1)\ldots\bar{\psi}(x_n)\psi(y_1)\ldots\psi(y_n)\bar{\phi}(w_1)\ldots\bar{\phi}(w_m)$$
$$\times\ \phi(z_1)\ldots\phi(z_m))|0\rangle \tag{22}$$

By construction J generates locally a U(1) transformation on the fields ϕ and $\bar{\phi}$. Therefore, from (21) and (22) we can define the charge operators

$$q = \int dx^1\, j_0(x)$$

(23)

$$Q = \int dx^1\, J_0(x)$$

Both are conserved and have commutation relations

$$[q,\psi] = -\psi$$

$$[q,\phi] = -\frac{g}{2\pi}[Q,\phi] = \frac{g}{2\pi}\phi$$

(24)

$$[Q,\psi] = 0$$

$$[Q,\phi] = -\phi$$

and their hermitian conjugates. q generates a more complicated transformation as one would expect from Lagrangian (1).

We consider now the axial currents. An analysis similar as that for the vector currents gives

$$\partial^\mu\langle 0|\varepsilon_{\mu\nu}T\big(j^\nu(x)B\big)|0\rangle\Big|_{1^{st}} = i\partial^\mu\varepsilon_{\mu\nu}\int d^2y\,\langle 0|T\big(j^\nu(x)j^\rho(y)\big)|0\rangle$$

$$g\langle 0|T\big(J_\rho(y)B\big)|0\rangle = \frac{g}{2\pi}\,\partial^\mu\varepsilon_{\mu\nu}\langle 0|T\big(J^\nu(x)B\big)|0\rangle \qquad ,$$

i.e.,

(25) $$\partial^\mu\varepsilon_{\mu\nu}\Big(j^\nu(x) - \frac{g}{2\pi}J^\nu(x)\Big) = 0$$

The new current

$$(26) \qquad \hat{j}_{5\mu}(x) \equiv \varepsilon_{\mu\nu}\left(j^{\nu}(x) - \frac{g}{2\pi} J^{\nu}(x)\right)$$

generates the usual chiral transformations on the field ψ and $\bar{\psi}$. Since the current is conserved, the theory is global chiral invariant.

The anomaly (25) suggests a new conserved current associated with the transformations (4):

$$(27) \qquad \hat{\mathcal{J}}^{\mu} = (\lambda g^{\mu\nu} + \tilde{\lambda}\varepsilon^{\mu\nu})\left(j_{\nu} - \frac{g}{2\pi} J_{\nu}\right) \quad .$$

An analysis similar to that in figure 1 shows that indeed the new current generates the required transformation.

Finally the consistency of our renormalization scheme given by the current in (6) and the Lagrangian (20) is shown by (21), (22) and the analogous relation involving the divergence of J. In fact, like in QED, these relations can be used to show the equality between vertex and wave function renormalization constants.

Conclusions

The model we have briefly discussed shows that the absence of mass for the neutrino allows for a local conservation law (14) and (27). The new interesting feature of this invariance is that it is absent in the tree approximation. In fact renormalization and anomaly (25) are essential in proving that the current (27) is conserved and generates the transformations (4).

Acknowledgments

The author is particularly grateful to A. Z. Capri for stimulating exchange of correspondence on the subject, and to P. Butera and G. Cicuta for interesting discussions. He thanks Professor Sidney Drell for the kind hospitality at SLAC. The support of a NATO fellowship is thankfully acknowledged.

This work was supported in part by the Department of Energy under contract number DE-AC03-76SF00515.

References

† If all neutrinos had a finite mass, one could define a parity operator P on the asymptotic states such that it leaves invariant all the internal quantum numbers and has the standard communication relations with the elements of the Poincaré group. In particular $PU(a_0,\underline{a}) = U(a_0, -\underline{a})P$, where U(a) is the translations operator, i.e., $PHP^{-1} = H$, where H is the Hamiltonian.

‡ A parity-violating interaction can be handled in a similar way as the Lagrangian in equation (1).

1. J. F. Ashmore, Nuovo Cimento Letters 4, 289 (1972); C. G. Bollini and J. J. Giambiagi, Phys. Letters 40B, 566 (1972); G. M. Cicuta and E. Montaldi, Nuovo Cimento Letters 4, 329 (1972); G. 't Hooft and M. Veltman, Nucl. Phys. B44, 189 (1972);

2. J. Schwinger, Phys. Rev. Letters 3, 269 (1959).

3. Y. Frishman, Lecture Notes in Physics 32, 118 (1975).

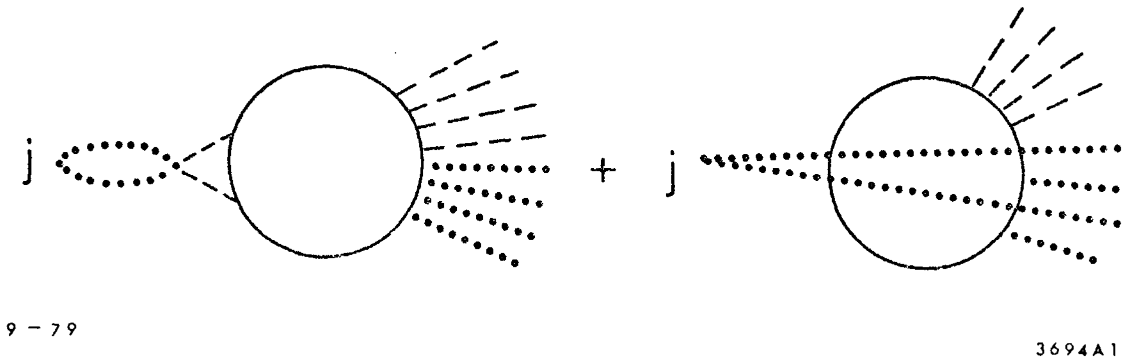

Fig. 1