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THE BOSE FORM OF TWO-DIMENSIONAL QUANTUM CHROMODYNAMICS *

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ABSTRACT

The Bose form of QCD_2 is presented. Weak ($m \ll g$) and strong ($m \gg g$) coupling regimes are briefly analyzed. The former corresponds to the quark phase with bound states exhibiting the string picture. Mesons and baryons appear as longitudinal modes of strings represented by electric vortex lines. The latter describes the Bose phase with a spectrum consisting of N colorless scalars and their bound states.

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In this note I would like to show that SU(N) quantum chromodynamics in one space-one time dimension (QCD_2) is equivalent to a local quantum field theory of N interacting scalar bosons.

The bosonization of two dimensional fermion-field theories was initiated by Schwinger [1] who demonstrated that massless QED_2 is equivalent to the massive scalar free field theory. Subsequently the massive QED_2 was also bosonized and extensively studied in refs. [2] to [5]. Recently the interest in the subject has been revived due to the discovery that seemingly very different Sine-Gordon and Thirring models are actually equivalent [6,7]. In the context of this equivalence, Coleman made an important conjecture that quantum solitons are in fact fermions [6]. These results have beem amplified and further developed by Dashen, et al., [8] Mandelstam [9] and others [10].

The present work is an attempt to extend the bosonization program to non-Abelian gauge field theories.

QCD_2 is obtained from the classical Hamiltonian density

$$\mathscr{H} = \sum_{i,j=1}^{N} \left\{ g^2 \left| E^i_j \right|^2 + i\overset{*}{\psi}{}^i \gamma_1 \left(\delta^j_i \partial - iA^j_i \right) \psi_j + m\delta^j_i \overset{*}{\psi}{}^i \psi_j \right\} \tag{1}$$

where canonical variables $\left\{ E^i_j, A^i_j \right\}$ and $\left\{ \overset{*}{\psi}{}^i, \psi_i \right\}$ are constrained by Gauss' law

$$G^i_j \equiv \partial E^i_j - J^i_j = 0 \quad , \tag{2a}$$

$$J^i_j = i[A,E]^i_j + \tfrac{1}{2}\left[\overset{*}{\psi}{}^i \psi_j - \frac{1}{N} \delta^i_j \overset{*}{\psi}{}^k \psi_k \right] \tag{2b}$$

We have adopted the Weyl basis for gauge fields which is related to the more standard basis by $E^i_j = (\lambda^a)^i_j\, E^a$ with the Gell-Mann matrices λ^a, $a = 1, \ldots, N^2 - 1$ being normalized as $\mathrm{Tr}(\lambda^a \lambda^b) = \frac{1}{2}\delta^{ab}$.

The following considerations will motivate our subsequent strategy. The maximal Abelian subgroup of SU(N) is a direct product of U(1) components $U_N(1) \equiv U_1(1) \otimes \ldots \otimes U_{N-1}(1)$. The $U_N(1)$ local gauge theory of quarks can be directly obtained from eqs. (1) and (2) after dropping all off-diagonal currents $j^i_{\mu k} = \bar{\psi}^i \gamma_\mu \psi_k$, $i \neq k$. In the Coulomb gauge $A^i_1 = 0$ Gauss' law allows us to eliminate gauge fields from the Hamiltonian. The quantum version of the resulting Hamiltonian can be easily bosonized by identifying the gradient of bose fields with the diagonal currents $j^i_\mu = \bar{\psi}^i \gamma_\mu \psi_i$. In particular various color diagonal quark densities, e.g., $\bar{\psi}^i \not{\partial} \psi_i$, $\bar{\psi}^i \psi_i$, etc., can be expressed in terms of the bose fields as it will be explained below. Thus in the case of QCD_2 the gauge choice must be such that it should lead to an effective Hamiltonian which is a function of color diagonal quark densities only. Obviously the Coulomb gauge $A^i_j = 0$, $i,j = 1, \ldots, N$ does not belong to this class of gauges since the corresponding Hamiltonian contains diagonal as well as off-diagonal currents coupled non-locally

$$\mathscr{H}_C = -\frac{1}{8}\int dx dy\, j^i_{0k}(x)\,|x-y|\,j^k_{0i}(y) + \int dx\, \bar{\psi}^i(x)(i\gamma_1\partial + m)\psi_i(x) \quad . \tag{3}$$

The hybrid gauge proves to be best suited for the above purpose [11]:

$$A^i_i = 0, \quad i = 1, \ldots, N\text{-}1 \tag{4a}$$

$$E^i_k = 0, \quad i,k = 1, \ldots, N, \quad i \neq k \tag{4b}$$

Equations (4) reduce Gauss' law (2) to

$$\partial e_i = \sqrt{\pi}\ \overset{*}{\psi}{}^i\ \psi_i \equiv \sqrt{\pi}\ j_0^i \tag{5a}$$

$$i(e_i - e_k)\ A_i^k = \sqrt{\pi}\ j_{0i}^k\ ,\ i \neq k \tag{5b}$$

For later convenience, the diagonal components of the electric field have been redefined as follows

$$2\ \sqrt{\pi}\ E_i^i = -\left(e_i - \frac{1}{N}\sum_k e_k\right)\ ,\quad i = 1,\ \ldots\ ,\ N \tag{6}$$

Equations (4) allow us to eliminate gauge fields A_j^i from the Hamiltonian (1). The resulting expression can be cast into the desired form. Indeed, after a simple rearrangement of quark fields one arrives at the Hamiltonian which depends on color diagonal quark densities only:

$$\mathscr{H} = \mathscr{H}_0 + \mathscr{H}_1 \tag{7}$$

$$\mathscr{H}_0 = \sum_i \left\{ i\left(\bar{\psi}^i \gamma_1 \partial \psi_i\right) + \frac{m}{2}\left(M_+^i + M_-^i\right) \right\}\ , \tag{8a}$$

$$\mathscr{H}_1 = \frac{g^2}{8\pi N} \sum_{i,j} \left(e_i - e_j\right)^2 + \tfrac{1}{2}\sqrt{\pi}\ i \sum_{i \neq j} M_+^i\ M_-^j \left(e_i - e_j\right)^{-1} \tag{8b}$$

with

$$M_\pm^i = \bar{\psi}^i\left(1 \pm \gamma_5\right)\ \psi_i\ ,\quad \gamma_5 = \gamma_0\gamma_1$$

and e_i's being given by eq. (5a). The sign of the last term in eq. (8b) accounts for the Fermi statistics of quark fields. Thus the problem has been reduced to the quantization of the classical Hamiltonian (7,8). I will first postulate equal time anticommutation relations for canonical quark fields (cf. ref. 11):

$$\left\{\overset{*}{\psi}{}^{i}(x,t),\ \psi_j(y,t)\right\} = \delta^i_j\ \delta(x-y) \quad ; \tag{9}$$

the remaining anticommutators vanish. Next I should regularize the product of operators which appear in eqs. (8). This may be accomplished by the standard point splitting method. However, it is much more convenient to tackle this problem after eq. (8) has been cast in the bose form. Therefore I will proceed to bosonization of eq. (8) using algebraic properties of its individual terms.

Let us complement eqs. (9) by the commutators of diagonal currents

$$\left[j^i_\mu(x,t),\ j^k_\mu(y,t)\right] = 0,\ \mu=0,1 \tag{10a}$$

$$\left[j^i_0(x,t),\ j^k_1(y,t)\right] = \frac{i}{\pi}\ \delta^{ik}\ \partial_x\ \delta(x-y) \tag{10b}$$

Note that the R.H.S. of eq. (10b) is the canonical Schwinger term unaffected by the superrenormalizable interactions of QCD_2. By means of eqs. (9) one can evaluate commutators of diagonal currents with various operators in eqs. (8)

$$\left[j^i_{0(1)}(t,s),\ i\bar{\psi}^k(t,y)\gamma_1\partial_y\psi_k(t,y)\right] = i\delta^{ik}j^i_{1(0)}(t,x)\partial_x\delta(x-y) \tag{11}$$

$$\left[j^i_0(t,x),\ M^k_\pm(t,y)\right] = 0 \tag{12}$$

$$\left[j^i_1(t,x),\ M^k_\pm(t,y)\right] = \pm\ 2\ \delta^{ik}M^i_\pm(t,x)\ \delta(x-y) \tag{13}$$

Now the Bose realization of these operators may be found by a standard trick. Equations (10) have a simple realization in terms of N pair of canonically conjugate Bose variables ϕ^i, π^i, $i = 1. ,,, , N$,

$$\sqrt{\pi}\, j_0^i(t,x) = \partial_x \phi^i(t,x), \qquad \sqrt{\pi}\, j_1^i(t,x) = \pi^i(t,x) \tag{14}$$

$$\left[\pi^i(t,x),\ \phi^k(t,y)\right] = -i\,\delta^{ik}\,\delta(x-y) \tag{15}$$

Hence one easily derives the following operator solutions to eqs. (11) to (13),

$$i\,\bar{\psi}^i \gamma_1 \partial_1 \psi_i = \tfrac{1}{2}(\pi_i^2 + \phi_i^2)\ , \tag{16}$$

$$M_\pm^i = -\Lambda \exp(\mp 2\sqrt{\pi}\,\phi_i)\ . \tag{17}$$

Here the constant Λ is a scale parameter which is independent of the color indices because operators $M_\pm^i$ belong to the U(N) multiplet $\bar{\psi}^i(1 \pm \gamma_5)\psi_k$. Without loss of generality Λ may be assumed to be real (see below).

Finally, a comparison of eqs. (5a) and (14) leads to the identification

$$e_i = \phi^i \tag{18}$$

In general the electric fields e_i in eq. (18) are determined up to additive constants e_i^0. However it can be argued that there is no such freedom in the present case [12].

It remains to substitute eqs. (16), (17) and (18) into eqs. (8) to arrive at the Bose representation of the Hamiltonian (7):

$$\mathscr{H}_0 = \tfrac{1}{2}\sum_i \left\{\pi_i^2 + (\partial\phi_i)^2 + 2m\Lambda(1 - \cos 2\sqrt{\pi}\,\phi_i)\right\} \tag{19a}$$

$$\mathcal{H}_1 = \frac{g^2}{8\pi N} \sum_{i,k} (\phi_i - \phi_k)^2 + (\sqrt{\pi}\,\Lambda^2) \sum_{i,k} \left\{ 1 - \int_0^1 d\gamma \cos 2\sqrt{\pi}\,\gamma(\phi_i - \phi_k) \right\} \qquad (19b)$$

For later convenience the trivial constant terms have been added to $\mathcal{H}_{0,1}$. Observe that eq. (19) is symmetric under permutations of ϕ_i's, i.e., it possesses Z_N symmetry.

The above expressions are not well defined and need to be regularized. All local composite operators $0(x)$ in eq. (13) should be interpreted in terms of a finite part of products of Bose fields $\phi(x_i)$ in the limit $x_i \to x$. For example the monomial $\phi^n(x)$ should be interpreted as

$$N\left\{\phi^n(x)\right\} \equiv \text{Finite}\left\{\lim_{x_i \to x} \prod_{i=1} \phi(x_i)\right\} \qquad (20)$$

Evidently this definition contains a great deal of ambiguity since an arbitrary finite counterterm may be added to its R.H.S. Of course, there exist simple regularizations such as those given by one parameter families of regulators N_μ; these regulators define the Wick normal ordering with respect to Hamiltonian of a free Bose field of mass μ. However, there is no reason within the scheme discussed so far to prefer one form of regularization over the other.

Apparently the theory (3) is equivalent to the regularized form of (19) with a given parameter Λ and particular set of counterterms all being fixed in terms of the coupling g and the quark mass m.

The correct regularization of $\mathcal{H}_0$ is trivial. Indeed the relation $:\bar{\psi}^i\psi_i: = c\,\Lambda\, N_\Lambda(\cos 2\sqrt{\pi}\,\phi_i)$ obtained in the free Dirac theory [6], remains unaffected by gauge interactions. Note that the quark mass term is actually independent of Λ.

Here I will not try to identify the counterterms to $\mathscr{H}_1$. Instead I will turn to qualitative predictions of the model (19) when the explicit knowledge of the counterterms is unnecessary.

Two extreme cases of the weak ($g^2 \ll m^2$) and strong ($g^2 \gg m^2$) coupling regimes will be analyzed. I will begin with the first case. Here the scale parameter Λ may be taken $\Lambda \sim g$ since the interaction energy (19b) is expected to be of the order of g^2.

Equation (19a) represents a direct sum of N Sine-Gordon Hamiltonians $H_{S-G}(\beta)$ with the mysterious small parameter $\beta = 2\sqrt{\pi}$. The spectrum of $H_{S-G}(2\sqrt{\pi})$ is known to consist of (anti)quarks which in the leading order in β are described by (anti)solitons [8,13].

Quantum fluctuations about multisoliton solutions of the Hamiltonian (19a) define a systematic β-expansion. I will only determine the energy of multiquark systems to the leading order in $(g/m)^2$ and β. For simplicity the (anti)soliton will be approximated by

$$s_1^{(*)}\left(x|x_0\right) = \overset{+}{(-)} \sqrt{\pi}\ \theta\left(x - x_0\right) \tag{21}$$

Then multisolitons may be described by a simple superposition of (21). In this approximation a soliton-antisoliton pair becomes

$$s_2\left(x|x_{1,2}\right) = \pm \sqrt{\pi}\ \theta\left(x_2 - x\right)\ \theta\left(x - x_1\right), \quad m\left|x_1 - x_2\right| \gg 1 \qquad 22)$$

Actually, eqs. (21) and (22) are quite accurate; although one expects them to fail in the vicinity of the end points $x = x_i$ where quark self energies are localized. Note that in eq. (22) I have ignored a weak dependence on the realtive velocity of the soliton-antisoliton pair.

Equation (19b) permits two classes of finite energy configurations

$$\phi_i^{(1)}(x) = \sum_{k=1}^{n_i} \left\{ S_1\left(x|x_k\right) + S_1^*\left(x|x_k\right) \right\}, \quad 1, \ldots, N \tag{23a}$$

$$\phi_i^{(2)}(x) = S_1\left(x|x_i\right) + \phi_i^{(1)}(x), \quad i = 1, \ldots, N \tag{23b}$$

where $\left\{n_i\right\}$ is an arbitrary set of integers. Observe that in these configurations the electric field vanishes at $x = \pm\infty$ (cf. eqs. (6) and (18)) and therefore the corresponding states are colorless. This immediately follows from Gauss' law (2a) and the definition of the global color charge $Q_i^j(t) = \int J_i^j(x,t)dx$.

The configurations (23a) and (23b) describe mesons (M) and baryons (B) respectively. As $g \ll m$, lowest energy M and B states are determined by the simplest configurations

$$M: \quad \phi_i(x) = S_2\left(x|x_{1.2}\right), \quad \phi_{j\neq i} = 0 \tag{24a}$$

$$B: \quad \phi_i(x) = S_1\left(x|x_i\right) \tag{24b}$$

By means of eqs. (19b), (21), (22) and (24) one directly evaluates the interaction energy stored in the M and B states

$$M: \quad V(x_{1,2}) = \left((g^2/4N) + 2\sqrt{\pi}\,\Lambda^2\right)(N-1)\,|x_1 - x_2| \tag{25a}$$

$$B: \quad V(x_{1\ldots N}) = \left((g^2/4N) + 2\sqrt{\pi}\,\Lambda^2\right)\sum_{i\neq j} |x_i - x_j| \tag{25b}$$

These expressions should not come as a surprise since for $\Lambda^2 = g^2/8\sqrt{\pi}$ they simply reproduce the potential in the color singlet sector of the theory (3). However the result (25) has an important virtue, i.e., it is amenable to an elegant interpretation. Indeed, the stringlike field configurations (24a) are just electric vortex lines between points x_1 and x_2. Hence the linear spectrum of the potential (25a) (see, e.g., ref.[5]) representa longitudinal modes of the uniform electric strings labeled by a dummy index. Similarly baryons according to (24b) represent N different electric strings converging at infinity. This interpretation is analogous to that of the Bose form of the massive Schwinger model. It is gratifying that one is able to describe QED_2 as well as QCD_2 in terms of electric strings.

It should be emphasized that the simple string picture has been unraveled for $g^2 \ll m^2$ in the leading order in β, i.e., when quarks are being treated semiclassically.

As one enters the moderate coupling regime $g \leq m$, the β-expansion apparently breaks down since the first term $\mathscr{H}_1^{(1)}$ in (19b) becomes $\mathscr{O}(1/\beta^4)$ formally dominating over $\mathscr{H}_0 = \mathscr{O}(1/\beta^2)$. In the large N-limit, when $g^2 N = \text{const}$ [14] the β-expansion may be regained since $\mathscr{H}_1^{(1)} \sim (1/N) \to 0$ and $\mathscr{H}_1 = \mathscr{O}(N^0)$ on the basis of the above formula $\Lambda^2 = g^2/8\sqrt{\pi}$. Now however various configurations (23), that describe in general non-uniform electric strings, should be treated on equal footing with (24).

Finally I turn to the strong coupling regime. For $g^2 \gg m^2$ the potential $\mathscr{H}_1 + m\Lambda \sum_i (1 - \cos 2\sqrt{\pi}\,\phi_i)$ has an absolute minimum at $\phi_i = 0$. Small oscillations about the minimum have N modes given by $\phi_D = \sum_i (\lambda^D)_i^i\, \phi_i$ where the matrices λ^D, $D = 1, \ldots, N$ are diagonal and orthogonal $2\mathrm{Tr}(\lambda^A \lambda^B) = \delta^{AB}$. The fields $\{\phi_D\}$ represent one scalar with a mass square $M_0^2 \sim m\Lambda$ and a multiplet of $N-1$ scalars of equal mass M. Note that the former decouples for $m=0$. All these particles are colorless since they are sources of an electric field which vanishes at infinity as $\exp(-M|x|)$.

It is easy to verify that due to interaction terms the permutation symmetry Z_N undergoes a complete spontaneous breakdown. Interaction couplings being $\sim\Lambda$ can be as large as M. By means of a nonrelativistic reasoning—strictly valid in the weak coupling regime $\Lambda \ll M$—one may gain an idea about the spectrum of bound states for $\Lambda \lesssim M$. Repeating Coleman's analysis of the massive Schwinger model [5] in this case, one finds n-body bound states with unequal masses corresponding to various combinations of ϕ_D's.

The above discussion suggests that QCD_2 describes two distinct phases: the quark ($g \ll m$) and Bose ($g \gg m$) phases with very different spectrum of excitations and the broken Z_N symmetry in the latter case [15].

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11. According to Dirac's terminology the Hamiltonian (1) describes a system with first class constraints $\phi^a = G^i_j (\lambda^a)^j_i = 0, a=1,\ldots,N^2-1 \equiv M$. For such systems Poisson brackets $\{A_1, A_2\}$ of gauge invariant quantities $A_{1,2}(p,q)$ defined in the canonical basis $\{p_a, q_a, a=1,\ldots, K+M\}$ can be truncated, i.e., $\{A_1, A_2\} = \sum_{a=M+1}^{K+M} \left[(\partial\tilde{A}_1/\partial p_a)(\partial\tilde{A}_2/\partial q_a) - (1 \leftrightarrow 2)\right]$, where $\tilde{A}_{1(2)}$ is obtained from $A_{1(2)}$ by eliminating q_a's in favor of $\phi^a = \phi^a(p,q), a=1,\ldots,M$. In other words the variables $\{p_a, q_a, a=M+1,\ldots,K+M\}$ form a canonical basis; in the text they have been identified with quark fields $\overset{*}{\psi}{}^i(x), \psi_i(x)$. Remaining variables $\{p_a, a=1,\ldots,M\}$ may be specified arbitrarily. The gauge (4) corresponds to the choice $\{p_a=0;\ a=1,\ldots,M\}$.

12. The parameters $\left\{e_i^0\right\}$ may be interpreted as a background electric field induced by charges on the boundaries, i.e., at $x = \pm\ \infty$. In the case the gauge group is simple these charges are quantized. Therefore, they are annihilated by the vacuum polarization when quarks are in the fundamental representation. These arguments for the massive Schwinger model have been advanced by Coleman [5] and extended to QCD_2 by E. Witten, Harvard preprint, HUTP-78/A058.

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