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RADIATION SPECTRA AND ANGULAR DISTRIBUTION OF EMITTED QUANTA FOR PLANAR CHANNELLED PARTICLES: RADIATION OF SINGLE PARTICLE *

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ABSTRACT

A method of calculating the radiation characteristics for the motion in an arbitrary one-dimensional potential is developed. Expressions for channelling radiation frequencies, polarization angles, and the number of emitted photons as functions of quanta angles, particle energy, amplitude of oscillations, and divergence in a plane parallel to the trapping crystal planes for any given harmonic number are found.

The problem is treated in the classical approximation. Numerical examples of the application of the derived formulae for channelling in the (1,1,0) direction of a Silicon crystal are given. The results are presented both for electrons and positrons. Comparison of the calculation results for two choices of continuum potential for positrons is also given.

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I. Introduction

Radiation of channelled particles trapped between planes of a crystal is a powerful tunable source of polarized radiation in an interesting range of frequencies. It can also be used to study the properties and characteristics of the crystal itself. For both applications of the phenomena, one needs knowledge of the radiation spectra and angular distribution of emitted quanta. All the known theoretical results[1-5] for these radiation characteristics are obtained in the linear approximation, where only the first (quadratic in coordinate) term of the power series expansion of the potential in which the particle moves is retained and considered. In the work of Pantell and Alguard[6] the next term (proportional to the fourth power of coordinate) is also taken into account using first order perturbation theory to get anharmonic corrections. In this way the authors were able to find the satellite lines (due to small anharmonicity) accompanying the main one and also to estimate the rate of radiation on the third harmonic. This rate appears to be very small as a direct consequence of the assumption of anharmonicity smallness.

In our own work[7] a method was developed which permits one to calculate the average radiation intensity for general nonlinear motion. This approach is extended now in this work to allow us to find radiation characteristics for the motion in an arbitrary one-dimensional potential. By using this method one can calculate all the characteristics of the channeling radiation without any assumption on the value of anharmonicity. In particular, expressions for radiation frequencies, polarization angles, and the number of emitted photons as functions of quanta angles, particle energy, amplitude of oscillations, and divergence in a plane parallel to the trapping crystal planes for any given harmonic number are found. For planar channeling, the motion of the particle is essentially two-dimensional. Hence, it is possible to simplify the calculations by choosing the appropriate coordinate frame. That, however, will make the effective potential different for different motion planes. To avoid this and keeping in mind the necessity to average the radiation over the divergence in the particle beam, we tie our coordinate frame to the axes of the crystal rather than to the particle trajectory.

The problem is treated in the classical approximation, the validity of which can be found elsewhere.[8] In Section II we solve the equations of motion in a given potential. Section III is devoted to calculations of intensity of radiation and number of emitted photons per interval of solid angle. In Section IV we apply our results to the parabolic potential to show how the derived formulae turn into known ones in this case. In Section V one can find a useful formula for the polarization angle of the emitted photon. Numerical results for electrons and positrons as well as comparison of results for different potentials for positrons can be found in Section VI.

II. Motion of Channelled Particle

Let a relativistic particle with energy E be trapped between the planes of a crystal. We choose a coordinate frame in which the crystal planes are parallel to the yz plane. The particle is assumed to have large relative velocity component β_z along the z axis, ($\beta_z \sim 1$), the other two components being small:

$$\beta_x << 1 \tag{2.1}$$

$$\beta_y << 1 \tag{2.2}$$

The length of one "oscillation" of the relativistic channelled particle, as we shall show, is much longer than any lattice periods. Consequently, the force exerted on the channelled particle can be derived from a time independent one-dimensional potential V(x) (for the "oscillations" occurring in the x-z plane) which is the planar average (over y and z) of the true electrostatic potential V(x,y,z) within the lattice. We choose further for convenience $V(0) = 0$.

The equations of motion in this case:

$$dp_x \,/\, dt = - dV \,/\, dx \quad , \tag{2.3}$$

$$dp_y \,/\, dt = 0 \quad , \tag{2.4}$$

$$dp_z \,/\, dt = 0 \quad , \tag{2.5}$$

give immediately p_y = constant, p_z = constant. After neglecting terms of the order of magnitude $(V/E)^{3/2}$, the first integral of equation (2.3) gives

$$\beta_x = \sqrt{2\left[V(x_m) - V(x)\right]/E} \quad , \tag{2.6}$$

where x_m is the value of x at the point $\beta_x = 0$. x_m is the maximum excursion of the particle from the plane $x = 0$ and we call this quantity the "amplitude" of (nonlinear) oscillations.

From Equation (2.4) we get $\beta_y = p_y(c/E)\left(1+(V/E)\right)$ or, since β_y is small, approximately

$$\beta_y = \frac{p_y c}{E} = \text{const.} \tag{2.7}$$

with the same accuracy ($\gamma = E/mc^2$)

$$\beta_z = \bar{\beta}_z + \langle (V_m - V_x)/E \rangle - (V_m - V_x)/E \quad , \tag{2.8}$$

$$\bar{\beta}_z = 1 - \tfrac{1}{2}\gamma^2 - \beta_y^2/2 - \langle (V_m - V_x)/E \rangle \quad , \tag{2.9}$$

where brackets <> mean taking the average over time. To make the formulae less cumbersome we use the following abbreviations here and for the rest of the paper:

$$V_x = V(x), \qquad V_m = V(x_m)$$

It is easy to verify that the full energy of the particle

$$E = mc^2 \Big/ \sqrt{1 - (\beta_z^2 + \beta_y^2 + \beta_x^2)} + V(x) \tag{2.10}$$

is constant with accuracy V^2/E^2 for β_z, β_x and β_y from (2.6 - 2.9).

From (2.6 - 2.9) we find the following solution for the particle trajectory:

$$ct = ct_0 + \int_0^x dw \Big/ \sqrt{2(V_m - V_w)/E} \quad , \tag{2.11}$$

$$z = z_0 + ct\bar{\beta}_z + \frac{\langle V_m - V_x \rangle}{\sqrt{2}\ E} \int_0^x \frac{dw}{\sqrt{(V_m - V_w)/E}} - \frac{1}{\sqrt{2}} \int_0^x \sqrt{(V_m - V_w)/E}\ dw \, , \tag{2.12}$$

$$y = y_0 + \beta_y ct \quad , \tag{2.13}$$

where z_0, y_0 and t_0 are the initial arbitrary constants.

From (2.11) we can define the "frequency" of oscillations $\Omega = 2\pi/\oint dw/\sqrt{2c^2(V_m - V_w)/E}$, where the sign $\oint$ means integration over the full period of oscillation. For the case of potential symmetric in $x(V_{-x} = V_x)$ we get:

$$\Omega = \pi c \Big/ 2\int_0^{x_m} dx \Big/ \sqrt{2(V_m - V_x)/E} \quad . \tag{2.14}$$

III. Radiation Characteristics

We start the calculation of the intensity of radiation with the expression for the Fourier harmonic of the vector-potential of the radiation field:[9]

$$\vec{A}_\omega = \frac{e^{iKR}}{2\pi R}\int_{-\infty}^{\infty} dt\ \vec{\beta}(t)\ \exp(i[\omega t - \vec{K}\cdot\vec{r}(t)]) \tag{3.1}$$

Here $\vec{K}$ is the wave vector of radiation, R is the distance to the observation point, $\vec{\beta}(t)$ and $\vec{r}(t)$ are the relative velocity and radius vector of the particle with a charge e at time t. Calling θ the azimuthal angle of the radiation direction with the z axis and φ the polar angle between its projection on the x-y plane and the x axis, the vector $\vec{K}$ has the following components:

$$K_{x,y,z} = K\ n_{x,y,z} = \frac{\omega}{c}\,(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta) \tag{3.2}$$

Now we substitute $\vec{\beta}(t)$ and $\vec{r}(t)$ from (2.6-2.9) and (2.11-2.13) into (3.1):

$$\vec{A}_\omega = e\,\frac{e^{i\frac{\omega}{c}R}}{2\ R}\int_{-\infty}^{\infty} dt\ \vec{a}(t)\ \exp\,[i\omega(1 - \bar{\beta}_z\cos\theta - \beta_y\sin\theta\cos\varphi)t]\ , \tag{3.3}$$

where inside the crystal with thickness L, the function $\vec{a}(t)$ is periodic with period $2\pi/\Omega$ and zero elsewhere:

$$\vec{a}(t) = \begin{cases} \vec{\beta}(t) \exp\left\{-i\frac{\omega}{c}\left[x(t)\sin\theta\cos\varphi + \delta z(t)\cos\theta\right]\right\}, & \text{if } |t| \le L/2\bar{\beta}_z c \\ 0, & \text{if } |t| > L/2\bar{\beta}_z c \end{cases} \quad (3.4)$$

Here we denote

$$\delta z(t) = \frac{\langle V_m - V_x \rangle}{\sqrt{2}\;E} \int_0^{x(t)} \frac{dw}{\sqrt{(V_m - V_w)/E}} - \frac{1}{\sqrt{2}} \int_0^{x(t)} \sqrt{(V_m - V_w)/E}\; dw\;. \quad (3.5)$$

Let us now expand $\vec{a}(t)$ in a Fourier series on the interval $|t| \le L/2\bar{\beta}_z c$:[10]

$$\vec{a}(t) = \sum_{k=-\infty}^{\infty} \vec{a}_k \exp(-ik\Omega t),\; \vec{a}_k = \frac{\Omega}{2\pi} \int_0^{2\pi/\Omega} dt\; \vec{a}(t) \exp(ik\Omega t) \quad (3.6)$$

Using this expansion we take the integral in (3.3) to find the spectral distribution of the energy radiated by the particle in a given direction (θ,φ) in an interval of solid angle do:

$$\frac{d\varepsilon}{d\omega do} = \frac{R^2\omega^2}{c}\left|\vec{n}\times(\vec{n}\times\vec{A}_\omega)\right|^2 = \frac{\omega^2 e^2}{\pi^2 c}\left|\sum_k \vec{n}\times(\vec{n}\times\vec{a}_k)\,\frac{\sin\pi N\sigma_k}{\Omega\sigma_k}\right|^2, \quad (3.7)$$

where

$$\sigma_k = \frac{\omega}{\Omega}\left(1 - \bar{\beta}_z\cos\theta - \beta_y\sin\theta\sin\varphi\right) - k \quad (3.8)$$

and $N = \Omega L/2\pi\bar{\beta}_z c$ is the number of oscillations over the length L. It is seen from (3.7) that for $N \gg 1$ the radiation occurs in the form of a line spectrum.

The center of the line is positioned at the frequency:

$$\omega_k = \frac{k\,\Omega}{1 - \bar{\beta}_z \cos\theta - \beta_y \sin\theta \sin\varphi} \quad . \tag{3.9}$$

Its half-width is

$$\Delta\omega_k / \omega_k \simeq 1/N\,k \tag{3.10}$$

With increasing N, more energy is radiated in decreasing frequency interval, so as $N \to \infty$ we get the line spectrum with the line frequencies ω_k and the spectral intensity of the radiation:

$$\frac{dI}{d\omega do} = \lim_{N\to\infty} \frac{\Omega}{2\pi N} \frac{d\varepsilon}{d\omega do} = \sum_{k=1}^{\infty} \frac{dI_k}{do} \; \delta(\omega - \omega_k) \quad , \tag{3.11}$$

where the intensity of the radiation of the k^{th} harmonic is:

$$\frac{dI_k}{do} = \frac{e^2 \omega_k^3}{2\pi c \Omega k} \left| \vec{n} \times (\vec{n} \times \vec{a}_k) \right|^2 \quad . \tag{3.12}$$

To evaluate $\vec{a}_k$ we change the integration over t in (3.5) to integration over x:

$$\begin{aligned} a_{kx} = \frac{i^k \Omega}{\pi c} \Bigg\{ & \int_0^{x_m} dx \sin\left(\int_x^{x_m} Q(w)\,dw \right) \exp\left(-ix \frac{\omega_k}{c} \sin\theta \cos\varphi\right) \\ & - (-1)^k \int_{-x_m}^{0} dx \sin\left(\int_{-x_m}^{x} Q(w)\,dw \right) \exp\left(-ix \frac{\omega_k}{c} \sin\theta \cos\varphi\right) \Bigg\} \quad , \end{aligned} \tag{3.13}$$

$$a_{ky} = \frac{i^{k+1}\Omega}{\pi c} \beta_y \left\{ \int_0^{x} \frac{dx}{\sqrt{(2/E)(V_m - V_x)}} \cos\left(\int_0^{x_m} Q(w)\,dw \right) \exp\left(- ix\frac{\omega_k}{c}\sin\theta\,\cos\varphi\right) \right.$$

$$\left. + (-1)^k \int_{-x_m}^{0} \frac{dx}{\sqrt{(2/E)(V_m - V_x)}} \cos\left(\int_{-x_m}^{0} Q(w)\,dw \right) \exp\left(- ix\frac{\omega_k}{c}\sin\theta\,\cos\varphi\right) \right\}. \tag{3.14}$$

The following notation is used in these formulae:

$$Q(w) = \frac{k\Omega - \omega_k \cos\theta \, \langle (V_m - V_x)/E \rangle}{c\sqrt{2(V_m - V_w)/E}} + \frac{\omega_k}{c}\cos\theta \sqrt{(V_m - V_w)/2E} \quad . \tag{3.15}$$

The expression for a_{kz} is the same as for a_{ky} with the change of β_y to β_z. The formulae for $\vec{a}_k$ simplify for a symmetric potential $V_{-x} = V_x$. In this case:

$$a_{kx} = \frac{2i^k\Omega}{\pi c} \int_0^{x_m} dx \, \sin\left(\int_x^{x_m} Q(w)\,dw \right) \begin{cases} -i\sin\left(x\frac{\omega_k}{c}\sin\theta\,\cos\varphi\right) & \text{if } k = 2p \\ \cos\left(x\frac{\omega_k}{c}\sin\theta\,\cos\varphi\right) & \text{if } k = 2p+1 \end{cases} , \tag{3.16}$$

$$a_{ky} = \frac{2i^{k+1}\Omega}{\pi c} \beta_y \int_0^{x_m} \frac{dx}{\sqrt{(2/E)(V_m - V_x)}} \cos\left(\int_x^{x_m} Q(w)\,dw \right)$$

$$\times \begin{cases} \cos\left(x\frac{\omega_k}{c}\sin\theta\,\cos\varphi\right) & \text{if } k = 2p \\ -i\sin\left(x\frac{\omega_k}{c}\sin\theta\,\cos\varphi\right) & \text{if } k = 2p+1 \end{cases} . \tag{3.17}$$

We now use conditions (2.1, 2.2) and consider the angular range $\theta \ll 1$, into which the main part of the radiation of the ultrarelativistic particle ($\gamma \gg 1$) goes. Equation (3.9) in this case gives:

$$\omega_k = \frac{\pi c \gamma k B}{x_m T_1 (1 + rB^2 + D^2 + \Theta^2 - 2\Theta D \sin\varphi)} , \tag{3.18}$$

where we introduce the following useful variables:

$$B = \sqrt{2V_m \gamma / mc^2} , \tag{3.19}$$

$$D = \beta_y \gamma , \tag{3.20}$$

$$\Theta = \theta\gamma , \tag{3.21}$$

and the notation

$$r = S_1/T_1 . \tag{3.22}$$

The radiation frequency ω_k (3.18) appears to be proportional to $\gamma^{3/2}$ for $B \ll 1$ and proportional to $\gamma^{1/2}$ for $B \gg 1$.

The definitions of T_1 and S_1 are shown below; see Equations (3.30) and (3.31), where one should substitute $w = 1$. In these variables the number of photons emitted on the k^{th} harmonic from the crystal length L equals:

$$\frac{dN_k}{\Theta d\Theta d\varphi} = \frac{k L B F_k}{137\gamma x_m T_1^3 (1 + rB^2 + D^2 + \Theta^2 - 2\Theta D \sin\varphi)^2} . \tag{3.23}$$

We neglect here all the terms proportional to the small factor $1/\gamma^2$. In expression (3.23) the following notation is used:

$$F_k = B^2 \Phi_{kx}^2 + (\Theta^2 + D^2 - 2\Theta D \sin\varphi)\, \Phi_{ky}^2 - 2B\Theta\Phi_{kx}\Phi_{ky} \cos\varphi , \tag{3.24}$$

where

$$\Phi_{kx} = \int_0^1 dw \left\{ \begin{matrix} \cos \\ \sin \end{matrix} \left(k\pi R_w/2\right) \times \begin{matrix} \cos \\ \sin \end{matrix} \left(k\pi f_1 w/2T_1\right) \right\} \begin{matrix} \text{if } k = 2p+1 \\ \text{if } k = 2p \end{matrix} \tag{3.25}$$

$$\Phi_{ky} = \int_0^1 \frac{dw}{\sqrt{1-V_{wx_m}/V_m}} \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \left(k\pi R_w/2\right) \times \begin{matrix} \sin \\ \cos \end{matrix} \left(k\pi f_1 w/2T_1\right) \right\} \begin{matrix} \text{if } k = 2p+1 \\ \text{if } k = 2p \end{matrix} \tag{3.26}$$

$$R_w = (1-f_2)\, T_w/T_1 + f_2\, S_w/S_1 \quad , \tag{3.27}$$

$$f_1 = 2B\Theta \cos\varphi \big/ (1 + rB^2 + D^2 + \Theta^2 - 2\Theta D \sin\varphi) \quad , \tag{3.28}$$

$$f_2 = rB^2 \big/ (1 + rB^2 + D^2 + \Theta^2 - 2\Theta D \sin\varphi) \quad , \tag{3.29}$$

$$T_w = \int_0^w d\tau \Big/ \sqrt{1-V_{\tau x_m}/V_m} \tag{3.30}$$

and

$$S_w = \int_0^w d\tau \sqrt{1-V_{\tau x_m}/V_m} \tag{3.31}$$

In the forward direction ($\Theta = 0$) $f_1 = 0$. Hence, in this case $\Phi_{2px} = 0$, and a particle moving in the $x-z$ plane ($D = 0$) radiates only odd harmonics.

IV. Parabolic Potential

It is useful to see how our results turn into the known ones for the particular case of the potential function:

$$V_x = k_1 x^2 / 2 \tag{4.1}$$

In this case a particle executes linear oscillations along the x- and z-axes. From (2.11) and (2.12) we have:

$$x = x_m \sin\Omega t \tag{4.2}$$

$$\delta z = -(\Omega x_m^2/8c) \sin 2\Omega t \tag{4,3}$$

Formula (2.12) gives

$$\Omega = (c/x_m)\sqrt{2V_m/E} \tag{4.4}$$

Since the maximum value of x_m for a channeling particle is of the order of the distance between crystal planes, we see that the wave length of particle oscillations is much longer than the characteristic spatial periods of the crystal itself.

Formulae (3.30) and (3.31) give:

$$T_w = \arcsin w \quad , \tag{4.5}$$

$$S_w = \tfrac{1}{2}(\arcsin w + w\sqrt{1-w^2}) \quad . \tag{4.6}$$

and in particular $T_1 = \pi/2$, $S_1 = \pi/4$ $(r = 1/2)$. For the values Φ_{kx} and Φ_{ky} after some algebra one gets

$$\Phi_{kx} = \frac{\pi}{2f_1} \sum_{\ell=-\infty}^{\infty} \left(1 + \frac{2\ell}{k}\right) J_\ell(kf_2/2)\, J_{k+2\ell}(kf_1) \quad , \tag{4.7}$$

$$\Phi_{ky} = \frac{\pi}{2} \sum_{\ell=-\infty}^{\infty} J_\ell(kf_2/2)\, J_{k+2\ell}(kf_1) \quad , \tag{4.8}$$

J_k being the Bessel function of the first kind.

The spectral intensity and frequency of the k^{th} harmonic of radiation in the case of linear oscillations coincides with the results of the work of Alferov, Bashmakov and Bessonov[10] if we put $D = 0$ (the case considered in that paper).

For numerical calculations of the radiation our formulae (3.25, 3.26) have an advantage over formulae (4.7, 4.8) even for linear oscillations, since the computation by means of integrations in the first ones is much more accurate and faster than that by means of summations of the Bessel functions of high order.

V. Polarization of the Radiation

To determine the polarization of the radiation of the k^{th} harmonic, we define two auxiliary complex quantities:

$$a_{k\xi} = \frac{i\omega}{c} \vec{a}_k \cdot (\vec{n} \times \vec{k}) \tag{5.1}$$

and

$$a_{k\rho} = \frac{i\omega}{c} \vec{a}_k \cdot \{\vec{n} \times (\vec{n} \times \vec{k})\} \tag{5.2}$$

where $\vec{k}$ and $\vec{n}$ are unit vectors along the z-axis and the radiation direction respectively. The polarization ellipse is defined[11] by its semiaxes

$$q_{1,2} = \sqrt{|a_{k\xi}|^2 + |a_{k\rho}|^2 + 2|a_{k\xi}|\,|a_{k\rho}|\sin\delta} \pm \sqrt{|a_{k\xi}|^2 + |a_{k\rho}|^2 - 2|a_{k\xi}|\,|a_{k\rho}|\sin\delta} \tag{5.3}$$

and by the angle ψ between the axis q_1 and the vector $\vec{n}\times\vec{k}$:

$$\tan 2\psi = \frac{2\,|a_{k\xi}|\,|a_{k\rho}|\cos\delta}{|a_{k\xi}|^2 - |a_{k\rho}|^2} \quad . \tag{5.4}$$

Angle δ is the phase difference between $a_{k\xi}$ and $a_{k\rho}$ and is determined by the following expression:

$$\exp(i\delta) = \frac{a_{k\rho}\,|a_{k\xi}|}{|a_{k\rho}|\,a_{k\xi}} \quad . \tag{5.5}$$

For planar channeling $\delta = 0$ and the radiation is linearly polarized ($q_2 = 0$). The angle α_k between the direction of polarization and the x axis can be found from the following expression:

$$\tan 2\alpha = \frac{2PQ\cos 2\varphi + (P^2 - Q^2)\sin 2\varphi}{2PQ\sin 2\varphi - (P^2 - Q^2)\cos 2\varphi} \tag{5.6}$$

where (for the range $\theta << 1$):

$$P = B\,\Phi_{kx}\sin\varphi - D\,\Phi_{ky}\cos\varphi \quad , \tag{5.7}$$

$$Q = B\,\Phi_{kx}\cos\varphi + (D\sin\varphi - \Theta)\,\Phi_{ky} \quad . \tag{5.8}$$

VI. Numerical Examples and Comparison of Different Potentials

We give here examples of the application of the derived formulae for channeling in the (1,1,0) direction of a Silicon (Si) crystal. The results are presented for both electrons and positrons. The comparison of the calculated results for two choices of continuum potential $V(x)$ for positrons is also given.

Summing up the contributions of adjacent crystal planes, we get the following expressions for the continuum potential of Lindhard[12] or Molière[13] at the distance x from the crystal plane:

$$V_L^e(u)=V_0\left(b+u-\sqrt{u^2+b^2}+\sum_{p=1}^{P}\left\{2\sqrt{b^2+4p^2}-\sqrt{b^2+(2p-u)^2}-\sqrt{b^2+(2p+u)^2}\right\}\right),$$

$$P\to\infty \tag{6.1}$$

$$V_M^e(u)=V_0\sum_{i=1}^{3}\frac{\alpha_i}{\beta_i}\;\frac{\sinh\,(\beta_i u)\,\sinh\,(2\beta_i-\beta_i u)}{\sinh\,(2\beta_i)}\quad, \tag{6.2}$$

where $u=2x/d$, $V_0=\pi Ze^2nd^2$, $b=2\sqrt{3}\,a/d$, $\alpha_i=(0.1,\ 0.55,\ 0.35)$, $\beta_i=(14.82,\ 2.964,\ 0.741)$; n is the number of atoms with atomic number Z per unit volume, d is the distance between crystal planes and a is the screening length of the electron-atom interaction for the Thomas-Fermi atom model. Expressions (6.1) and (6.2) are valid in the region $0<u<1$. For other values of u, one can use the relations $V(-u)=V(u)$ and $V(u+2p)=V(u)$, $p=0,\ \pm1,\ \pm2,\ \ldots$. For the (1,1,0) direction in Si these constants have the following values: $V_0=117$ eV ($Z=14$, $a=0.194$ Å, $d=1.920$ Å, $n=4.994\times10^{22}\ \mathrm{cm}^{-3}$, $b=0.350$). Practically speaking, only the first few terms of the sum in expression (6.1) contribute for any given value of u.

In Figure 1 we present the functions $V_L^e(u)\big/V_0$ and $V_M^e(u)\big/V_0$ for P=5 (that corresponds to summing over eleven nearest crystal planes).

Figures 2 and 3 present as functions of u the frequency Ω (2.14) of channelled electron oscillations and the maximum angle $B = \beta_m \gamma$ (3.19) of the electron trajectory with z-axis for $\gamma = 4.5 \times 10^3$.

All the functions were recalculated also with $P = 10$ with practically identical results. All the following calculations are performed with $P = 5$.

Formulae analogous to (6.1) and (6.2) for a positron are the following:

$$V_L^P(u) = V_0 \sum_{p=1}^{P} \left\{ \sqrt{(2p-1-u)^2 + b^2} + \sqrt{(2p-1+u)^2 + b^2} - 2\sqrt{(2p-1)^2 + b^2} \right\}, \quad P \to \infty \quad , \tag{6.3}$$

and

$$V_M^P(u) = V_0 \sum_{i=1}^{3} \frac{\alpha_i \sinh^2(\beta_i u)}{\beta_i \sinh(2\beta_i)} \quad , \tag{6.4}$$

with the same constants as in (6.1) and (6.2).

In Figure 1 the potentials $V_L^P(u)/V_0$ and $V_M^P(u)/V_0$ are also plotted. The oscillation frequency Ω and the maximum trajectory angle B with the z-axis of a channelled positron are drawn in Figures 2 and 3, respectively.

Figures 4(7) give the angular dependence of several first harmonics of radiation spectra for an electron (positron). Plotted is the number of emitted quanta per interval of solid angle $dN_k/\Theta d\Theta d\varphi$ (3.23) as a function of the emission angle $\Theta = \theta\gamma$ (3.21). The crystal thickness L was assumed to be equal to 0.1 cm in all the calculations. In Figure 7 we also present comparison of the spectra for two choices of continuum potential. Figures 5(8) and 6(9) give the spectra of the first harmonic for different values of the oscillation amplitude u, divergence angle

$D = \beta_y\gamma$ (3.20) of the electron (positron), polar angle φ of the radiation and Lorentz factor γ of the particle.

It is interesting to note that the angular spectra for the directions near the x - z plane ($\varphi = 0$ and $\varphi = \pi$) go to zero at the values of Θ defined by equation $F_k(B, \Theta, \varphi) = 0$ $\{$cf., (3.24)$\}$.

VIII. Conclusion

The method of calculating the frequency and angular spectra of channelling radiation developed here gives us the possibility of finding all the characteristics of the phenomenon for any single particle and quanta parameters. These results can further be used to obtain spectra averaged over the particle distribution in transverse phase space of a beam for any given geometry of experiment.

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Figure Captions

1. Continuum potential of the Si crystal for a channelled particle as a function of $u = 2x_m/d (0 \le u \le 1)$: 1(2) - Lindhard (Molière) potential for electron (right scale): 3(4) - Lindhard (Molière) potential for positron (left scale).

2. Oscillation frequency Ω (2.14) of a channelled motion in units 10^{15} 2π/sec. Notations are the same as in Figure 1. $\gamma = 4.5 \times 10^3$.

3. Maximum oscillation angle B (3.19). Notations are the same as in Figure 1. $\gamma = 4.5 \times 10^3$.

4. Angular ($\Theta = \theta\gamma$) dependence of several first harmonics of the radiation of electron with amplitude $u = 0.1$ in (1,1,0) direction of Si crystal 0.1 cm thick. $\gamma = 4.5 \times 10^3$; (1) $k = 1$, (2) $k = 2$, (3) $k = 3$, et cetera.

5. The angular dependence of the first harmonic ($k = 1$) of electron radiation for amplitude $u = 0.1$ and different divergence angle D (3.20) and polar angle φ:

(1) $D = 0.0$, $\varphi = 0$ and π

(2) $D = 0.0$, $\varphi = \pi/2$ and $3\pi/2$

(3) $D = 0.5$, $\varphi = 0$ and π

(4) $D = 0.5$, $\varphi = \pi/2$

(5) $D = 0.5$, $\varphi = 3\pi/2$

(6) $D = 1.0$, $\varphi = 0$ and π

(7) $D = 1.0$, $\varphi = \pi/2$

(8) $D = 1.0$, $\varphi = 3\pi/2$

Lindhard potential, $\gamma = 4.5 \times 10^3$

6. The dependence of the first harmonic ($k=1$) of the channelled radiation on the emission angle Θ for different amplitudes u of the electron trajectory and different Lorentz factors γ for $\varphi=0$:

(1)	$u=0.1$,	$\gamma=4.5\times10^2$
(2)	$u=0.1$,	$\gamma=4.5\times10^3$
(3)	$u=0.1$,	$\gamma=4.5\times10^4$
(4)	$u=0.5$,	$\gamma=4.5\times10^2$
(5)	$u=0.5$,	$\gamma=4.5\times10^3$
(6)	$u=0.5$,	$\gamma=4.5\times10^4$

Lindhard potential.

7. Angular ($\Theta=\theta\gamma$) dependence of several first harmonics of the radiation of positron with amplitude $u=0.9$ in (1,1,0) direction of Si crystal 0.1 cm thick. $\gamma=4.5\times10^3$; (1) $k=1$, (2) $k=2$, (3) $k=3$, et cetera. Also presented is the comparison of radiation spectra calculated for Moliére (solid lines) and Lindhard (dashed lines) potentials.

8. The angular dependence of the first harmonic ($k=1$) of positron radiation for amplitude $u=0.9$ and different divergence angle D (3.20) and polar angle φ:

(1)	$D=0.0$,	$\varphi=0$ and π
(2)	$D=0.0$,	$\varphi=\pi/2$ and $3\pi/2$
(3)	$D=0.5$,	$\varphi=0$ and π
(4)	$D=0.5$,	$\varphi=\pi/2$
(5)	$D=0.5$,	$\varphi=3\pi/2$
(6)	$D=1.0$,	$\varphi=0$ and π
(7)	$D=1.0$,	$\varphi=\pi/2$
(8)	$D=1.0$,	$\varphi=3\pi/2$

Moliére potential.

9. The dependence of the first harmonic ($k=1$) of the channelled radiation on the emission angle Θ for different amplitudes u of the positron trajectory and different Loretz factors γ for $\varphi=0$:

(1) $u=0.9$, $\gamma=4.5\times 10^2$

(2) $u=0.9$, $\gamma=4.5\times 10^3$

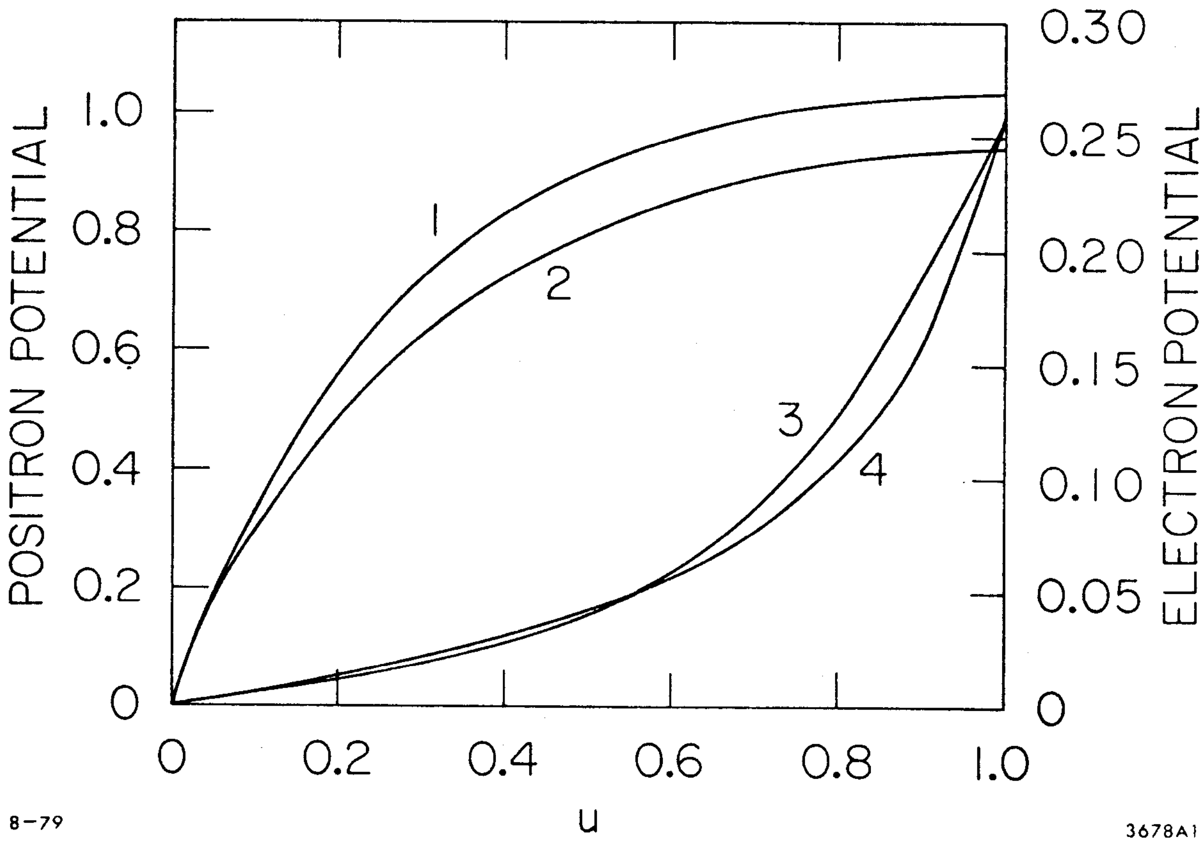
(3) $u=0.9$, $\gamma=4.5\times 10^4$

(4) $u=0.5$, $\gamma=4.5\times 10^2$

(5) $u=0.5$, $\gamma=4.5\times 10^3$

(6) $u=0.5$, $\gamma=4.5\times 10^4$

Moliére potential.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ELECTRON POTENTIAL
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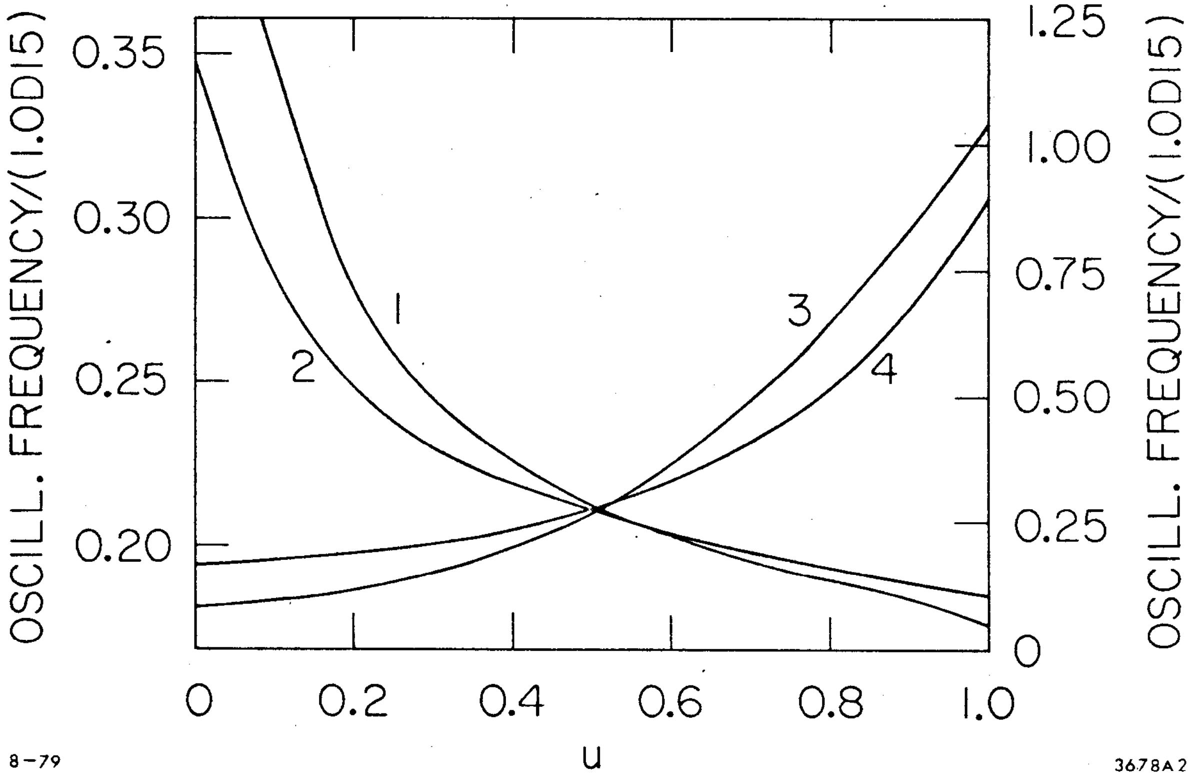
0.6
0.4
0.2
0
0.30
0.25
0.20
0.15
0.10
0.05
0
0.2
0.4
0.6
0.8
1.0
u
1
2
3
4
8-79
3678A1

Fig. 1


OSCILL. FREQUENCY/(1.0D15)
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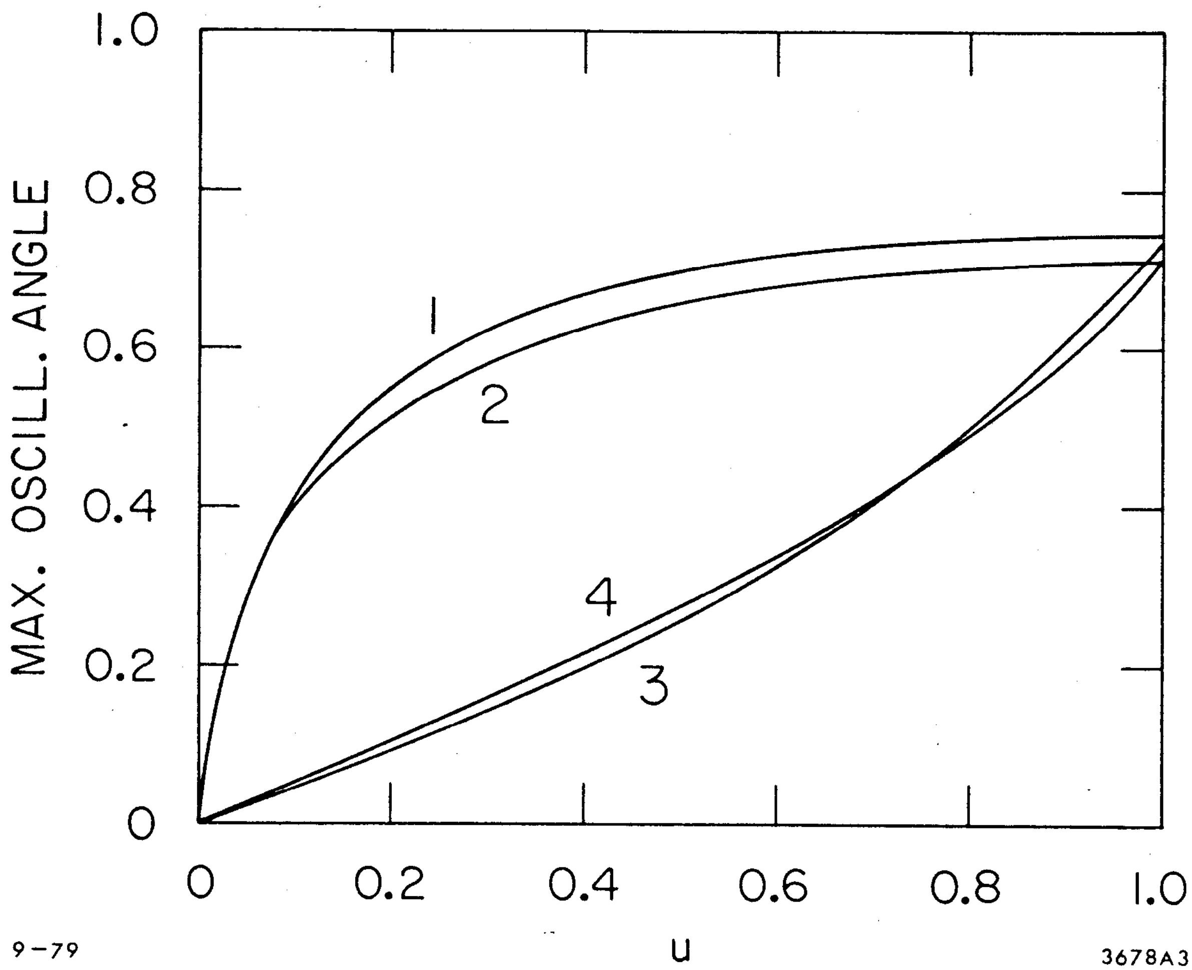
0.30
0.25
0.20
1.25
1.00
0.75
0.50
0.25
0
1
2
3
4
0
0.2
0.4
0.6
0.8
1.0
u
8−79
3678A2

Fig. 2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0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1.0
u
1
2
3
4
9-79
3678A3

Fig. 3

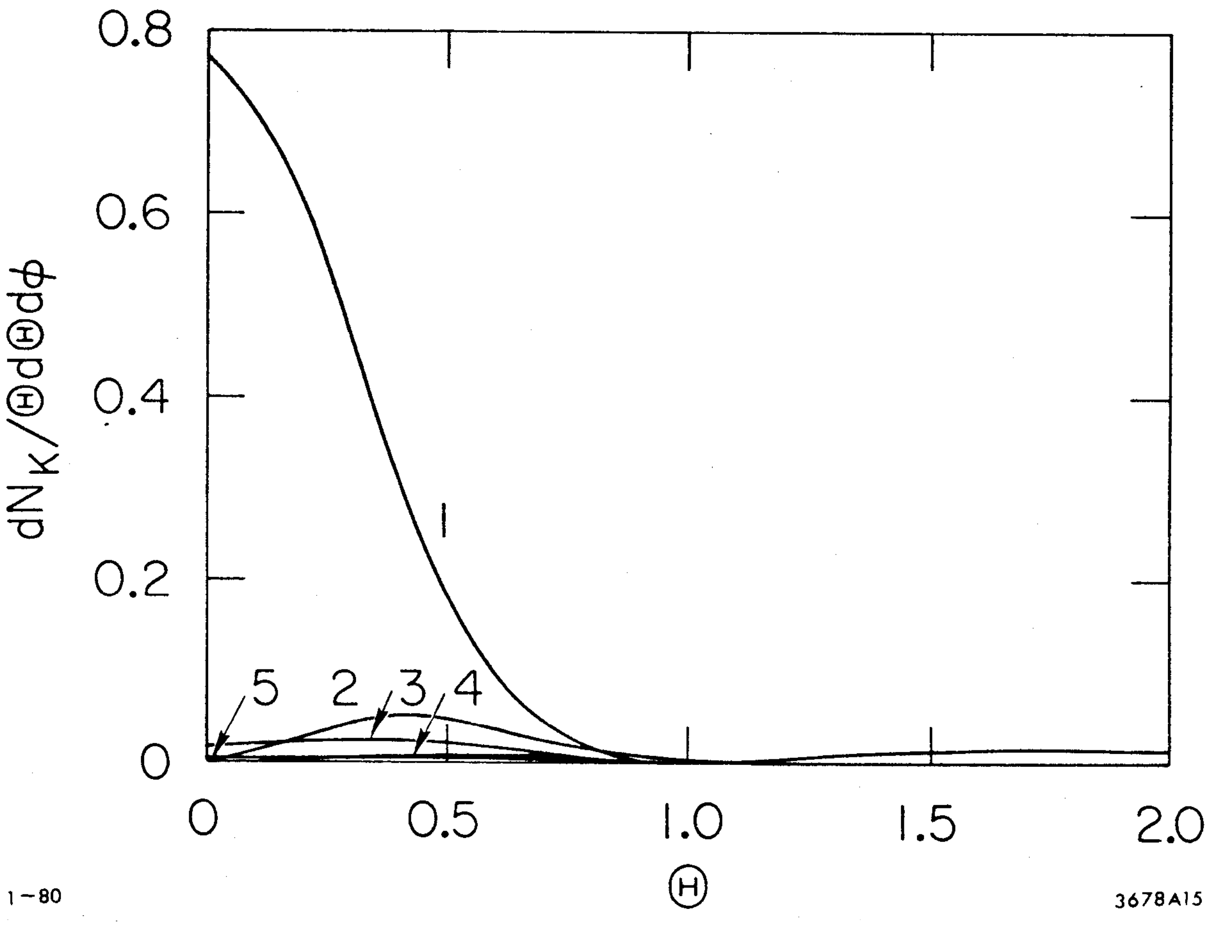

0.8
0.6
0.4
0.2
0
$dN_K/\Theta d\Theta d\phi$
1
5
2
3
4
0
0.5
1.0
1.5
2.0
Θ
1-80
3678A15

Fig. 4

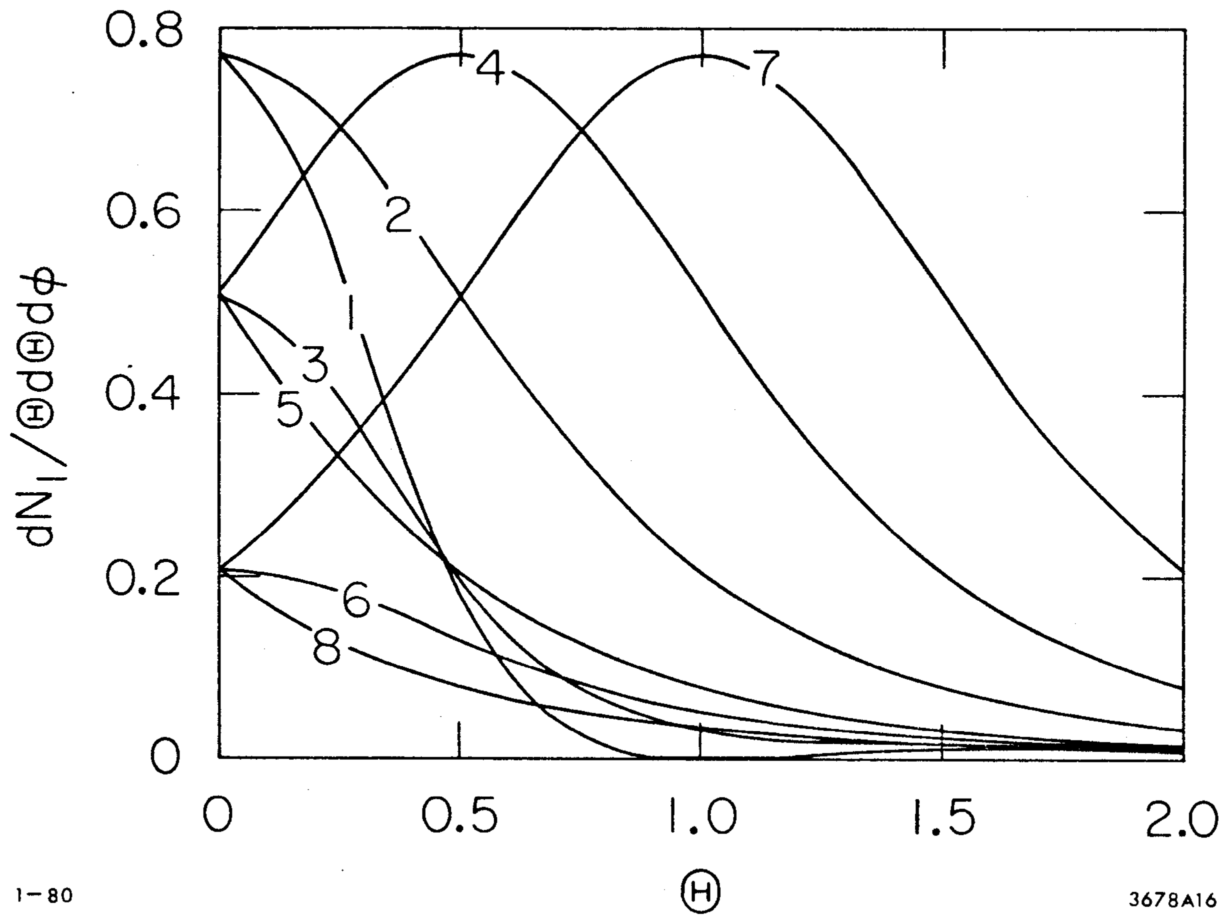


Fig. 5

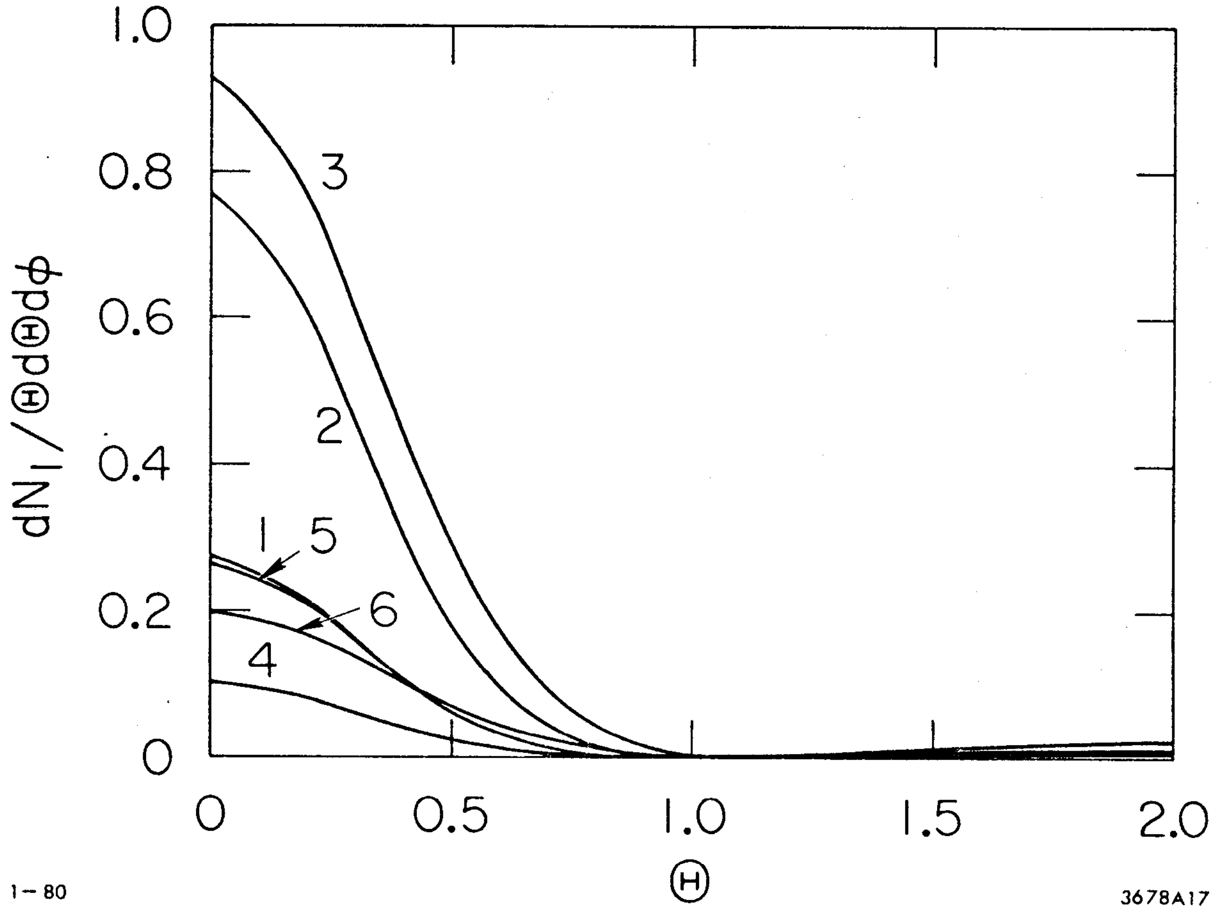
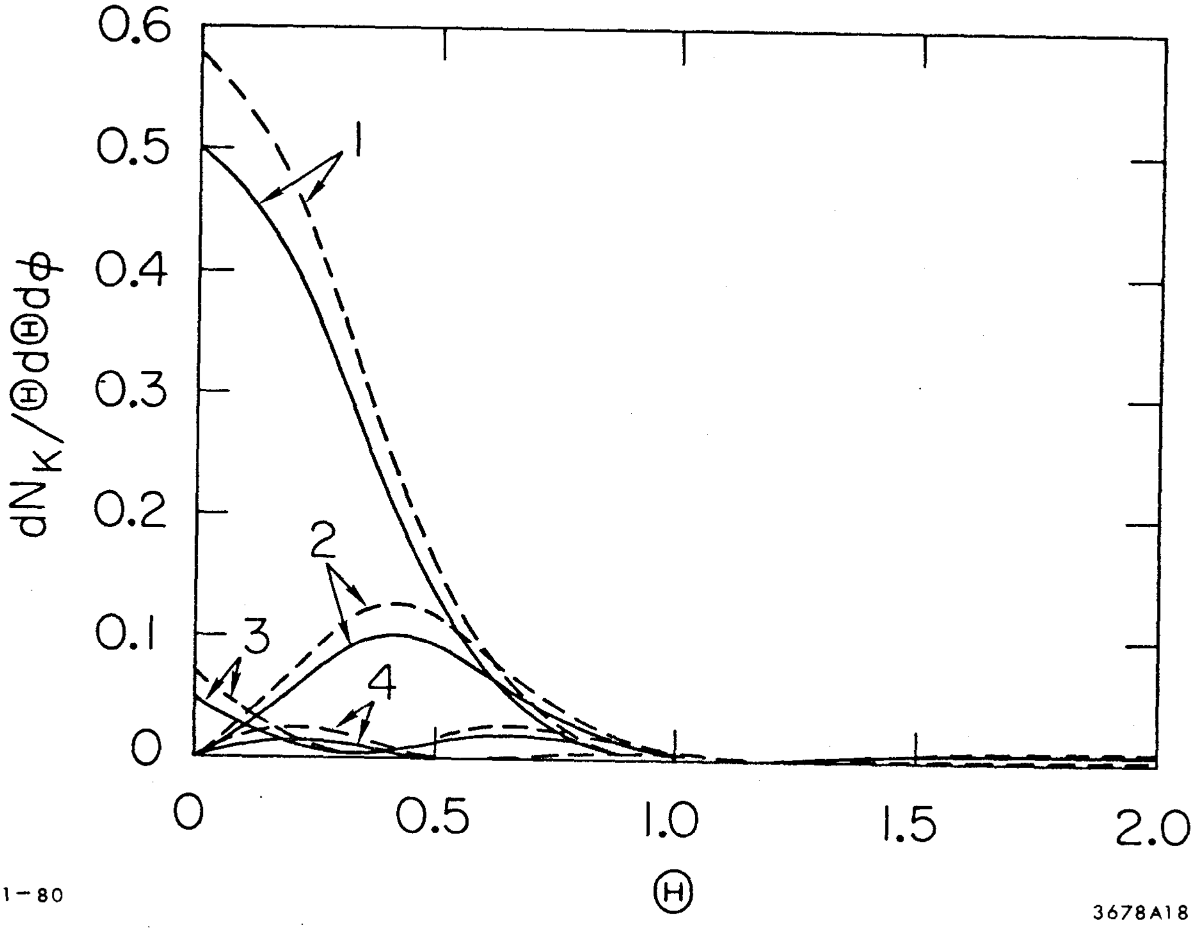

1.0
0.8
0.6
0.4
0.2
0
0
0.5
1.0
1.5
2.0
$dN_1/\Theta d\Theta d\phi$
Θ
1
2
3
4
5
6
1-80
3678A17

Fig. 6



1-80

3678A18

Fig. 7

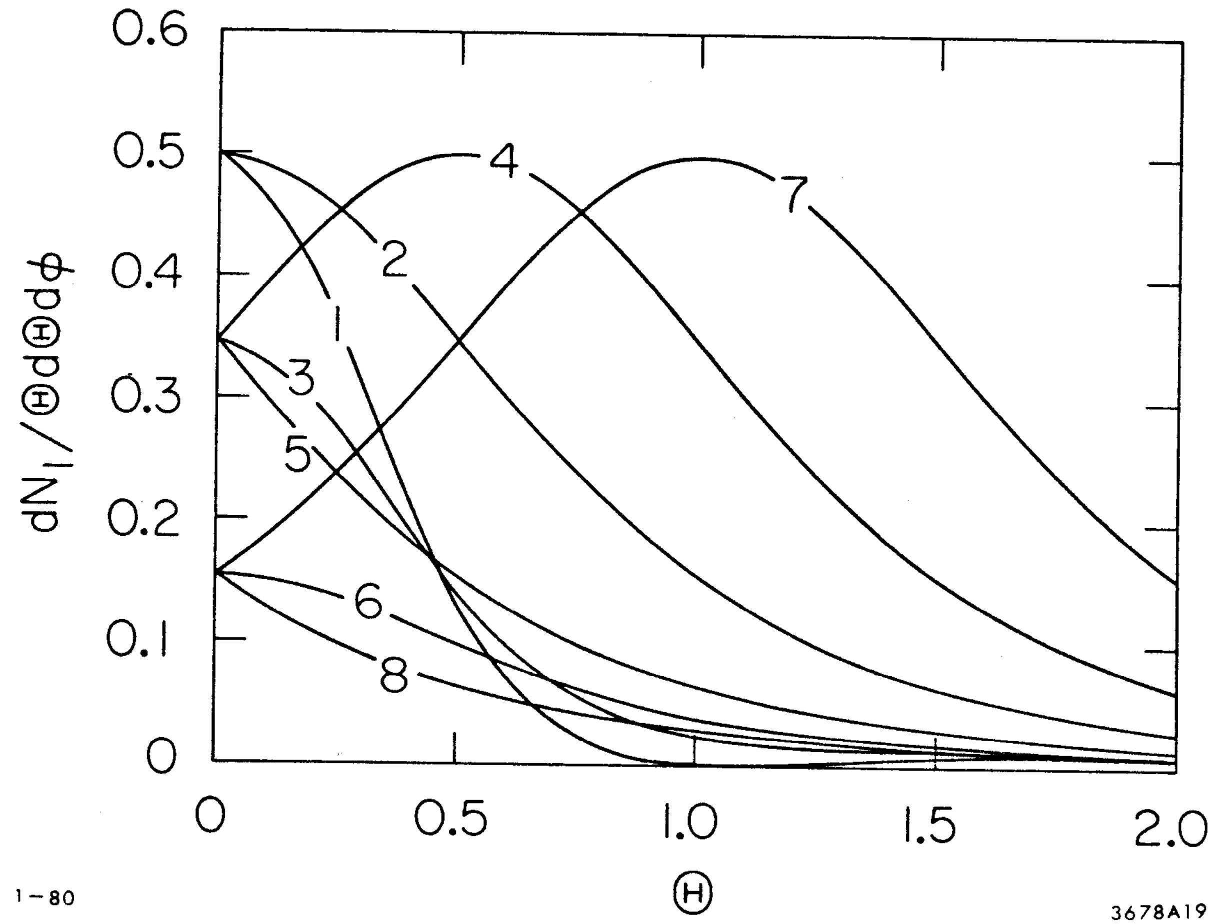
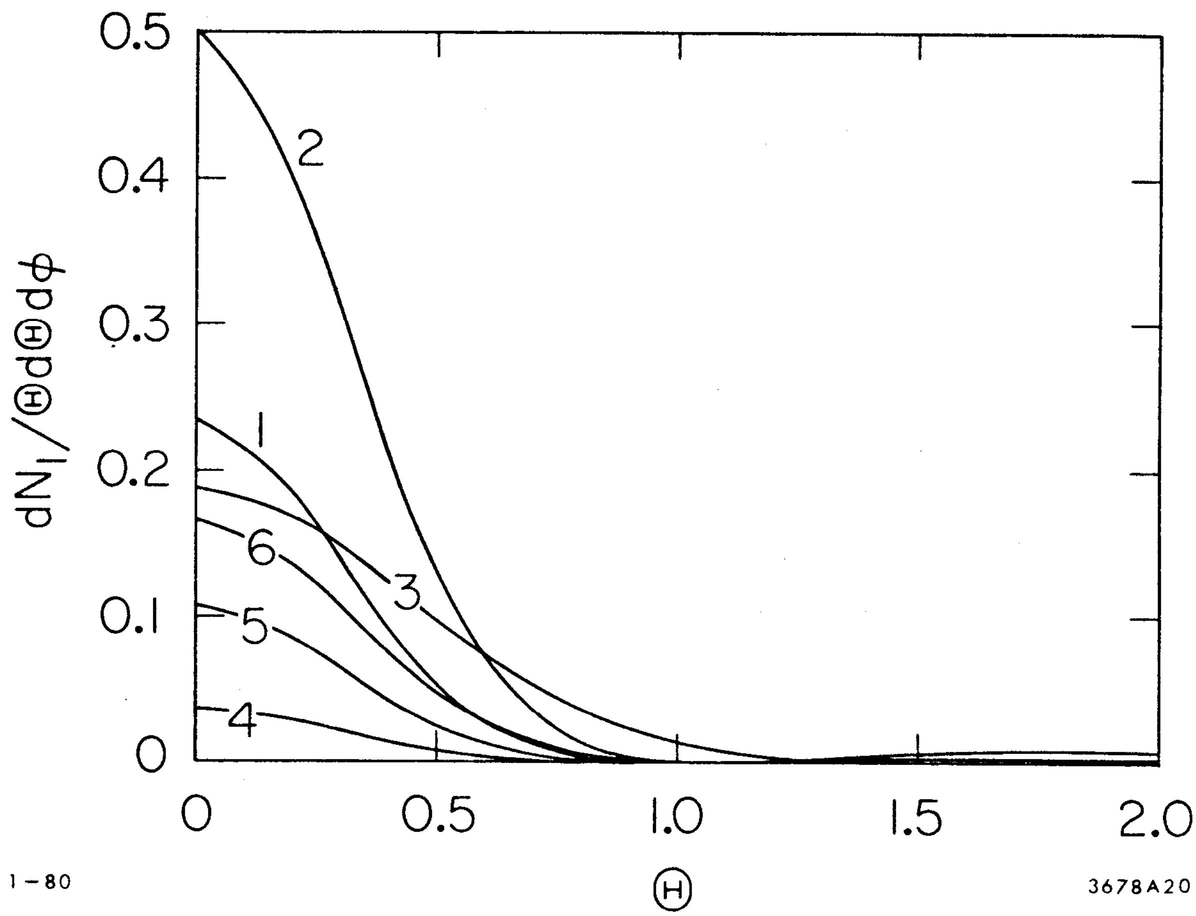


Fig. 8



0.5
0.4
0.3
0.2
0.1
0
$dN_1/\Theta d\Theta d\phi$
2
1
3
6
5
4
0
0.5
1.0
1.5
2.0
1-80
Θ
3678A20

Fig. 9