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BEAM ENVELOPE MATCHING FOR BEAM GUIDANCE SYSTEMS*

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ABSTRACT

Ray optics and phase ellipse optics are developed as tools for designing charged particle beam guidance systems. Specific examples of basic optical systems and of phase ellipse matching are presented as illustrations of these mathematical techniques.

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1. Introduction

Traditionally ray optics expanded in a Taylor series of linear and higher order terms about a central trajectory has been used in the design of single pass charged particle optical systems such as spectrometers, spectrographs and beam analyzing systems [1,2]. Phase ellipse optics, on the other hand, has been used primarily for systems that can be described adequately by linear theory and where knowledge of the phase shift is paramount to an understanding of the system performance, such as in circular particle accelerators [3]. Both methods may be used to design single pass beam transport systems, but there are applications for which the conceptual understanding and/or the mathematical description favors one of the two approaches. By combining the best features of each technique further simplifications result which make many problems easier to solve and understand. It is the purpose of this report to develop the theory and to present some specific examples of these methods.

The basic mathematical formalism for linear ray optics and linear phase ellipse optics is summarized below for monoenergetic trajectories in one transverse plane. The notation used for the ray optics is that of the TRANSPORT program [2]. And the notation for the phase ellipse optics follows that of the traditional circular machine theory introduced by Courant, Snyder, Twiss, and others [3,4].

Linear ray optics may be described by a transfer matrix R expressing the amplitude and angle of an arbitrary trajectory at position 2 as a linear function of the amplitude and angle at position 1, where the amplitudes and angles are measured relative to the optical axis of the system. In TRANSPORT notation this becomes

$$\begin{pmatrix} x_2 \\ \theta_2 \end{pmatrix} = \mathbf{R}\begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix} \quad \text{with} \quad \mathbf{R} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \quad , \tag{1}$$

where for static magnetic fields $|\mathbf{R}| = 1$.

In linear phase ellipse optics an ensemble of particles enclosed by an arbitrary ellipse at position 1 in a beam guidance system will be enclosed by another ellipse of the same area at position 2. The Twiss parameters α,β,γ and the beam emittance ε specify the beam ellipse at each position. This is illustrated in fig. 1. The area of the ellipse is $A = \pi\varepsilon$. The maximum spatial extent of the ellipse (the beam envelope) is $x_{max} = \sqrt{\beta\varepsilon}$ and the maximum angular divergence of the beam within the phase ellipse is $\theta_{max} = \sqrt{\gamma\varepsilon}$. The parameters α or r_{21} define the orientation of the ellipse relative to the x and θ axes.

Given a matrix

$$\mathbf{T} = \begin{bmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{bmatrix} \quad \text{with} \quad |\mathbf{T}| = \left(\beta\gamma - \alpha^2\right) = 1 \quad , \tag{2}$$

an ellipse of area $A = \pi\varepsilon$ is generated by the matrix equation

$$\mathbf{X}^T\mathbf{T}^{-1}\mathbf{X} = \varepsilon \quad \text{where} \quad \mathbf{X} = \begin{pmatrix} x \\ \theta \end{pmatrix}$$

or in algebraic form, the equation of the ellipse is

$$\gamma x^2 + 2\alpha x\theta + \beta\theta^2 = \varepsilon \tag{3}$$

The transformation of the Twiss parameters defining an ellipse at position 1 to those those defining an ellipse at position 2 is given by the matrix equation

$$\mathbf{T}_2 = \mathbf{R}\,\mathbf{T}_1\mathbf{R}^T \tag{4}$$

as derived in ref. [2]. This result may also be written in the familiar form [4]

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \left[\begin{array}{c|c|c} R_{11}^2 & -2R_{11}R_{12} & R_{12}^2 \\ \hline -R_{11}R_{21} & R_{11}R_{22}+R_{12}R_{21} & -R_{12}R_{22} \\ \hline R_{21}^2 & -2R_{21}R_{22} & R_{22}^2 \end{array}\right] \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix} \tag{5}$$

In eq. (5), the transformation of the Twiss parameters is expressed as a function of the ray optics matrix elements R_{ij}. It is equally useful to express the matrix R describing the transformation of the ray optics from position 1 to position 2 as a function of the Twiss parameters. To do this an additional variable is required. Courant and Snyder introduced for this purpose the phase shift, $\Delta\psi$, measured between positions 1 and 2 and defined as follows:

$$\Delta\psi = \int_{s_1}^{s_2} \frac{ds}{\beta(s)} \tag{6}$$

where s is the distance measured along the optical axis of the system and $\beta(s)$ is the Twiss parameter β evaluated at position s. The final result is the following [4],

$$\mathbf{R} = \begin{bmatrix} \sqrt{\frac{\beta_2}{\beta_1}}\left(\cos\Delta\psi + \alpha_1 \sin\Delta\psi\right) & \sqrt{\beta_1\beta_2}\sin\Delta\psi \\ -\left[\frac{(1+\alpha_1\alpha_2)\sin\Delta\psi + (\alpha_2-\alpha_1)\cos\Delta\psi}{\sqrt{\beta_1\beta_2}}\right] & \sqrt{\frac{\beta_1}{\beta_2}}\left(\cos\Delta\psi - \alpha_2 \sin\Delta\psi\right) \end{bmatrix} \tag{7}$$

where the subscripts 1 and 2 correspond to the initial and final positions of the beam transfer section.

Several useful observations can be derived from eq. (7).

a) Given the transformation matrix **R**, the phase shift $\Delta\psi$ may also be expressed as

$$\sin\Delta\psi = \frac{R_{12}}{\sqrt{\beta_1\beta_2}} \quad , \quad \tan\Delta\psi = \frac{R_{12}}{R_{11}\beta_1 - R_{12}\alpha_1} = \frac{R_{12}}{R_{22}\beta_2 + R_{12}\alpha_2} \tag{8}$$

b) Furthermore

If	Then
$R_{11} \neq 0$, $R_{12} \neq 0$	$\Delta\psi$ is a function of both α_1 and β_1
$R_{11} = 0$, $R_{12} \neq 0$	$\tan \Delta\psi = -\frac{1}{\alpha_1}$ independent of β_1
$R_{12} = 0$	$\Delta\psi = N\pi$ independent of α_1 and β_1
R_{12} and β_1 are constants and $R_{12} \neq 0$	β_2 (minimum value) $= \frac{R_{12}^2}{\beta_1}$ when $\sin\Delta\psi = 1$. (9)

It should be noted that a beam transport system, characterized by the matrix **R**, is completely determined by the array of optical elements from which it is constructed, i.e., the lenses and drift distances making up the system. The numerical values of the matrix elements, R_{ij}, are therefore independent of the particular phase space ellipse configuration that exists at the beginning of the system. However, for design purposes, it is often useful to specify a particular optical condition at the beginning and at the end of a system for the purpose of 'inventing' or devising an optical array. In particular, if it is assumed that the phase ellipses at the beginning and at the end of a system are identical, then the mathematical expressions describing the matrix **R** become particularly simple. The properties of the resulting system may then be studied for other initial and final values of the Twiss parameters. It then remains to devise an actual optical array of physical elements that possesses the 'assumed' properties. Specific examples using this design procedure are given below.

If the phase ellipses at the beginning and at the end of a transfer section are identical, i.e., $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$, and we define $\Delta\psi = \mu$, then eq. (7) reduces to the well-known form used in circular machine theory [3,4],

$$\mathbf{R} = \left[\begin{array}{c|c} \cos\mu + \alpha \sin\mu & \beta \sin\mu \\ \hline -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{array}\right] \tag{10}$$

where as before $(\beta\gamma - \alpha^2) = 1$, and now

$$\text{Trace } \mathbf{R} = 2 \cos\mu \tag{11}$$

For N such unit cells in sequence, as in a matched repetitive lattice of a circular machine, the total transfer matrix is given by

$$\mathbf{R}_T = \mathbf{R}^N = \left[\begin{array}{c|c} \cos N\mu + \alpha \sin N\mu & \beta \sin N\mu \\ \hline -\gamma \sin N\mu & \cos N\mu - \alpha \sin N\mu \end{array}\right] \tag{12}$$

and

$$\text{Trace } \mathbf{R}^N = 2 \cos N\mu \tag{13}$$

An equally interesting simplification occurs when $\alpha_1 = \alpha_2 = \alpha$ and $\Delta\psi = \mu$ are constant from cell to cell, whereas the beam envelope size is allowed to change by a constant ratio r from cell to cell, i.e.,

$$\sqrt{\frac{\beta_2}{\beta_1}} = \sqrt{\frac{\beta_3}{\beta_2}} = \sqrt{\frac{\beta_{N+1}}{\beta_N}} = r \tag{14}$$

For the first cell in such a series, the transfer matrix, **R**, is

$$\mathbf{R} = \left[\begin{array}{c|c} r(\cos\mu + \alpha\sin\mu) & r\beta_1 \sin\mu \\ \hline -\dfrac{\gamma_1 \sin\mu}{r} & \dfrac{\cos\mu - \alpha\sin\mu}{r} \end{array}\right] \tag{15}$$

where $\gamma_1 = \dfrac{1+\alpha^2}{\beta_1}$. The transfer matrix, $\mathbf{R}_N$, for the Nth cell is

$$\mathbf{R}_N = \left[\begin{array}{c|c} r(\cos\mu + \alpha\sin\mu) & r^{2N-1}\beta_1 \sin\mu \\ \hline -\dfrac{\gamma_1 \sin\mu}{r^{2N-1}} & \dfrac{\cos\mu - \alpha\sin\mu}{r} \end{array}\right] \tag{16}$$

and the total matrix, $\mathbf{R}_T$, for a sequence of N such cells is

$$\mathbf{R}_T = \mathbf{R}_N \cdots \mathbf{R}_1 = \left[\begin{array}{c|c} r^N(\cos N\mu + \alpha\sin N\mu) & r^N \beta_1 \sin N\mu \\ \hline -\dfrac{\gamma_1 \sin N\mu}{r^N} & \dfrac{\cos N\mu - \alpha\sin N\mu}{r^N} \end{array}\right] \tag{17}$$

This completes the discussion of the basic mathematics for linear optics as is needed here. We shall now make use of it to develop some specific examples.

2. Optical Building Blocks

Monoenergetic first-order (linear) optical systems are basically composed of combinations of 'thin' lenses interspersed with drift distances. It seems appropriate, therefore, to explore the properties of some of the basic elements before formulating more complex systems.

A. *Drift Distances*

A drift distance is characterized, in ray optics, by the fact that the angle of any arbitrary trajectory relative to the optical axis remains unchanged. Stated in terms of an ensemble of trajectories enclosed by an ellipse, the angular divergence of the beam, $\theta_{max} = \sqrt{\gamma\varepsilon}$, is a constant whereas the beam envelope, $x_{max} = \sqrt{\beta\varepsilon}$, and the orientation α of the ellipse are changing.

A typical drift distance and its basic properties are illustrated in fig. 2. The following characteristics are to be noted:

a) Since the phase space area is conserved, it follows that $\gamma = \frac{1}{\beta_w}$, where β_w is the value of β at a beam waist.

b) The phase shift through a drift region depends not only on the length of the drift, but also on the value of the initial Twiss parameters β_1 and α_1 at the beginning of the drift.

c) If a thin lens of variable strength precedes a drift, it may be adjusted to provide a minimum beam size $x_{min} = \sqrt{\beta_2(\text{min})\varepsilon}$ at the end of the drift. The magnitude of $\beta_2(\text{min})$ is derivable from eq. (9) by setting $\sin\Delta\psi = 1$. We conclude that

$$\beta_2(\text{min}) = \frac{R_{12}^2}{\beta_1} = \frac{L^2}{\beta_1} \quad ,$$

where $x_1 = \sqrt{\beta_1 \varepsilon}$ is the beam envelope size at the thin lens. By measuring x_{min} and x_1, the emittance of the beam is uniquely determined and given by

$$\varepsilon = \frac{x_1 x_{min}}{L} \quad .$$

d) If it is desired to transmit a beam of particles through a constant aperture, e.g., the gap of a magnet, then the minimum aperture required is also readily obtained from eq. (9) by equating $\beta_1 = \beta_2 = \beta$ and requiring that β be at a minimum value. That is, the beam envelope should have the same size at the beginning and at the end of the system and be at a minimum value. Under these circumstances a beam waist, β_w, occurs at the midpoint. The result is

$$\beta = L \text{ and } \beta_w = \frac{1}{\gamma} = \frac{L}{2} \quad .$$

The minimum aperture required to transmit the beam is

$$x_1(\text{min}) = \sqrt{\beta\varepsilon} = \sqrt{L\varepsilon}$$

and the ratio of the beam envelope size at the two ends of the system to the size of the waist at the midpoint is

$$\left(\frac{x_1}{x_w}\right) = \sqrt{\frac{\beta}{\beta_w}} = \sqrt{2} \quad .$$

B. A Thin Lens

A thin lens changes the direction but not the position of a particle trajectory as the particle passes through the lens. In linear theory the change in angle is $\Delta\theta = -x/F$, where x is the amplitude of the trajectory and F is the focal length of the lens. Stated in terms of the phase ellipse formalism, the Twiss parameter α changes by $\Delta\alpha = \beta/F$, while the beam envelope, $\sqrt{\beta\varepsilon}$, remains unchanged. The net phase shift, $\Delta\psi$, is zero as can be seen from the Courant-Snyder definition of phase shift given in eq. (6), and from the fact that the matrix element R_{12} for a thin lens is zero. These properties of a thin lens are illustrated in fig. 3.

C. A Thin Quadrupole Lens

A quadrupole focuses in one transverse plane while defocusing in the other plane. For a thin lens quadrupole it is assumed that the two planes differ only by the sign of the focal length F as is illustrated in fig. 4. (In realistic systems the absolute value of the focal length in the two planes is not the same but this assumption is a good approximation for many purposes.) As with the simple thin lens discussed in the previous paragraph, the phase shift, $\Delta\psi$, vanishes in both planes. If we define θ to be the particle direction in the x plane and ϕ its direction in the y plane, then $\Delta\theta = -x/F$ and $\Delta\phi = y/F$. Stated in terms of the phase ellipse formalism, $\Delta\alpha_x = \beta_x/F$ and $\Delta\alpha_y = -\beta_y/F$, while x, y, β_x and β_y remain constant in a 'thin' lens.

Very often in beam guidance systems a segment of the system may be composed of a periodic array of identical elements or 'unit cells', such that the phase ellipse in the two transverse planes x and y is similar or even identical at specific locations in the periodic array. One such situation occurs at the midpoint between two quadrupoles of a matched periodic array where the quadrupoles are of equal strength but of opposite sign, i.e., a FODO array. The beam envelopes, at this location, have the same magnitude but the phase ellipses in the x and y planes are mirror images of each other about the θ and ϕ axes, i.e., $\beta_x = \beta_y$ but $\alpha_x = -\alpha_y$. If a thin lens quadrupole is positioned at this location, any adjustment of its focal length preserves the above symmetry. The beta functions β_x and β_y remain unchanged, the absolute values of $|\alpha_x|$ and $|\alpha_y|$ change, but the mirror symmetry property $\alpha_x = -\alpha_y$ is maintained. This is illustrated at the bottom of fig. 4. This characteristic is a very useful feature for phase ellipse matching between two dissimilar systems as will be demonstrated in some of the examples.

D. A Telescope

Another basic optical module is the telescope. For a one-dimensional system it consists of two thin lenses, separated by a distance equal to the sum of their focal lengths, as illustrated in fig. 5. The telescope has the unique property of simultaneous parallel to parallel and point to point imaging. This is equivalent to saying that the R_{21} and R_{12} matrix elements are zero. Since $R_{12} = 0$, the phase shift is always a multiple of π, independent of the initial phase ellipse configuration. The fact that $R_{21} = 0$ coincident with $R_{12} = 0$ requires that $\alpha_2 = \alpha_1$, i.e., the parameter α is the same at the beginning and end of the telescope. (This is the condition imposed upon eqs. (14-17) and the significance of this will become evident later.)

A two-dimensional telescopic system, using four quadrupoles, is illustrated in fig. 6. It is an obvious extension of the one-dimensional example of fig. 5. It has the added advantage that the magnifications of the beam envelopes in the two transverse planes may either be the same or different. This property allows such an array of lenses to be used for matching systems with different properties in the x and y planes.

3. Beam Envelope Matching

A common task in beam optics is to match the phase space ellipse of one beam guidance system to that of another one by an appropriate transition section. We shall now describe some solutions to this problem that have evolved from the theory and techniques discussed in the previous paragraphs.

In general, it is almost always possible to match two dissimilar systems by using one or more telescopic arrays similar to those shown in figs. 5 and 6. In particular, the system illustrated in fig. 6 has the flexibility of simultaneously matching different phase ellipses in the x and y planes. Six variables are needed to achieve a match in both transverse planes. Typically the variables used are the strengths of the four quadrupoles and the two drift distances ℓ_3 and ℓ_4, though other combinations of six variables are permissible. It should also be noted that the endpoint (position 2) for the two planes need not coincide, this provides additional flexibility to the range of possible solutions. This system, however, has the disadvantage that the position of the quadrupoles as well as their strengths change when the match requirements change. It is therefore desirable to explore solutions where only a variation of the strengths of the lenses and not their positions is sufficient to establish a phase space match. This is possible, and systems having this property are developed in the following paragraphs.

A thin lens varies the Twiss parameter α. If it is placed at the beginning of an arbitrary beam transfer section, characterized by the matrix **R**, it is observed from eq. (5) that as it varies α_1 it also varies β_2 and α_2, provided the matrix elements R_{11} and R_{12} are non-zero. A second thin lens positioned at the end of the system will vary α_2. Thus β_2 and α_2 may be continuously adjusted by varying only the strengths of the two lenses. Their positions remain fixed. The range of variation of β_2 is obtained from eq. (9) and is

$$\beta_2 \geqq \frac{R_{12}^2}{\beta_1} \quad ,$$

where $\beta_2 = R_{12}^2/\beta_1$ is the minimum value of β_2 allowed.

By using quadrupoles for the two variable lenses, it is possible to simultaneously match the phase ellipses in both transverse planes. Let us assume that the desired phase ellipse in the two transverse planes x and y at both positions 1 and 2 possesses the following symmetry: $\beta_x = \beta_y$ and $\alpha_x = -\alpha_y$, but that in general $\beta(2) \neq \beta(1)$ and $\alpha(2) \neq \alpha(1)$. Under these circumstances it is possible to achieve a match between the two positions with two quadrupoles, one placed at the beginning of a transfer section and the other positioned at the end provided that the transfer section, described by the matrix **R**, has the following properties: a) The absolute value of the matrix elements of **R** is the same in the x and y planes. b) Either R_{11} and R_{22} should change sign from the x to the y planes and R_{12} and R_{21} remain unchanged, or c) R_{12} and R_{21} change sign and R_{11} and R_{22} remain unchanged. Under these circumstances the two quadrupole singlets may be used to simultaneously match both planes. As an example of matrices that possess the above properties we cite the following:

$$\mathbf{R}_{x,y} = \begin{bmatrix} \underline{\underline{R_{11}}} & R_{12} \\ R_{21} & \underline{\underline{R_{22}}} \end{bmatrix} \text{ or } \mathbf{R}_{x,y} = \begin{bmatrix} R_{11} & \underline{\underline{R_{12}}} \\ \underline{\underline{R_{21}}} & R_{22} \end{bmatrix}$$

where the underlined matrix elements change sign from the x plane to the y plane. The consequence of this is to change the sign of the underlined matrix elements in eq. (5) as shown below:

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \left[\begin{array}{c|c|c} R_{11}^2 & \underline{\underline{-2R_{11}R_{12}}} & R_{12}^2 \\ \hline \underline{\underline{-R_{11}R_{21}}} & R_{11}R_{22} + R_{12}R_{21} & \underline{\underline{-R_{12}R_{22}}} \\ \hline R_{21}^2 & \underline{\underline{-2R_{21}R_{22}}} & R_{22}^2 \end{array}\right] \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix}$$

where this Twiss transformation applies to both the x and y transverse planes. The absolute value of each of the matrix elements in the Twiss transformation is the same in both planes. The consequences of this are the following: If $\beta_1(x) = \beta_1(y)$ and $\alpha_1(x) = -\alpha_1(y)$ and if it is desired to match another system where $\beta_2(x) = \beta_2(y)$ and $\alpha_2(x) = -\alpha_2(y)$, then a thin lens quadrupole positioned at 1 changes α by $\Delta\alpha_1(x) = -\Delta\alpha_1(y)$ as its strength is adjusted. This variation, combined with the properties of the Twiss transformation, varies β_2 and α_2 such that the symmetry conditions $\beta_2(x) = \beta_2(y)$ and $\alpha_2(x) = -\alpha_2(y)$ are always preserved. A second quadrupole positioned at 2 varies α_2 in a similar manner. Hence a combination of the two quadrupoles plus the transfer section characterized by the **R** matrix permits a match to be made, provided

$$\beta_2 \geqq \frac{R_{12}^2}{\beta_1} .$$

We now wish to formulate a specific beam envelope matching system having the above properties. To do this we make use of the linear phase ellipse theory developed in the earlier paragraphs. Consider a unit cell such that $\beta_1 = \beta_2 = \beta$, $\alpha_1 = \alpha_2 = \alpha$ and $\Delta\psi = \pi/2$ for both transverse planes, but where the sign of α changes from the x plane to the y plane. By a simple substitution of the above conditions into eq. (10), it immediately follows that the matrix **R** must have the following form:

$$\mathbf{R}_{x,y} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} \pm\alpha & \beta \\ -\gamma & \pm\alpha \end{bmatrix}$$

where the upper sign of α corresponds to the x plane and the lower to the y plane. Thus R_{11} and R_{22} change sign from x to y whereas R_{12} and R_{21} do not change sign and the absolute value of each of the matrix elements is the same for both transverse planes. The above matrix describes an optical system that is the same from left to right in the x plane as it is from right to left in the y plane. The question is, what optical system is described by this matrix? One possible answer is the quadrupole doublet shown in fig. 7. The proof that this is so is outlined in the figure. The focal length and spacing of the elements must now be chosen such that $\Delta\psi = \mu = \pi/2$ for the "matched" condition $\beta_1 = \beta_2 = \beta$ and $\alpha_1 = \alpha_2 = \alpha$. This condition for μ is satisfied when

$$\sin\frac{\mu}{2} = \frac{\sqrt{\ell\ell_s}}{2F} = \frac{1}{\sqrt{2}}$$

Having chosen the parameters F, ℓ, and ℓ_s to correspond to a matched phase shift of $\mu = \pi/2$, the optical design of the cell is established and remains fixed. The transformation properties of the Twiss parameters for <u>any</u> initial and final condition are then given by substitution of the matrix $R_{x,y}$ into eq. (5). The result is

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \left[\begin{array}{c|c|c} \alpha^2 & \mp 2\alpha\beta & \beta^2 \\ \hline \pm\alpha\gamma & -(1+2\alpha^2) & \pm\alpha\beta \\ \hline \gamma^2 & \mp 2\alpha\gamma & \alpha^2 \end{array}\right] \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix}$$

We note that the above equation has the desired transformation properties; that if

$$\beta_1(x) = \beta_1(y) \text{ and } \alpha_1(y) = -\alpha_1(x)$$

then it must follow that

$$\beta_2(x) = \beta_2(y) \text{ and } \alpha_2(y) = -\alpha_2(x).$$

We now use this module, fig. 7, to formulate the beam envelope matching system shown in fig. 8. A quadrupole Q_1 is added at the beginning of the system and a second quadrupole Q_2 at the end. The final result is a beam matching system having the properties outlined in fig. 8. Energizing Q_1 varies α_1 and hence α_2 and β_2. Energizing Q_2 varies α_2. Therefore, both α_2 and β_2 may be adjusted by varying the strengths of the two quadrupoles Q_1 and Q_2 without moving their positions. As before, the range of adjustment of β_2 is restricted by eq. (9) and is

$$\beta_2 \geq \frac{R_{12}^2}{\beta_1} \text{ or } \beta_2 \geq \frac{\beta^2}{\beta_1}$$

In fig. 9 and fig. 10, two additional examples of phase ellipse matching are given. These examples use the same design concepts as were used in the preceding example, but the choice of the matrix $\mathbf{R}_{x,y}$, and hence the optical module corresponding to it, is very different.

In fig. 9 we use the telescope as the basic module. A unity magnification telescopic system is devised such that the x plane image precedes the y plane image by a distance 2L. The endpoint of the system, position 2, is chosen to be midway between the x and y images. The matrix $\mathbf{R}_{x,y}$ describing the linear ray optics between positions 1 and 2 is

$$\mathbf{R}_{x,y} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} -1 & \pm L \\ 0 & -1 \end{bmatrix}$$

Here R_{12} and R_{21} change sign between the x and y planes, while R_{11} and R_{22} do not change sign. The fact that $R_{21} = 0$ is unimportant to the final result. Again a quadrupole Q_1 is positioned at the beginning of the system and a second quadrupole Q_2 at the end. By energizing Q_1 and Q_2 a phase ellipse match becomes possible. The results are summarized in fig. 9.

Figure 10 illustrates still another system. Here the basic module is a segment of a periodic FODO array of quadrupoles. The system is designed such that all of the quadrupoles have the same focal length F and the spacing between the quadrupoles is L = F. When Q_2 is turned off and Q_1 is set to a focal length F = L, the ray optics matrix becomes

$$\mathbf{R}_{x,y} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} \mp 1 & 2L \\ 0 & \mp 1 \end{bmatrix}$$

In this example R_{11} and R_{22} change sign between x and y, but R_{12} and R_{21} remain unchanged. As before the absolute value of the matrix elements is the same in both transverse planes. As can be seen from the Twiss transformation given in the figure, varying Q_1 varies α_1 and hence α_2 and β_2. Similarly varying Q_2 varies α_2. Hence, as before, a phase space match is possible. The details are summarized in the figure.

Another approach to phase space matching is that described by eqs. (14-17) where the beam envelope is increased by a constant ratio r from cell to cell and several cells are used to complete the transition. The advantage of this method is related to the mathematical ease with which second-order aberrations may be analyzed and controlled [5]. To illustrate the concept, we choose a system with a phase shift per cell of $\mu = \pi/2$. The Twiss parameter α is held constant from cell to cell but the function β is allowed to change. A practical realization of such a system is illustrated in fig. 11. As can be seen, it is a sequence of telescopic systems such that the spacing between any two adjacent lenses is equal to the sum of their focal lengths. The starting point (position 1) of the system is arbitrary as long as $\ell \leqq L_1$. The total transfer matrix is derived by setting $\mu = \pi/2$ and $N\mu = 2\pi$ in eq. (17). Such systems are being studied as possible candidates for matching low beta interaction regions to the main lattice in large storage rings [5]. The advantage gained is the ease with which global cancellation of chromatic aberrations may be achieved.

4. Summary

Phase space matching between two dissimilar optical systems has been a time consuming task for optics designers in the past. In this report we have presented the mathematics for, and examples of, phase space matching techniques that have proved useful to the author and many of his colleagues. It is hoped that the reader may benefit from our experience.

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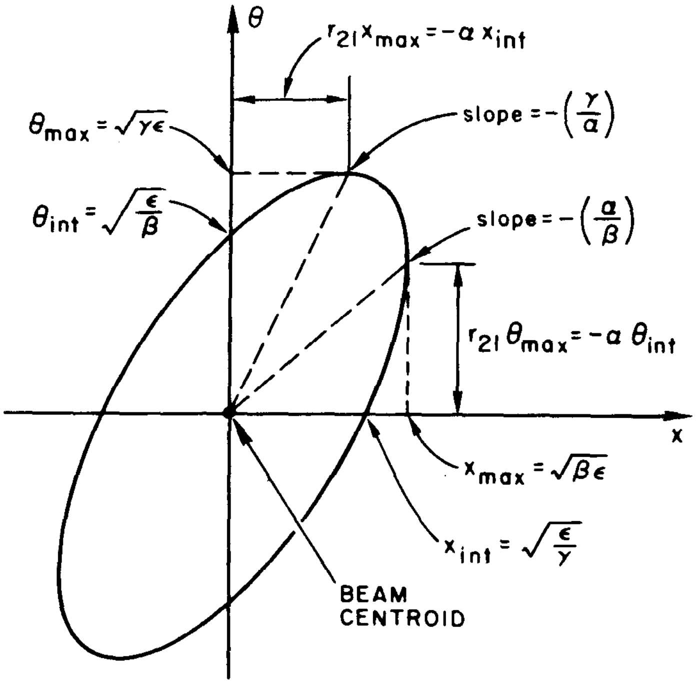

A TWO-DIMENSIONAL BEAM PHASE ELLIPSE

The area of the ellipse is given by:

$$A = \pi(\det \sigma)^{\frac{1}{2}} = \pi x_{max}\,\theta_{int} = \pi x_{int}\,\theta_{max} = \pi\varepsilon$$

The equation of the ellipse is:

$$\gamma x^2 + 2\alpha x\theta + \beta\theta^2 = \varepsilon$$

where

TRANSPORT NOTATION		COURANT-SNYDER NOTATION

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \varepsilon \begin{bmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{bmatrix}$$

and

$$\beta\gamma - \alpha^2 = 1$$

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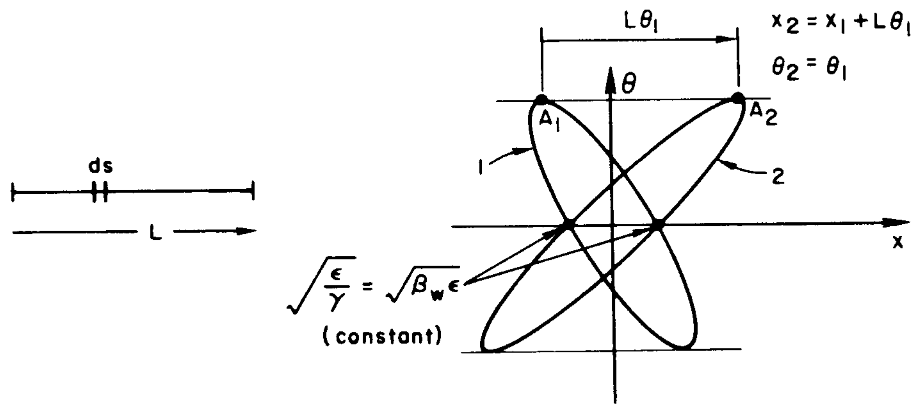
Fig. 1

A DRIFT



$$\mathbf{R} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{\beta_2}{\beta_1}}\left(\cos\Delta\psi + \alpha_1 \sin\Delta\psi\right) & \sqrt{\beta_1\beta_2}\,\sin\Delta\psi \\ -\left[\frac{(1+\alpha_1\alpha_2)\sin\Delta\psi + (\alpha_2 - \alpha_1)\cos\Delta\psi}{\sqrt{\beta_1\beta_2}}\right] & \sqrt{\frac{\beta_1}{\beta_2}}\left(\cos\Delta\psi - \alpha_2 \sin\Delta\psi\right) \end{bmatrix}$$

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \frac{1}{\beta_w} \end{pmatrix} = \begin{bmatrix} 1 & -2L & L^2 \\ 0 & 1 & -L \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \frac{1}{\beta_w} \end{pmatrix}$$

<u>Conclusions</u>:

$\gamma = \frac{1}{\beta_w}$ is a constant for a drift

$$\sin\Delta\psi = \frac{L}{\sqrt{\beta_1\beta_2}} \quad , \quad \tan\Delta\psi = \left(\frac{\alpha_1 - \alpha_2}{1+\alpha_1\alpha_2}\right) = \frac{1}{\left(\frac{\beta_1}{L}\right) - \alpha_1} = \frac{1}{\left(\frac{\beta_2}{L}\right) + \alpha_2}$$

$$\Delta\alpha = -\left(\frac{L}{\beta_w}\right) \quad , \quad \beta_2(\text{minimum}) = \frac{L^2}{\beta_1} \quad \text{when} \quad \Delta\psi = \frac{\pi}{2}$$

If $\beta_1 = \beta_2$ then $\alpha_2 = -\alpha_1$

If $\beta_1 = \beta_2 = \beta$ and $\Delta\psi = \frac{\pi}{2}$, then β is at a minimum value

and $\beta = L$, $\alpha_1 = -\alpha_2 = 1$, $\left(\frac{\beta}{\beta_w}\right) = \beta\gamma = (1+\alpha^2) = 2$

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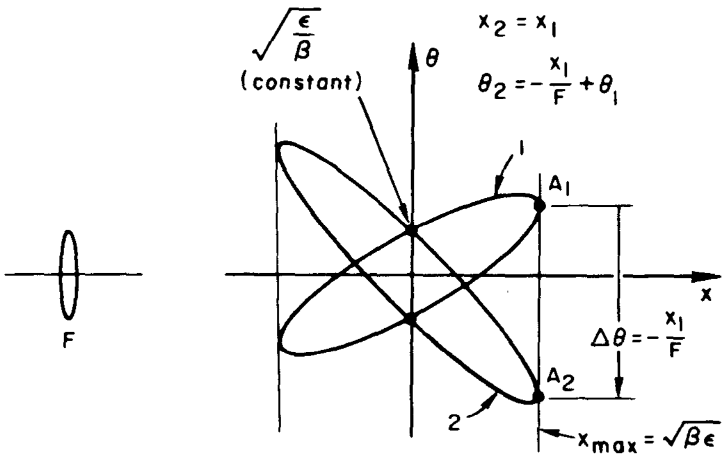
Fig. 2

A THIN LENS



$$\mathbf{R} = \begin{bmatrix} \sqrt{\frac{\beta_2}{\beta_1}}\left(\cos\Delta\psi + \alpha_1 \sin\Delta\psi\right) & \sqrt{\beta_1\beta_2}\,\sin\Delta\psi \\ -\left[\frac{(1+\alpha_1\alpha_2)\sin\Delta\psi + (\alpha_2-\alpha_1)\sin\Delta\psi}{\sqrt{\beta_1\beta_2}}\right] & \sqrt{\frac{\beta_1}{\beta_2}}\left(\cos\Delta\psi - \alpha_2 \sin\Delta\psi\right) \end{bmatrix}$$

or

$$\mathbf{R} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{F} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{\Delta\alpha}{\beta} & 1 \end{bmatrix}$$

and

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{F} & 1 & 0 \\ \frac{1}{F^2} & \frac{2}{F} & 1 \end{bmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix}$$

Conclusions:

β = constant for a lens

$$\Delta\psi = \int_1^2 \frac{ds}{\beta(s)} = \sin^{-1} R_{12} = 0$$

$$\beta_1 = \beta_2 = \beta \quad , \quad \Delta\alpha = \left(\frac{\beta}{F}\right)$$

$$x_1 = x_2 = x \quad , \quad \Delta\theta = -\left(\frac{x}{F}\right)$$

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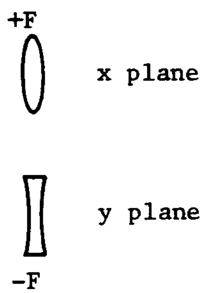
Fig. 3

A THIN LENS QUADRUPOLE



$$R = \begin{bmatrix} 1 & 0 \\ \mp\frac{1}{F} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{\Delta\alpha}{\beta} & 1 \end{bmatrix}, \quad \Delta\alpha = (\alpha_2 - \alpha_1)$$

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline \pm\frac{1}{F} & 1 & 0 \\ \hline \frac{1}{F^2} & \pm\frac{2}{F} & 1 \end{array}\right] \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix}$$

when ± or ∓ signs appear, top sign = x plane and bottom sign = y plane

Conclusions:

$$\beta_2 = \beta_1 \;, \quad \Delta\alpha_x = \frac{\beta_x}{F} \;, \quad \Delta\alpha_y = -\frac{\beta_y}{F} \;, \quad \Delta\psi_{x,y} = \sin^{-1} R_{12} = 0$$

Special Case:

If $\beta_x = \beta_y = \beta$ and $\alpha_x(1) = -\alpha_y(1)$

Then $\beta^{(2)}_{x,y} = \beta^{(1)}_{x,y} = \beta$ and $\alpha_x(2) = -\alpha_y(2)$, $\Delta\alpha_x = \frac{\beta}{F}$

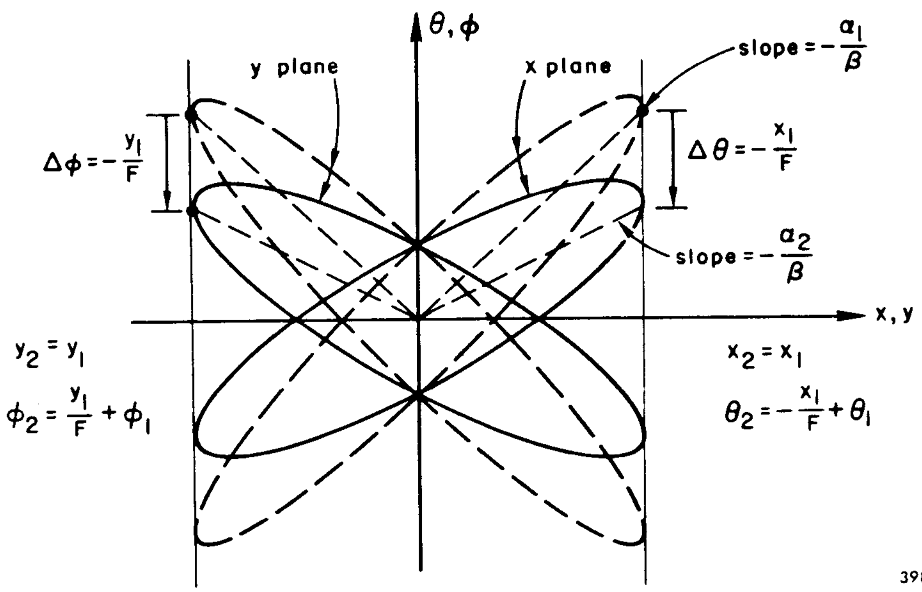


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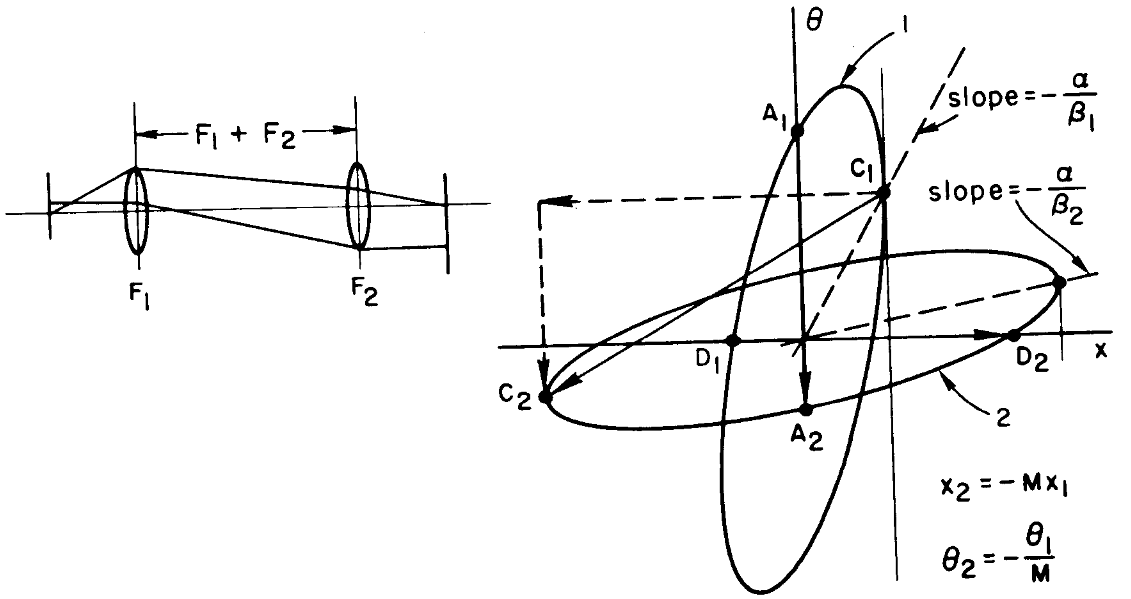
Fig. 4

A TELESCOPIC SYSTEM



$$R = \begin{bmatrix} -M & 0 \\ 0 & -\frac{1}{M} \end{bmatrix} = \begin{bmatrix} -\sqrt{\frac{\beta_2}{\beta_1}} & 0 \\ 0 & -\sqrt{\frac{\beta_1}{\beta_2}} \end{bmatrix} = \begin{bmatrix} -\frac{F_2}{F_1} & 0 \\ 0 & -\frac{F_1}{F_2} \end{bmatrix}$$

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{bmatrix} M^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{M^2} \end{bmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix}$$

Conclusions:

$$\alpha_2 = \alpha_1 = \alpha \quad \text{is a constant}$$

$$\left(\frac{\beta_2}{\beta_1}\right) = M^2 \quad , \quad \Delta\psi = \sin^{-1} R_{12} = \pi$$

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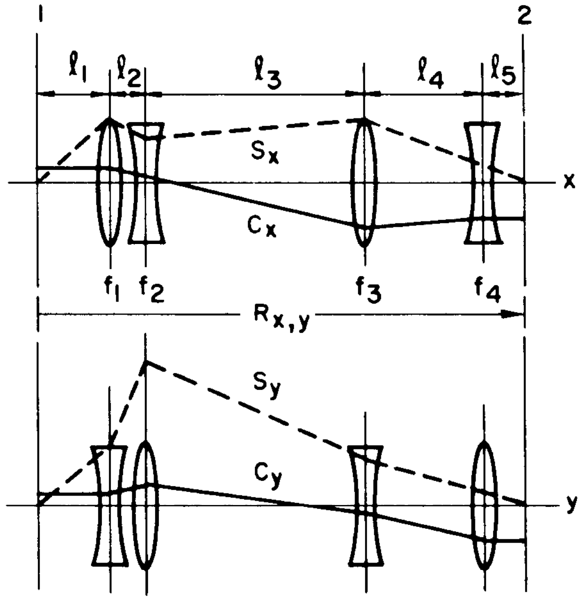
Fig. 5

A TWO-DIMENSIONAL TELESCOPIC TRANSFORMER USING QUADRUPOLE LENSES



$$
R_{x,y} = \begin{bmatrix} -M_x & 0 & 0 & 0 \\ 0 & -\frac{1}{M_x} & 0 & 0 \\ 0 & 0 & -M_y & 0 \\ 0 & 0 & 0 & -\frac{1}{M_y} \end{bmatrix} = \begin{bmatrix} -\sqrt{\frac{\beta_{2x}}{\beta_{1x}}} & 0 & 0 & 0 \\ 0 & -\sqrt{\frac{\beta_{1x}}{\beta_{2x}}} & 0 & 0 \\ 0 & 0 & -\sqrt{\frac{\beta_{2y}}{\beta_{1y}}} & 0 \\ 0 & 0 & 0 & -\sqrt{\frac{\beta_{1y}}{\beta_{2y}}} \end{bmatrix}
$$

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Fig. 6

A $\mu = \pi/2$ UNIT CELL MATCHED FOR β AND α

$$\beta_1 = \beta_2 = \beta \quad , \quad \alpha_1 = \alpha_2 = \alpha \quad , \quad \Delta\psi_{x,y} = \mu = \frac{\pi}{2}$$

Then

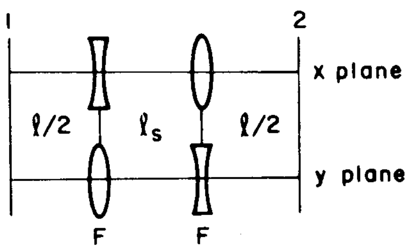
$$R_x = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix} \quad \text{and choose} \quad R_y = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} R_x^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -\alpha & \beta \\ -\gamma & \alpha \end{bmatrix}$$

or

$$R_{x,y} = \begin{bmatrix} \pm\alpha & \beta \\ -\gamma & \mp\alpha \end{bmatrix} \quad \text{where} \quad \begin{matrix} \text{upper sign} = x \text{ plane} \\ \text{lower sign} = y \text{ plane} \end{matrix}$$

This describes an optical system that is the same from left to right in the x plane as it is from right to left in the y plane.

One example of such a system is the Quadrupole Doublet



$$\sin\left(\frac{\mu_{x,y}}{2}\right) = \frac{\sqrt{\ell\ell_s}}{2F}$$

$$\mu_{x,y} = \frac{\pi}{2}$$

The Twiss transform is

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{bmatrix} \alpha^2 & \mp 2\alpha\beta & \beta^2 \\ \pm\alpha\gamma & -(1+2\alpha^2) & \pm\alpha\beta \\ \gamma^2 & \mp 2\alpha\gamma & \alpha^2 \end{bmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix}$$

Notice that if $\beta_1(x) = \beta_1(y)$ and $\alpha_{1y} = -\alpha_{1x}$

then $\beta_2(x) = \beta_2(y)$ and $\alpha_{2y} = -\alpha_{2x}$

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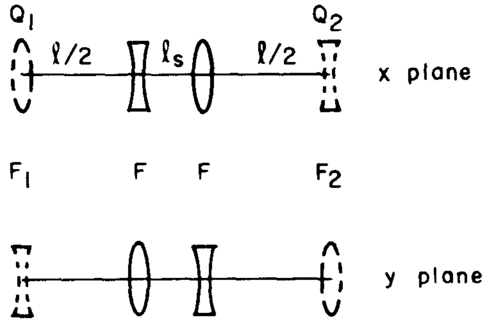
Fig. 7

A VARIABLE $\pi/2$ PHASE ELLIPSE TRANSFORMER



$$\sin\frac{\mu}{2} = \frac{\sqrt{\ell \ell_s}}{2F}\ , \qquad \mu_{x,y} = \frac{\pi}{2}$$

$$R_{x,y} = \begin{bmatrix} \pm\alpha & \beta \\ -\gamma & \mp\alpha \end{bmatrix}, \quad \text{Trace } R = 2\cos\mu$$

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{bmatrix} \alpha^2 & \mp 2\alpha\beta & \beta^2 \\ \pm\alpha\gamma & -(1+2\alpha^2) & \pm\alpha\beta \\ \gamma^2 & \mp 2\alpha\gamma & \alpha^2 \end{bmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix}$$

The above equations apply when Q_1 and Q_2 are turned off.

Energizing Q_1 varies α_1 where $\Delta\alpha_1 = \pm\left(\frac{\beta_1}{F_1}\right)$

Q_2 varies α_2 $\Delta\alpha_2 = \mp\left(\frac{\beta_2}{F_2}\right)$

It follows that:

If $\alpha_{1x} = -\alpha_{1y}$ and $\beta_{1x} = \beta_{1y}$

Then $\alpha_{2x} = -\alpha_{2y}$ and $\beta_{2x} = \beta_{2y}$

The allowable range of variation for β_2 is:

$$\beta_2 \geq \frac{\beta^2}{\beta_1} \quad \text{i.e.,} \quad \beta_2(\text{minimum}) = \frac{R_{12}^2}{\beta_1}$$

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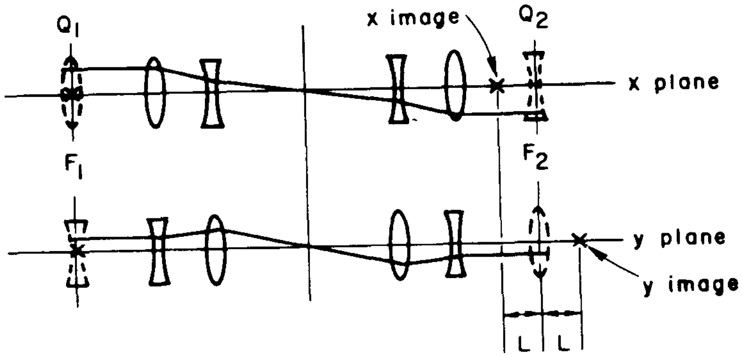
Fig. 8

VARIABLE TELESCOPIC PHASE ELLIPSE TRANSFORMERS



$$R_{x,y} = \begin{bmatrix} -1 & \pm L \\ 0 & -1 \end{bmatrix}$$

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{bmatrix} 1 & \pm 2L & L^2 \\ 0 & 1 & \pm L \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix}$$

The above equations apply when Q_1 and Q_2 are turned off.

Energizing Q_1 varies α_1 where $\Delta\alpha_1 = \pm\left(\frac{\beta_1}{F_1}\right)$

Q_2 varies α_2 $\Delta\alpha_2 = \mp\left(\frac{\beta_2}{F_2}\right)$

It follows that:

If $\alpha_{1x} = -\alpha_{1y}$ and $\beta_{1x} = \beta_{1y}$

Then $\alpha_{2x} = -\alpha_{2y}$ and $\beta_{2x} = \beta_{2y}$

The allowable range of variation for β_2 is:

$$\beta_2 \geq \frac{L^2}{\beta_1} \quad \text{i.e.,} \quad \beta_2(\text{minimum}) = \frac{R_{12}^2}{\beta_1}$$

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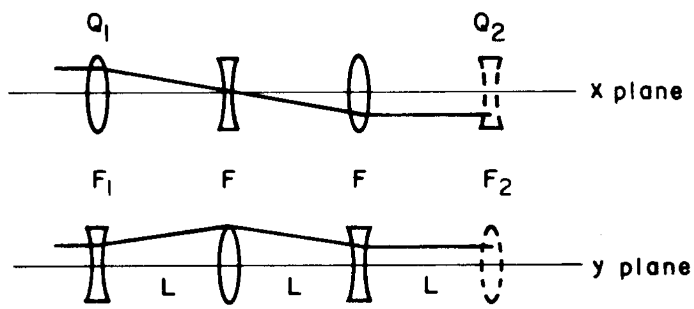
Fig. 9

A PERIODIC QUADRUPOLE ARRAY PHASE ELLIPSE MATCHING



$$R_{x,y} = \begin{bmatrix} \mp 1 & 2L \\ 0 & \mp 1 \end{bmatrix}$$

$L = F$

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_1 \end{pmatrix} = \begin{bmatrix} 1 & \pm 4F & 4F^2 \\ 0 & 1 & \pm 2F \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix}$$

The above equations apply when Q_2 is off and Q_1 is set to focal length $F = L$.

Varying Q_1 varies α_1

Varying Q_2 varies α_2

If $\alpha_{1x} = -\alpha_{1y}$ and $\beta_{1x} = \beta_{1y}$

Then $\alpha_{2x} = -\alpha_{2y}$ and $\beta_{2x} = \beta_{2y}$ when Q_1 and Q_2 are varied

The available range of variation for β_2 is

$$\beta_2 \geqq \frac{4L^2}{\beta_1} \quad \text{i.e.,} \quad \beta_2(\text{minimum}) = \frac{R_{12}^2}{\beta_1}$$

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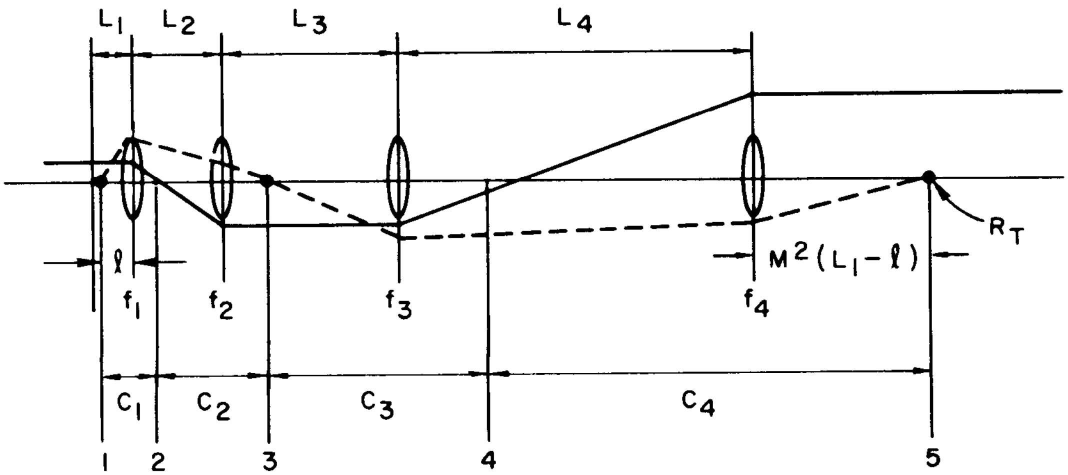
Fig. 10

A MAGNIFYING TELESCOPIC SYSTEM WHERE

THE PHASE SHIFT PER CELL $\mu_c = \pi/2$



$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha \quad , \quad \Delta\psi = 4\mu_c = 2\pi$$

$$L_N = \left(f_N + f_{N-1}\right) \quad , \quad \frac{1}{F_N} = \frac{1}{L_N} + \frac{1}{L_{N+1}}$$

$$\left(\frac{L_{N+1}}{L_N}\right) = \left(\frac{C_{N+1}}{C_N}\right) = \left(\frac{F_{N+1}}{F_N}\right) = \left(\frac{\beta_{N+1}}{\beta_N}\right) = r^2$$

$$R_T = \begin{bmatrix} r^4 & 0 \\ 0 & \frac{1}{r^4} \end{bmatrix} = \begin{bmatrix} M_T & 0 \\ 0 & \frac{1}{M_T} \end{bmatrix}$$

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Fig. 11