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DUALITY FOR NON-ABELIAN LATTICE FIELDS*

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ABSTRACT

It is shown that no non-Abelian spin system or gauge theory on a lattice is self-dual, unless the model can be rewritten as an Abelian system. Some examples of such effectively Abelian models are given. In addition, for both discrete groups and Lie groups with non-trivial centers dual local field theories are constructed. For the gauge theory the dual order parameter has the 't Hooft commutation relations with the Wilson loop.

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I. Introduction

In recent years many authors have investigated the duality properties of Z(N) spin systems and gauge theories on a lattice [1]-[7]. Duality transformations are powerful tools for examining the behavior of order parameters of systems. For a spin system the appropriate order parameter is the spin expectation value, while for a gauge theory it is the Wilson loop. In this paper duality transformations will be constructed for non-Abelian 1 + 1 dimensional spin systems and 3 + 1 dimensional gauge theories in the discrete space/continuous time Hamiltonian formulation. In particular it will be shown that systems based on finite non-Abelian groups cannot be self-dual unless they are effectively invariant under an Abelian symmetry, and a local dual field theory can be constructed whose order parameter is the Nielsen-Olsen-t' Hooft loop [8],[9].

Cardy [10], Elitzur, Pearson, and Shigemitsu [11], Horn, Weinstein, and Yankielowicz [12], and Ukawa, Windey, and Guth [13] have investigated the phases of the Z(N) models, and have found that as N increases their phase diagrams greatly resemble those of U(1) models. This suggests that finite non-Abelian gauge theories may parallel Yang-Mills theories when the order of the group is large enough (for example the polyhedral groups may tell us something about O(3)). For this reason it seems worthwhile to understand the duality properties of finite non-Abelian models, since this will shed some light on their confinement properties.

Mandelstam [9] suggested that there exists a field theory dual to Yang-Mills whose order parameter is the Nielsen-Olsen-t' Hooft loop. It will be shown here that this is a property of any <u>local</u> dual theory.

Duality transformations are a means of describing a model by a new system whose degrees of freedom are on the dual lattice. The dual of a 1-dimensional lattice has its sites midway between the sites of the original lattice. In 3-dimensions, the dual lattice has its sites at the centers of the cubes of the original lattice.

Consider a Hamiltonian

$$H(\Gamma,\Omega\,;\lambda) \;=\; \sum_x \Big\{ T[\Gamma(x)] + \lambda V[\Omega(x)] \Big\} \quad , \tag{1a}$$

where x denotes a point on the lattice (either a site or a link), $\Omega(x)$ is the field operator, and $\Gamma(x)$ is some other operator which does not commute with $\Omega(x)$ at the same point. Now suppose there is a second Hamiltonian

$$h(\gamma,\omega\,;\lambda) \;=\; \sum_x \Big\{ t[\gamma(x)] + \lambda V[\omega(x)] \Big\} \quad , \tag{1b}$$

and a unitary operator D such that

$$DT[\Gamma(x)]D^{+} \;=\; v[\omega(x)] \;, \quad DV[\Omega(x)]D^{+} \;=\; t[\gamma(x)] \;. \tag{2}$$

If T,V,t,v are all polynomial functions

$$T[D\Gamma(x)D^{+}] \;=\; v[\omega(x)] \;, \quad V[D\Omega(x)D^{+}] \;=\; t[\gamma(x)] \;. \tag{3}$$

Define the transformation

$$\mathscr{D} \;=\; dD \quad , \tag{4}$$

where d displaces quantities by a half-unit in each coordinate, for example

$$df(x) \;=\; f\left(x - \frac{\epsilon}{2}\right) \quad , \tag{5}$$

in 1-dimension. The dual variables are

$$\Gamma_D\left(x+\frac{\varepsilon}{2}\right) = \mathscr{D}\Gamma(x)\mathscr{D}^{-1} \quad , \quad \Omega_D\left(x+\frac{\varepsilon}{2}\right) = \mathscr{D}\Omega(x)\mathscr{D}^{-1} \quad , \tag{6}$$

and

$$h(\gamma,\omega\,;\lambda) = \lambda H(\Gamma_D,\Omega_D\,;1/\lambda) \quad . \tag{7}$$

If this is the case the two Hamiltonians (1a) and (1b) are said to be dual to one another. The practical value of (7) for statistical mechanical systems is that λ_c is a critical point for the model (1a) if and only if $1/\lambda_c$ is a critical point for (1b). If H and h are the same Hamiltonian, the system is called <u>self-dual</u>. Systems invariant under continuous compact groups, such as non-linear sigma models and Yang-Mills theories, are never self-dual, since their kinetic terms have discrete eigenvalues, whereas their potentials have continuous spectra.

In the next section Hamiltonians for spin systems and gauge theories with any finite group will be constructed. Section III is devoted to duality for both these models, and Lie group systems.

II. <u>Hamiltonians</u>

The most general globally invariant Lagrangian for a 2-dimensional spin system is a function of a group element field $g(\bar{x}) \in G$ and may be written as

$$\mathscr{L}(\bar{x}) = -\sum_{k=1}^{\ell}\sum_{\mu=0}^{1} \beta_\mu a_k \chi_F[g(x)^{-1}g(x+\varepsilon_\mu\hat{n}_\mu)]^k + \text{h.c.} \quad , \tag{8a}$$

where $\varepsilon_0,\hat{n}_0$ and $\varepsilon_1,\hat{n}_1$ are the lattice spacings and unit vectors in the time and space directions, respectively, ℓ is some finite integer, and χ_F is the character function (trace) in the lowest dimensional faithful

representation. The positive coefficients a_k are the relative couplings which characterize the theory, and β_μ are the time and space inverse temperatures. For the lattice gauge theory in 4-dimensions, the Lagrangian is

$$\mathscr{L}(\bar{x}) = -\sum_{k=1}^{\ell} \sum_{\substack{\mu,\nu=0 \\ \mu\neq\nu}}^{3} \beta_{\mu\nu} a_k \chi_F \Big[g(\bar{x},\hat{n}_\mu) g(\bar{x}+\varepsilon_\mu \hat{n}_\mu, \hat{n}_\nu) g(\bar{x}+\varepsilon_\mu \hat{n}_\nu, \hat{n}_\mu)^{-1} g(x,\hat{n}_\nu)^{-1} \Big]^k + \text{h.c.} \quad , \tag{8b}$$

where the group elements are now located on the links of the lattice, $\beta_{io} = \beta_{oi} = \beta_t$, $\beta_{ij} = \beta_s$ are the time and space temperatures, respectively, and $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$.

In order to find the Hamiltonians for (8a) and (8b), it is convenient to use the regular representation of the group. Suppose the elements are labeled by g_a, $a = 1,\ldots,N$. The <u>regular representation</u> is an N-dimensional representation defined by

$$U_a^{bc} = \begin{cases} 1, & g_b = g_a g_c \\ 0, & g_b \neq g_a g_c \end{cases} \quad , \quad U_a^T = U_a^{-1} \tag{9}$$

The spin system Lagrangian (8a) may be rewritten as

$$\mathscr{L}(\bar{x}) = -\frac{1}{N} \sum_{\mu=0}^{1} \sum_{a,b,c,d} \sum_{k=1}^{\ell} \beta_\mu \Big[a_k \chi_F (g_c)^k + \text{h.c.} \Big] U_c^{ba} U^{da}(\bar{x}) U^{db}(\bar{x}+\varepsilon_\mu \hat{n}_\mu)$$

$$= -\sum_{k} \sum_{k=1}^{\ell} \Big[a_k \chi_F (g_c)^k + \text{h.c.} \Big] \sum_{\mu=0}^{1} \beta_\mu \, \delta_{[g_c, g(\bar{x})^{-1} g(\bar{x}+\varepsilon_\mu \hat{n}_\mu)]} \quad , \tag{10}$$

which is a linear combination of double-valued quantities. If each one of these quantities is regarded as a Lagrangian, the corresponding

Hamiltonian may be constructed through the use of the transfer matrix [14]. It is not difficult to show that the transformation yielding Hamiltonians from Lagrangians (the discrete Legendre transformation) is linear. Consequently resumming gives the spin system Hamiltonian (discrete space/continuous time)

$$H = -\frac{1}{N}\sum_{c}\left[\sum_{k=1}^{\ell} a_k \chi_F(g_c)^k + \text{h.c.}\right]\sum_{x}\left[\sum_{f\neq c} G^f(x) + \lambda\sum_{abg} U_c^{ab}\,\Omega^{ga}(x)\,\Omega^{gb}(x+\varepsilon)\right]$$

$$= -\sum_{f} d^f \sum_{x} G^f(x) - \lambda \sum_{abg} A^{ab} \sum_{x} \Omega^{ga}(x)\,\Omega^{gb}(x+\varepsilon) \quad , \qquad (11)$$

where x_1, ε_1 have been relabeled by x, ε, the number λ is proportional to the inverse square of the coupling constant, and the local operators $G^f(x)$ and $\Omega^{ab}(x)$ are defined by

$$G^f|g_d\rangle = |g_f\, g_d\rangle \quad , \qquad (12a)$$

$$\Omega^{ab}|g_d\rangle = U_d^{ab}|g_d\rangle \quad , \qquad (12b)$$

where $|g_d\rangle$ is a local Hilbert space vector at a site. The operator $G^f(x)$ may be regarded as a "spin flipper".

To obtain the Hamiltonian for (8b), it is easiest to work in $A_0 = 0$ gauge ($g(\bar{x},\hat{n}_0) = I$). This forces the Lagrangian of a time-space oreinted plaquette to have the same form as (8a). Thus the kinetic term is the same as that in (11), and the Hamiltonian is

$$H = -\sum_{f} d^f \sum_{x,i} G^f(\bar{x},\hat{n}_i) - \lambda \sum_{abcde} A^{ab} \sum_{x,i\neq j} \Omega^{ac}(\bar{x},\hat{n}_i)\,\Omega^{cd}(\bar{x}+\varepsilon\hat{n}_j\,,\,\hat{n}_k)$$

$$\times\ \Omega^{ed}(\bar{x}+\hat{n}_k\,,\,\hat{n}_j)\,\Omega^{be}(\bar{x}\,,\,\hat{n}_k) \qquad (13)$$

For (11) a global gauge transformation is

$$\mathcal{G}^f = \prod_x G^f(x) \quad . \tag{14}$$

The physical states must be eigenstates of (14). The gauge theory Hamiltonian (13) must be augmented by Gauss' law (because of the gauge condition). This is just the statement of the invariance of physical states under spatial gauge transformations. The basic spatial gauge transformations are

$$\mathcal{G}^f(x) = \prod_i G^f(\bar{x}, \hat{n}_i)\, G^f(\bar{x}, -\hat{n}_i) \quad . \tag{15}$$

This establishes the strong coupling (small λ) states as loops and strings of excited links.

The form of (11) and (13) can be made more compact by defining

$$B^{ab} = \delta^{ab}\, d^a + \frac{\sum_c d^c}{\sum_e A^{ee}} \left[A^{ab} - \delta^{ab} A^{aa}\right] \quad , \tag{16}$$

and making the replacement

$$\lambda \to \frac{\sum_a d^a}{\sum_e A^{ee}}\, \lambda \quad , \tag{17}$$

so that (11) becomes

$$H = -\sum_{ab} B^{ab} \sum_x \left[\delta^{ab}\, G^a(x) + \lambda \sum_c \Omega^{ca}(x)\, \Omega^{cb}(x+\epsilon)\right] \quad , \tag{18}$$

and (13) is

$$H = -\sum_{ab} B^{ab} \left[\sum_{\text{links}} \delta^{ab}\, G^a + \lambda \sum_{\text{plaquettes}} \sum_{cde} \Omega^{ac}\, \Omega^{cd}\, \Omega^{ed}\, \Omega^{bc}\right]. \tag{19}$$

The order parameter for the spin system is the magnetization

$$M(\lambda) = {}_\lambda\langle 0|\sum_{ab} B^{ab}\,\Omega^{ab}(x)|0\rangle_\lambda \quad , \tag{20}$$

where $|0\rangle_\lambda$ is the ground state for a particular value of λ. Unless global gauge invariance is spontaneously broken $M(\lambda)$ is zero. This can be seen by averaging over global gauge transformations (14) and from the fact that the sum over all the group elements of the character of an irreducible representation vanishes. If $M(\lambda)$ is zero the system is said to be in the disordered phase. Otherwise it is (at least partially) in the ordered phase. The boundary between these phase λ_c is where a phase transition occurs (there may in general be more than two phases). For the gauge theory, the order parameter is the vacuum expectation value of the Wilson loop

$$M(c,\lambda) = {}_\lambda\langle 0|A(c)|0\rangle_\lambda = {}_\lambda\langle 0|\mathrm{Tr}\Big[B\prod_{m\in c}\Omega(m)\Big]|0\rangle_\lambda \quad , \tag{21}$$

where m is a link of the closed loop c. In strong coupling perturbation theory, the first non-vanishing term is of order λ^n where n is the minimal number of plaquettes of a surface enclosed by c. This is just the Wilson criterion for confinement.

III. Duality

It is now straight forward to test these models for self-duality. A necessary condition for self-duality is that the spectrum of the kinetic term is the same as that of the potential. Therefore there exists an orthogonal operator S such that

$$S^+\sum_a B^{aa}\,G^a\,S = \sum_{ab} B^{ab}\,\Omega^{ab} + \alpha\mathbb{1} \quad . \tag{22}$$

where α is some real constant. As an example of this situation, if the a_k's in (8a,b) are adjusted so that the Lagrangian is the trace in the regular representation, we have $B^{ab} = 1$ for all a,b and $= -1$. In the case of (8a) this is the Potts model [15,16]. Define

$$\Gamma^{ab} = S\,\Omega^{ab}\,S^{+} \quad . \tag{23}$$

For an Abelian group Γ^{ab} differs from $\delta^{ab}G^{a}$ by a constant, since all the G^{a}'s may be simultaneously diagonalized. The spin system Hamiltonian may be written from (22) and (23) as

$$H = -\sum_{a,b} B^{ab} \sum_{x}\left[\Gamma^{ab}(x) + \lambda \sum_{c} \Omega^{ca}(x)\,\Omega^{cb}(x+\varepsilon)\right] . \tag{24}$$

A general eigenstate of the potential term is

$$|\{g(x)\}, \lambda = \infty\rangle = \bigotimes_{x} |g(x)\rangle \quad , \tag{25a}$$

where $g(-\infty) = g(\infty) = I$ is imposed as a boundary condition. An eigenstate of the kinetic term with the same eigenvalue as (25a) is

$$|\{g(x)\ , \lambda = 0\rangle = \bigotimes_{x} S|g(x)^{-1}g(x)\rangle \quad . \tag{25b}$$

Now define

$$\mathscr{D} = d\left[\sum_{\{g(x)\}} \bigotimes_{x} S|g(x)^{-1}g(x+\varepsilon)\rangle\langle g(x)|\right] . \tag{26}$$

Using (26) it is simple to define variables on the dual lattice

$$\omega^{ab}\left(x + \frac{\varepsilon}{2}\right) = \mathscr{D}\Omega^{ab}(x)\mathscr{D}^{-1} = \prod_{x' \le x} \Gamma^{ab}(x') \ , \tag{27a}$$

$$\gamma^{ab}\left(x + \frac{\varepsilon}{2}\right) = \mathscr{D}\Gamma^{ab}(x)\,\mathscr{D}^{-1} \quad . \tag{27b}$$

Then

$$\sum_{abc} B^{ab}\ \omega^{ca}\left(x+\frac{\varepsilon}{2}\right)\omega^{cb}\left(x+\frac{\varepsilon}{2}\right) = \sum_{a} B^{aa}\ G^{a}(x) \quad . \qquad (28)$$

In general, unless the model is self-dual

$$\sum_{ab} B^{ab}\ \gamma^{ab}\left(x+\frac{\varepsilon}{2}\right) \neq \sum_{abc} B^{ab}\ \Omega^{ca}(x)\ \Omega^{cb}(x+\varepsilon) \quad , \qquad (29)$$

however a variable $\gamma'^{ab}\left(x+\frac{\varepsilon}{2}\right)$ may be defined,

$$\gamma'^{ab}\left(x+\frac{\varepsilon}{2}\right) = \mathscr{D}^{-1}\ \Gamma^{ab}(x+\varepsilon)\mathscr{D} = \sum_{c}\Omega^{ca}(x)\ \Omega^{cb}(x+\varepsilon), \qquad (30)$$

such that

$$\sum_{ab} B^{ab}\ \gamma'^{ab}\left(x+\frac{\varepsilon}{2}\right) = \sum_{abc} B^{ab}\ \Omega^{ca}(x)\ \Omega^{cb}(x+\varepsilon) \quad . \qquad (31)$$

Now suppose the theory is non-Abelian, then ω will not have local commutation relations with γ'. This is because the form (27a) is distinguishable from a global gauge transformation at points different from x. Consequently ω will not have local commutation relations with the potential term, unless the model is <u>effectively invariant under an Abelian group</u>. This means it is possible to rewrite the Lagrangian as an Abelian Lagrangian. However, Ω^{ab} certainly has local commutation relations with the kinetic term. Therefore self-duality implies that the model is effectively Abelian.

A trivial example of an effectively Abelian theory is the Potts model. This model is not characterized by anything pertaining to the group except its order (the action is just a Kronecker delta in the group indices). Consequently, if the group is non-Abelian, it may be replaced by an Abelian group. Bellissard [17], Drouffe, Itzykson and

Zuber [18], and Monastyrsky and Zamolodchikov [19] have shown that any spin system invariant under a solvable group is an effectively Abelian model. A group G is solvable if it contains a chain of subgroups

$$G = G_0 \supset G_1 \supset \ldots \supset G_k = I \tag{32}$$

such that each G_ℓ is normal in $G_{\ell-1}$ (that is, G_ℓ is invariant under similiarity transformations in $G_{\ell-1}$) and $G_{\ell-1}/G_\ell$ is an Abelian group. These groups include S(3), S(4) (the permutation groups of three and four objects respectively), the octahedral group, and any group of odd order. A spin system invariant under a solvable group may always be written as one based on

$$G' = G_0/G_1 \otimes \ldots \otimes G_{k-2}/G_{k-1} \otimes G_{k-1} \quad . \tag{33}$$

The duality transformation had earlier been found for A(4) = S(4)/Z(2) by Dotsenko [20].

For an Abelian system the criterion for self-duality is

$$\sum_a B^{aa} \langle \{g(x)'\}, \lambda = \infty | G^a(x) | \{g(x)\}, \lambda = \infty \rangle \tag{34}$$

$$= \sum_{abc} B^{ab} \langle \{g(x)'\}, \lambda = 0 | \Omega^{ca}(x)\ \Omega^{cb}(x+\epsilon) | \{g(x)\}, \lambda = 0 \rangle + \text{const.}$$

which is satisfied if $\left[S^2, \Omega^{ab}\right] = 0$.

These results simply generalize to the case where (22) is replaced by the condition that the kinetic term of a spin system has the same spectrum as the potential of another spin system based on the same group and vice-versa, which is neccessary for mutual duality. For the same reasons as those outlined above, intrinsically non-Abelian spin systems are not dual to other systems with the same group.

The above analysis applies to gauge theories as well. It is simplest to carry out the procedure if an additional gauge restriction $A_3(t = t_0) = 0$ is imposed. The problem of non-locality persists in the non-Abelian case, accompanied by further difficulties in requiring the dual states to satisfy Gauss' law as well as the Bianchi identity. Again if the model is effectively Abelian these troubles disappear.

For the solvable groups there are effectively Abelian gauge systems, although they exist only for certain values of the relative couplings. If the a_k's are chosen properly the model is explicitly broken to a system with the gauge group $G_{\ell-1}/G_\ell$. Two examples will be given here. Consider S(3), whose breakdown as in (32) is

$$S(3) \supset Z(2) \supset I \quad , \qquad S(3)/Z(2) = Z(3) \quad . \tag{35}$$

The Lagrangian (8b) for this model has $\ell = 3$, and χ_F is the trace in the two dimensional representation. For the choice $a_1 = a_3 = 0$, $a_2 = 1$, this is a Z(3) model. This becomes apparent if local degeneracies in the potential are summed over. The group has six elements $S(3) = \{I, \omega, \omega^2, r, r\omega, r\omega^2\}$, where ω is a Z(3) rotation and r is the reflection (this may be seen geometrically as the set of relative couplings the local Hilbert space states $|I\rangle$, $|r\rangle$, $|r\omega\rangle$, $|r\omega^2\rangle$ may be summed over. The second set of relative couplings which renders the invariance Abelian is $a_1 = a_2 = 0$, $a_3 = 1$. This is a Z(2) system. A more interesting example is the group of the tetrahedron

$$S(4) \supset A(4) \supset V(4) \supset I \quad , \qquad S(4)/A(4) = Z(2) \quad ,$$

$$A(4)/V(4) = Z(3) \quad , \tag{36}$$

where V(4) is the dihedral group of order 4, $V(4) = \{I, q_1, q_2, q_3\}$,

$q_i^2 = I$, $q_i q_j = q_k$ if $i \neq j \neq k \neq i$. In this case $\ell = 6$. The choices (i) $a_1 = a_2 = a_3 = a_4 = a_5 = 0$, $a_6 = 1$, (ii) $a_1 = a_3 = a_4 = a_5 = a_6 = 0$, $a_2 = 1$, and (iii) $a_1 = a_2 = a_4 = a_5 = a_6 = 0$, $a_3 = 1$, give (i) Z(2), (ii) Z(3), and (iii) V(4) invariances, respectively.

Though it is not possible to find a self-dual intrinsically non-Abelian model, it is still interesting to try to define dual fields which develope an expectation value in the disordered phase. A first guess might be to try to define objects like

$$\psi^a\left(x + \frac{\varepsilon}{2}\right) = \prod_{x' \leq x} G^a(x) \qquad (37)$$

as the dual fields for the spin system. The operator $\psi^a\left(x + \frac{\varepsilon}{2}\right)$ commutes with the kinetic term, and produces a soliton when acting on a classical state. Thus it develops an expectation value in disordered phase, and has a particle interpretation in the ordered phase. In fact, for Abelian systems these operators are just what is needed [3,7,16]. For non-Abelian systems, however, they do not possess local commutativity. Hence wave functionals of these operators are not defined. So even if the Hamiltonian can be rewritten in terms of $\psi^a\left(x + \frac{\varepsilon}{2}\right)$ the theory does not exist!

The correct dual fields must satisfy the following criteria:

i) They violate the boundary conditions of classical states.

ii) They have local commutation relations with the potential term.

iii) They have local commutation relations with each other.

iv) It must be possible to construct the kinetic term from them.

First consider spin systems. In order to satisfy i) and ii), the dual field operator must be indistinguishable from a global gauge transforma-

tion at large distances. Because of iii) the only acceptable global gauge transformations correspond to elements of an Abelian subgroup. Here the requirement iv) will be strengthened to:

iv)' The dual fields may be used to construct $G^f(x)$.

This is not essential for spin systems, but is needed for gauge theories, where Gauss' law must be imposed as a constraint.

It is impossible to satisfy i), ii), iii) and iv)' unless the group has a non-trivial center. With iv)', global gauge transformations not corresponding to the center are unacceptable. For simplicity, assume that the group is a subgroup of O(3) with center Z(2). Then the simplest choice of a dual field obeying the above conditions is

$$\phi^a\left(x + \frac{\varepsilon}{2}\right) = G^a(x) \prod_{x' < x} G^r(x') \quad , \tag{38}$$

where g_r is the reflection element. The Hamiltonian (18) in terms of these fields is

$$H = -\lambda \sum_{ab} B^{ab} \sum_x \left[V^{ab}\left(x + \frac{\varepsilon}{2}\right) + \frac{1}{\lambda}\, \delta^{ab}\, \phi^r\left(x - \frac{\varepsilon}{2}\right) \phi^a\left(x + \frac{\varepsilon}{2}\right) \right] \tag{39}$$

where

$$V^{ab}\left(x + \frac{\varepsilon}{2}\right) = \sum_c \Omega^{ca}(x)\, \Omega^{cb}(x+\varepsilon) \quad . \tag{40}$$

When viewed as an operator on wave functions of ϕ^a, the relevant feature of V^{ab} is its commutation relation with the dual field,

$$V^{ab}\left(x+\frac{\varepsilon}{2}\right)\phi^{c}\left(y+\frac{\varepsilon}{2}\right) = \delta_{xy}\sum_{d} U_{c}^{ad}\,\phi^{c}\left(y+\frac{\varepsilon}{2}\right)V^{db}\left(x+\frac{\varepsilon}{2}\right)$$

$$+\ \delta_{xy-\varepsilon}\sum_{de} U_{r}^{ae}\,U_{c}^{ed}\,\phi^{c}\left(y+\frac{\varepsilon}{2}\right)V^{db}\left(x+\frac{\varepsilon}{2}\right)\quad . \tag{41}$$

The physical states must satisfy a dual global invariance

$$\frac{1}{N}\,\mathrm{Tr}\left[B\prod_{x}V\left(x+\frac{\varepsilon}{2}\right)\right]\Big|\,\Psi\Big\rangle = |\,\Psi\,\rangle\quad . \tag{42}$$

For (19) the appropriate dual fields are

$$\phi_{i}^{a}\left[\bar{x}+\frac{\varepsilon}{2}\,(\hat{n}_{i}-\hat{n}_{j}+\hat{n}_{k})\,,\,\hat{n}_{j}\right] = G^{a}(\bar{x},\hat{n}_{k})\prod_{\ell=1}^{\infty}G^{r}(\bar{x}-\ell\varepsilon\hat{n}_{i}\,,\,\hat{n}_{k})\quad , \tag{43}$$

where (i,j,k) is an even permutation of (1,2,3). The Hamiltonian is

$$H = -\lambda\sum_{a,b}B^{ab}\sum_{x}\left\{\sum_{k}V^{ab}\left[x+\frac{\varepsilon}{2}\,(\hat{n}_{1}+\hat{n}_{2}+\hat{n}_{3}),\hat{n}_{k}\right]\right. \tag{44}$$

$$\left.+\ \frac{1}{2\lambda}\sum_{i\neq j}\phi_{i}^{r}\left[x+\frac{\varepsilon}{2}\,(\hat{n}_{1}+\hat{n}_{2}+\hat{n}_{3}),\hat{n}_{j}\right]\phi_{i}^{a}\left[\bar{x}+\frac{\varepsilon}{2}\,(\hat{n}_{1}+\hat{n}_{2}+\hat{n}_{3})+\varepsilon\hat{n}_{i},\hat{n}_{j}\right]\right\}$$

where

$$V^{ab}\left[\bar{x}+\frac{\varepsilon}{2}\,(\hat{n}_{i}+\hat{n}_{j}-\hat{n}_{k}),\hat{n}_{k}\right] = \sum_{cde}\Omega^{ac}(\bar{x},\hat{n}_{i})\;\Omega^{cd}(\bar{x}+\varepsilon\hat{n}_{i},\hat{n}_{j})$$

$$\times\,\Omega^{ed}(\bar{x}+\varepsilon\hat{n}_{j},\hat{n}_{i})\;\Omega^{be}(\bar{x},\hat{n}_{j})\quad , \tag{45}$$

and again (i,j,k) is an even permutation of (1,2,3). The commutation relations between V^{ab} and ϕ^{c} is similiar to (41),

$$V^{ab}\left[\bar{x} + \frac{\varepsilon}{2}(\hat{n}_1+\hat{n}_2+\hat{n}_3), \hat{n}_i\right] \phi^c\left[\bar{y} + \frac{\varepsilon}{2}(\hat{n}_1+\hat{n}_2+\hat{n}_3), \hat{n}_j\right]$$

$$= \delta_{ij} \sum_d \left(\delta_{\bar{x}\bar{y}}\, U_c^{ad} - \delta_{\bar{x}\bar{y}} - \varepsilon\hat{n}_k \sum_e U_r^{ae}\, U_c^{ed}\right) \phi^c\left[y + \frac{\varepsilon}{2}(\hat{n}_1+\hat{n}_2+\hat{n}_3), \hat{n}_j\right]$$

$$\times\ V^{ab}\left[\bar{x} + \frac{\varepsilon}{2}(\hat{n}_1+\hat{n}_2+\hat{n}_3), \hat{n}_i\right] \quad . \tag{46}$$

The Bianchi identity becomes the dual local invariance

$$\frac{1}{N} \mathrm{Tr}\left[B \prod_i V(\bar{x}, \hat{n}_i)\ V(\bar{x}, \hat{n}_j)\right] \Big| \Psi \Big\rangle = |\Psi\rangle \quad . \tag{47}$$

Gauss' law gives a constraint on the dual fields

$$\prod_{ijk} \phi_i^r\left[\bar{x} + \frac{\varepsilon}{2}(\hat{n}_1+\hat{n}_2+\hat{n}_3) - \varepsilon\hat{n}_i, \hat{n}_j\right] \phi_i^a\left[\bar{x} + \frac{\varepsilon}{2}(\hat{n}_1+\hat{n}_2+\hat{n}_3), \hat{n}_j\right] = 1. \tag{48}$$

The dual order parameter, or disorder parameter, is

$$N^a(c',\lambda) = {}_\lambda\langle 0|B^a(c')|0\rangle_\lambda = {}_\lambda\left\langle 0 \,\middle|\, \prod_{m'\varepsilon c'} \phi^a(m') \,\middle|\, 0\right\rangle_\lambda \tag{49}$$

where c' is a closed curve on the dual lattice. The object $B^a(c')$ is the simplest that can be defined which is invariant under (43). If the invariance (47) is not spontaneously broken, i.e., the phase is completely disordered, the states will contain loops and strings of excited vortices (magnetic confinement), produced by operators such as $B^a(c')$ acting on the ground state. If c and c' are two closed paths such that no link in c is a distance $\frac{\varepsilon}{2}$ from any link in c', then

$$A(c)\ B^a(c') = rB^a(c')\ A(c) \tag{50a}$$

if c' closes through c and

$$A(c)\ B^a(c') = B^a(c')\ A(c) \tag{50b}$$

if c' doesn't close through c. This results from requirements i), ii), iii), iv)'. Equations (50a,b) are the commutation relations of 't Hooft [8].

All of the discussion (37)-(50a,b) applies to continuous compact groups as well as finite groups. The only formal difference is the summation over group indices is replaced by group invariant integration over group parameters (coefficients of generators). This is needlessly cumbersome however, and the discussion here will be in a notation better suited to Lie groups.

For the O(3) non-linear sigma model the Hamiltonian is

$$H = -\frac{1}{2\varepsilon}\sum_x \left\{ \sum_\alpha \frac{g^2}{\mu^4 \varepsilon^4} G^\alpha + \frac{1}{g^2} \mathrm{Tr}\left[\Omega(x)^\dagger \, \Omega(x+\varepsilon)\right] \right\} + \text{h.c.} \,, \quad (51)$$

where $\alpha = 1,2,3$, Tr is the trace in the 3-dimensional representation, μ^2 is the mass scale, and

$$G^\alpha(x) = e^{-i\mu^2 \varepsilon / g (d/dA^\alpha(x))} \,, \quad (52a)$$

and

$$\Omega(x) = e^{-ig\varepsilon A^\alpha T^\alpha} \,, \quad (52b)$$

where T^α are the group generators. The dual field is

$$\phi^\alpha\left(x + \frac{\varepsilon}{2}\right) = G^\alpha(x) \prod_{x' < x} G^r(x') \equiv G^\alpha(x) \, \phi^r\left(x - \frac{\varepsilon}{2}\right) \quad (53)$$

where G^r is the same as before. Defining analogously to (40)

$$V\left(x + \frac{\varepsilon}{2}\right) = \Omega(x)^\dagger \, \Omega(x+\varepsilon) \,, \quad (54)$$

the Hamiltonian may be rewritten as

$$H = -\frac{1}{2\varepsilon g^2}\sum_x \left\{ \mathrm{Tr}\left[V\left(x + \frac{\varepsilon}{2}\right)\right] + \frac{g^4}{\mu^4 \varepsilon^4} \sum_\alpha \phi^r\left(x - \frac{\varepsilon}{2}\right) \phi^\alpha\left(x + \frac{\varepsilon}{2}\right) \right\} + \text{h.c.}, \quad (55)$$

with the commutation relations

$$V\left(x+\frac{\varepsilon}{2}\right)\phi^{\alpha}\left(y+\frac{\varepsilon}{2}\right) = (\delta_{xy}-\delta_{xy-\varepsilon})\, e^{-\mu^2\varepsilon^2 T^{\alpha}}\,\phi\left(y+\frac{\varepsilon}{2}\right)V\left(x+\frac{\varepsilon}{2}\right) . \tag{56}$$

The dual global invariance is

$$\frac{1}{3}\,\mathrm{Tr}\left[\prod_{x} V\left(x+\frac{\varepsilon}{2}\right)\right]\Big|\,\Psi\Big\rangle = |\,\Psi\rangle \quad . \tag{57}$$

The O(3) gauge Hamiltonian is

$$H = -\frac{1}{4\varepsilon}\sum_{x,ijk}\left\{\frac{g^2}{\mu^4\varepsilon^4}\,G^{\alpha}+\frac{1}{g^2}\,\mathrm{Tr}\Big[\Omega(\bar{x},\hat{n}_j)\;\Omega(\bar{x}+\varepsilon\hat{n}_j,\hat{n}_k)\right.$$
$$\left.\times\;\Omega(\bar{x}+\varepsilon\hat{n}_k,\hat{n}_j)^{\dagger}\;\Omega(\bar{x},\hat{n}_k)^{\dagger}\Big]\right\}+\mathrm{h.c.}\quad , \tag{58}$$

where, as before, (i,j,k) is an even permutation of (1,2,3). Defining

$$\phi_i^{\alpha}\left[\bar{x}+\frac{\varepsilon}{2}\,(\hat{n}_i-\hat{n}_j+\hat{n}_k),\hat{n}_j\right] = G^{\alpha}(\bar{x},\hat{n}_k)\prod_{\ell=1}^{\infty}G^{r}(\bar{x}-\varepsilon\ell\hat{n}_i,\hat{n}_k)$$
$$\equiv G^{\alpha}(\bar{x},\hat{n}_k)\;\phi^{r}\left[\bar{x}-\frac{\varepsilon}{2}\,(\hat{n}_i+\hat{n}_j-\hat{n}_k),\hat{n}_j\right] \quad , \tag{59}$$

and

$$V\left[\bar{x}+\frac{\varepsilon}{2}\,(\hat{n}_i+\hat{n}_j-\hat{n}_k),\hat{n}_k\right] = \Omega(\bar{x},\hat{n}_i)\;\Omega(\bar{x}+\varepsilon\hat{n}_i,\hat{n}_k)$$
$$\times\;\Omega(\bar{x}+\varepsilon\hat{n}_k,\hat{n}_j)^{\dagger}\;\Omega(\bar{x},\hat{n}_k)^{\dagger} \quad , \tag{60}$$

(60) becomes

$$H = -\frac{1}{8\varepsilon g^2}\sum_{x,ijk}\left\{2\mathrm{Tr}\;V\left[\bar{x}+\frac{\varepsilon}{2}\,(\hat{n}_i+\hat{n}_j-\hat{n}_k),\hat{n}_k\right]\right.$$
$$\left.+\frac{g^4}{\mu^4\varepsilon^4}\sum_{\alpha}\phi_i^{r}\left[\bar{x}+\frac{\varepsilon}{2}\,(\hat{n}_1+\hat{n}_2+\hat{n}_3),\hat{n}_j\right]\phi_i^{\alpha}\left[\bar{x}+\frac{\varepsilon}{2}\,(\hat{n}_1+\hat{n}_2+\hat{n}_3)+\varepsilon\hat{n}_i,\hat{n}_j\right]\right\}$$
$$+\,\mathrm{h.c.}\quad , \tag{61}$$

and the commutation relations are

$$V\left[\bar{x} + \frac{\epsilon}{2}(\hat{n}_1+\hat{n}_2+\hat{n}_3),\hat{n}_i\right]\phi^{\alpha}\left[\bar{y} + \frac{\epsilon}{2}(\hat{n}_1+\hat{n}_2+\hat{n}_3),\hat{n}_j\right]$$

$$= \delta_{ij}\left(\delta_{\bar{x}\bar{y}} - \delta_{\bar{x}\bar{y}-\epsilon\hat{n}_k}\right)e^{-\mu^2\epsilon^2 T^{\alpha}}\phi^{\alpha}\left[\bar{y} + \frac{\epsilon}{2}(\hat{n}_1+\hat{n}_2+\hat{n}_3),\hat{n}_j\right]$$

$$\times V\left[\bar{x} + \frac{\epsilon}{2}(\hat{n}_1+\hat{n}_2+\hat{n}_3),\hat{n}_i\right] \quad . \tag{62}$$

The dual local invariance is

$$\mathrm{Tr}\left\{\prod_i V\left[\bar{x} + \frac{\epsilon}{2}(\hat{n}_1+\hat{n}_2+\hat{n}_3),\hat{n}_i\right]V\left[\bar{x} + \frac{\epsilon}{2}(\hat{n}_1+\hat{n}_2+\hat{n}_3),-\hat{n}_i\right]\right\}\left|\Psi\right\rangle = |\Psi\rangle \tag{63}$$

Finally there is the constraint

$$\prod_{ijk}\phi_i^r\left[\bar{x} + \frac{\epsilon}{2}(\hat{n}_1+\hat{n}_2+\hat{n}_3) - \epsilon\hat{n}_i,\hat{n}_j\right]\phi_i^{\alpha}\left[\bar{x} + \frac{\epsilon}{2}(\hat{n}_1+\hat{n}_2+\hat{n}_3),\hat{n}_j\right] = 1 \; . \tag{64}$$

Now that the dual system is defined, the disorder parameter invariant under (63) can be given. It is

$$N^{\alpha}(c',\lambda) = {}_{\lambda}\langle 0|B^{\alpha}(c')|0\rangle_{\lambda} = {}_{\lambda}\left\langle 0\left|\prod_{m'\epsilon c'}\phi^{\alpha}(m')\right|0\right\rangle_{\lambda} \quad . \tag{65}$$

IV. Concluding Remarks

It has been shown here that intrinsically non-Abelian spin systems and gauge theories cannot be self-dual. Even so, there are many spin systems and a few gauge theories, based on the solvable groups, which have the same duality properties as Abelian systems. This should be of some aid in finding phase diagrams. In particular, the phases of the polyhedral groups should be investigated. The smallest Z(N) group which

has some resemblance to U(1) is Z(5) [10-13]. Consequently it may be that the only polyhedral group which is relevant to O(3) is the group of the dodecahedron, whose faces are pentageous. Unfortunately, this group is not solvable.

The utility of the operators defined in (38), (43), (53) and (61) is not clear. They are not the Hamiltonian analogues of the dual fields founded by Ukawa, Windey and Guth [13]. Their approach has been to factor out the Z(N) piece of an SU(N) Lagrangian and perform the Z(N) duality transformation. This prescription does not switch the kinetic and potential terms, although a loop satisfying (50a,b) can be made with the resulting variables.

Since iv)' need not be satisfied by spin systems, it might be interesting to investigate duality with iv) instead. It may be possible to extend the duality transformations found here to the continuum. However, there will be difficulties in defining the dual fields and the Hamiltonian in a non-singular way. A less ambitious task would be to understand the behavior of Fermion field operators in the O(3) non-linear sigma model, constructed from the original and dual fields. These are very useful in the Ising model [14]. To investigate either of these possibilities will rely heavily on the use of operator product expansions.

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