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UNIQUE, UNITARY THREE PARTICLE EQUATIONS USING ONLY TWO PARTICLE OBSERVABLES*

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ABSTRACT

Three particle equations containing only the phase shifts and binding energies of the two particle subsystems are derived from two particle zero range boundary conditions on the three particle wave function and shown to be unitary under the restriction that the two particle amplitudes have no left hand cuts; this excludes nuclear forces.

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If we understood the forces between two hadrons, we should at least be able to use this knowledge to calculate the behavior of three hadron systems. We know that currently we lack this much understanding. For example, the nucleon-nucleon scattering amplitudes are known up to, and in some cases well beyond, pion production threshold; but from this knowledge we cannot predict the binding energy of the triton or He^3, or their electromagnetic form factors. Even the n-p capture cross section at threshold differs by 10% from the model-independent[1] Bethe-Longmire prediction. The reason is, as we learned long ago from Wick,[2] that the coupling of the uncertainty principle to special relativity entails the creation of mesonic degrees of freedom at short distance. Nuclear physicists usually assume that these hidden degrees of freedom can be approximated by a "potential", but there is no unique way to define such a potential once the short range non-locality implied by the Wick-Yukawa mechanism is taken seriously.

Faced with this ambiguity, it is important to have clear experimental criteria for determining what new information is contained in three hadron observables which is not already predictable using two hadron observables. Starting from the "Fixed Past-Uncertain Future" interpretation of quantum mechanics,[3] it has been proposed that such a reference theory might be provided by calculating three particle amplitudes using only two particle on shell scatterings.[4] Once a way of doing this has been developed, the theoretically ambiguous mixture of mesonic effects - designated but not defined by the terms "off shell effects" and "three body forces" - could be uniquely parameterized by adding an on shell three particle direct scattering term to the model. A specific attempt to articulate this program[5] failed because it could not be proved unitary.

Recently, it has been shown[6] that the zero range limit of the Karlsson-Zeiger equations[7] define a unique three particle theory of the type sought. Most requisite physical properties were established, including time reversal invariance, but not unitarity of the three particle amplitude. In this communication we derive these equations directly as a consequence of the usual zero range boundary condition on the two particle subsystems applied directly to the full three particle wave function, and prove that the resulting amplitude is unitary, provided the two particle amplitudes have no singularities at negative energy other than bound state poles.

Since we wish our theory to be independent of any assumptions about the "interactions" which conventionally are thought to lie behind the scatterings, we start from a free particle wave function with the correct boundary conditions. For N_A particles of defined momenta in and N_B particles out, this can be derived by introducing the coordinates X_A where the initial particles disappear, the coordinates Y_B where the final particles appear, and summing over all possibilities in such a way that these unobservable "points" cannot enter the theory as hidden variables. The result[8] is the standard Goldberger-Watson asymptotic wave function[9] with the Phipps phase factor,[10] but with the transition matrix T_{BA} now an arbitrary function describing any conceivable quantum scattering process. T_{BA}, so viewed, is purely kinematic and need not be the matrix element of some "interaction"; it need not even be unitary. Our task is to supply dynamical equations for T_{BA} and to prove that this T is unitary.

In the phenomenological approach presented here, our basic dynamical assumption is that for each pair the two particle wave function in each partial wave satisfies the free particle Schrödinger equation with outgoing wave boundary conditions and at short distance the condition

$\lim\limits_{y\to 0} \left(y^{\ell}u(y)\right)'/\left(y^{\ell}u(y)\right) = k\,\mathrm{ctn}\,\delta$. For simplicity, we consider here only the wave function $\Psi(x,y) = U(x,y)/xy$ for a state with total angular momentum zero with pairwise scattering only in s-waves; the generalization to arbitrary finite numbers of angular momenta is immediate. Using the Bollé-Osborn coordinates,[11] we project out a one variable coordinate space radial wave function by Fourier transformation of the spectator coordinate x into the spectator momentum $\bar{p}$ using the definition $U_p(y) = (2/\pi)\int_0^\infty dx\, U(x,y)\, \sin px/p$ to obtain, with $k_\beta = \left[2\mu_\beta\left(W - \tilde{p}_\beta^2\right)\right]^{\frac{1}{2}}$,

$$U_{p_\beta}(y_\beta) = \frac{\delta\left(p_\beta - p_\beta^{(0)}\right)}{p_\beta\, p_\beta^{(0)}}\,\frac{\sin k_\beta^{(0)}\, y_\beta}{k_\beta^{(0)}} - \sum_\alpha \pi\mu_\beta\, M_{\beta\alpha}\left(p_\beta, p_\alpha^{(0)}; Z\right) e^{ik_\beta y_\beta}$$
$$- \sum_{\gamma\alpha} \frac{\bar{\delta}_{\beta\gamma} V^{o}_{\beta\gamma}}{p_\beta} \int_0^\infty p_\gamma\, dp_\gamma\, M_{\gamma\alpha}\left(p_\gamma, p_\alpha^{(0)}; Z\right) \int_{q^-_{\beta\gamma}}^{q^+_{\beta\gamma}} q_\gamma\, dq_\gamma\, \frac{\sin \bar{q}_{\beta\gamma}\, y_\beta}{\bar{q}_{\beta\gamma}\left(\tilde{p}_\gamma^2 + \tilde{q}_\gamma^2 - Z\right)} \qquad (1)$$

where

$$V^{o}_{\beta\gamma} = \left(m_\beta + m_{\beta\mp}\right)/2m_\beta; \quad \bar{\delta}_{\beta\gamma} = 1 - \delta_{\beta\gamma}; \quad \gamma = \beta\pm; \quad \beta_-, \beta, \beta_+ \text{ cyclic}$$

$$\bar{q}^2_{\beta\gamma} = 2\mu_\beta\left(\tilde{p}_\gamma^2 + \tilde{q}_\gamma^2 - \tilde{p}_\beta^2\right); \quad q^\pm_{\beta\gamma} = \left|p_\beta \pm m_\beta p_\gamma/\left(m_\beta + m_{\beta\mp}\right)\right| \qquad (2)$$

If we now apply our two particle boundary condition at $y = 0$ to this equation, we find that for each value of p_β

$$\frac{\delta\left(p_\beta - p_\alpha^{(0)}\right)\delta_{\beta\alpha}}{p_\beta p_\alpha^{(0)}} + ik_\beta\left(-\pi\mu_\beta M_{\beta\alpha}\right) - \sum_\gamma \frac{\bar{\delta}_{\beta\gamma} V^{o}_{\beta\gamma}}{p_\beta} \int_0^\infty p_\gamma\, dp_\gamma\, M_{\gamma\alpha}\left(p_\gamma, p_\alpha^{(0)}; Z\right)$$
$$\times \int_{q^-_{\beta\gamma}}^{q^+_{\beta\gamma}} \frac{q_\gamma\, dq_\gamma}{\tilde{p}^2 + \tilde{q}^2 - Z} = k_\beta\, \mathrm{ctn}\, \delta_\beta\left(-\pi\mu_\beta M_{\beta\alpha}\right) \qquad (3)$$

Hence, solving for $M_{\beta\alpha}$, we find $M_{\beta\alpha}$ proportional to $\left[\left(-\pi\mu_\beta\right)\left(k_\beta\, \mathrm{ctn}\, \delta_\beta - ik_\beta\right)\right]^{-1}$, which is simply the on shell limit of $t_\beta\left(q,\bar{q}\,;\,z-\tilde{p}_\beta^2\right)$. Comparison with Ref. 11, Eq. (3.7) shows that in this way we have in fact derived the on shell limit of the Faddeev equations without any reference to interactions.

Since we can carry out this derivation using any choice of spectator variables, we can derive two equations, symbolically represented by

$$\left[\delta_{\beta\alpha} - \sum_\gamma M_{\beta\gamma}\, G_0\, V_{\gamma\alpha}\, \bar{\delta}_{\gamma\alpha}\right] t_\alpha = M_{\beta\alpha} = t_\beta\left[\delta_{\beta\alpha} - \sum_\gamma \bar{\delta}_{\beta\gamma}\, V_{\beta\gamma}\, G_0\, M_{\gamma\alpha}\right] \qquad (4)$$

The essential three particle on shell unitarity relation to be proved is

$$M_{\beta\alpha} - M^*_{\beta\alpha} = -2\pi i \sum_{\gamma'\gamma''} M_{\beta\gamma'}\, \delta_W\, V_{\gamma'\gamma''}\, M^*_{\gamma''\alpha} \qquad (5)$$

where δ_W is the on shell restriction $2\pi i\delta\left(\tilde{p}^2 + \tilde{q}^2 - W\right) = G_0 - G_0^*$ and $V_{\gamma'\gamma''}$ is the real geometric recoupling matrix connecting different choices for spectator and distinguished pair. Since two particle on shell unitarity (i.e., real phase shifts) gives us $-2\pi i\, t_\gamma\, t_\gamma^*\, \delta_W = \left(t_\gamma - t_\gamma^*\right)\delta_W$ for $0 \le \tilde{p}^2 \le W$, the diagonal term in the sum is

$$-2\pi i\sum_\gamma M_{\beta\gamma}\, \delta_W\, M^*_{\gamma\alpha} = \sum_\gamma \left[\delta_{\beta\gamma} - \sum_{\gamma'} M_{\beta\gamma'}\, G_0\, V_{\gamma'\gamma}\, \bar{\delta}_{\gamma'\gamma}\right]\left(t_\gamma - t_\gamma^*\right)$$

$$\times\, \delta_W\left[\delta_{\beta\alpha} - \sum_{\gamma''} \bar{\delta}_{\gamma\gamma''}\, V_{\gamma\gamma''}\, G_0^*\, M_{\gamma''\alpha}\right] = M_{\beta\alpha} - M^*_{\beta\alpha} - \sum_{\gamma\gamma''} \bar{\delta}_{\gamma\gamma''}\, M_{\beta\alpha}\, V_{\gamma\gamma''}\, G_0^*\, M^*_{\gamma''\alpha}\, \delta_W$$

$$+ \sum_{\gamma\gamma'} \bar{\delta}_{\gamma\gamma'}\, M_{\beta\gamma'}\, G_0\, V_{\gamma'\gamma}\, M^*_{\gamma\alpha}\, \delta_W \qquad (6)$$

We can now establish the unitarity relation (Eq. (5)) by noting that the two terms beyond $M_{\beta\alpha} - M^*_{\beta\alpha}$ in Eq. (6) exactly cancel the off diagonal terms in Eq. (5), i.e., $-2\pi i \sum_{\gamma'\gamma''} \bar{\delta}_{\gamma'\gamma''}\, M_{\beta\gamma'}\, \delta_W\, V_{\gamma'\gamma''}\, M^*_{\gamma''\alpha}$, when careful attention is paid to the variables of integration, which allow us to reconstitute $G_0 - G_0^*$ as an appropriate δ_W. So far this proof only holds for $0 \le p^2 \le W$ on the left hand side of Eq. (5), and is independent of whether or not there are bound state poles in t_β thanks to the fact that the range of the p integration is restricted by $\int_0^\infty q^2\, dq\, \delta\left(\tilde{p}^2 + \tilde{q}^2 - W\right) \propto \left[\theta\left(\tilde{p}^2\right) - \theta\left(W - \tilde{p}^2\right)\right]$. To show that the relation also holds for the unique values $\tilde{p}_\beta^{K2} = \tilde{K}_\beta^2 + W$, we need simply define $M_{\beta\alpha}^{K_\beta}\left(\tilde{p}_\alpha;W\right) = \lim_{p_\beta \to p_\beta^K} \left(\tilde{p}_\beta^2 - \tilde{K}_\beta^2 - W\right) \times M_{\beta\alpha}^{K_\beta}\left(p_\beta, p_\alpha;W\right)$, and, since Eq. (4) factors these poles out explicitly thanks to the two particle relation $\lim_{p_\beta \to p_\beta^K} \left(\tilde{p}_\beta^2 - \tilde{K}_\beta^2 - W\right) t_\beta\left(W - \tilde{p}_\beta^2\right) = -N_\beta^2$, discover that the same proof goes through unaltered at these specific values.

Although unitarity holds on shell, independent of any assumptions other than the fact that t_β can be represented by a phase shift in the physical region of two particle scattering, and has the poles discussed above if there are bound states, the equation for $M_{\beta\alpha}$ is not well defined if $t_\beta\left(W - p^2\right)$ has singularities for negative real arguments other than these poles. To avoid this difficulty, we extend t_β by the on shell dispersion relation (see discussion at end of paper)

$$t_\beta\left(W - \tilde{p}_\beta^2\right) = \int_0^\infty \frac{dq\, \sin^2\delta_q}{\left(\tilde{p}_\beta^2 + \tilde{q}^2 - W - i\varepsilon\right)} - \frac{N_\beta^2}{\tilde{p}_\beta^2 - \tilde{K}_\beta - W - i\varepsilon} \tag{7}$$

With this definition we can then show, by writing

$$M_{\beta\alpha}\left(p_\beta, p_\alpha;W\right) = \delta_{\beta\alpha} t_\beta\left(W - \tilde{p}_\beta^2\right) + t_\beta\left(W - \tilde{p}_\beta^2\right) Z_{\beta\alpha}\left(p_\beta, p_\alpha;W\right) t_\alpha\left(W - \tilde{p}_\alpha^2\right) \tag{8}$$

that the on shell limit of the KZ equations can be derived from our equation for M, and hence recover the equations for $Z_{\beta\alpha}$ previously presented.[7] These are

$$Z_{\beta\alpha}\left(p_\beta, p_\alpha^{(0)}; Z\right) = -\frac{\bar{\delta}_{\beta\alpha}}{2}\int_{-1}^{1} d\xi \left(\tilde{p}_\alpha^{(0)\,2} + \tilde{q}_\alpha^{(2)\,2}(\xi) - Z\right)^{-1}$$

$$+\sum_\gamma \bar{\delta}_{\beta\gamma}\int_0^\infty p_\gamma^2\, dp_\gamma\, K_{\beta\gamma}\left(p_\beta, p_\gamma; Z\right) Z_{\gamma\alpha}\left(p_\gamma, p_\alpha^{(0)}; Z\right) \tag{9}$$

with the kernel given by

$$K_{\beta\gamma}\left(p_\beta, p_\gamma; Z\right) = \frac{1}{2}\int_{-1}^{1} d\xi \left\{ N_\gamma^2\left(\tilde{p}_\gamma^2 - \tilde{K}_\gamma^2 - Z\right)^{-1}\left(\tilde{q}_\beta^{(2)\,2}(\xi) + \tilde{K}_\gamma^2\right)^{-1}\right. \tag{10}$$

$$\left. + \int_0^\infty dq\left(\tilde{p}_\gamma^2 + \tilde{q}_\gamma^2 - Z\right)^{-1}\left[\frac{\sin\delta_\gamma \cos\delta_\gamma}{q_\gamma}\,\delta\left(q_\gamma - q_\beta^{(2)}(\xi)\right) + \frac{2\mathscr{P}}{\pi}\frac{\sin^2\delta_\gamma}{q_\beta^{(2)\,2} - q_\gamma^2}\right]\right\}$$

Note that the kernal depends explicitly only on real functions of two body observables in the physical region. Note also that if we insert the solution of this equation in $M_{\beta\alpha}$, thanks to having the primary singularities in $t_\beta\left(z - \tilde{p}_\beta^2\right)$ explicitly separated out, we can identify the elastic, rearrangement, breakup, coalesence, and 3-3 amplitudes immediately; calculation of Z suffices to define all of them at once.

In order to actually solve these equations, it will be convenient to isolate the moving singularity in the kernel for $0 \le \tilde{p}^2 \le W$, and the coefficients of the primary singularities, from the non-singular parts which do not contribute to the asymptotic wave function. We do this by splitting Z into an exterior and interior piece by defining $Z = \theta\left(W - \tilde{p}^2\right)Z^E + \theta\left(\tilde{p}^2 - W\right)Z^I$. For Z^E we make a change of variable appropriate to the finite interval, e.g., $\tilde{p}^2 = W\sin^2\omega$, and expand in terms of an appropriate complete set, in this case $\sin 2n\omega/\sin 2\omega$. The logarithmic singularity in the kernel can then be integrated analytically, leaving a non-singular integral to be done to define the matrix coefficients in nn'; if we use empirical input for the two body observables, this last integral has to be done numerically in any case. Our splitting guarantees that the kernels for Z^I are non-singular. This leads to coupled equations of the form

$$
\begin{aligned}
Z_n^E &= \sum_{n'} K_{nn'}^{EE} Z_{n'}^E + \sum_b K_{nb}^{EB} Z_b^E + \int_W^\infty K_n^{EI}(p')\, Z^I(p')\, d\tilde{p}'^2 \\
Z_b^E &= \sum_n K_{bn}^{BE} Z_n^E + \sum_{b'} K_{bb'}^{BB} Z_{b'}^E + \int_W^\infty K_b^{BI}(p')\, Z^I(p')\, d\tilde{p}'^2 \qquad (15) \\
Z^I(p) &= \sum_n K_n^{IE}(p)\, Z_n^E + \sum_b K_b^{IB}(p)\, Z_b^E + \int_W^\infty K^{II}(p,p')\, Z^I(p')\, d\tilde{p}'^2
\end{aligned}
$$

Here we exhibit the bound state indices b as a reminder that the value of Z at these singularities (elastic and rearrangement amplitudes) should be explicitly separated out, but have suppressed the Faddeev indices for simplicity. We see that for finite n the matrix for Z^E can be explicitly inverted and substituted into the continuum equation for Z^I making that equation also explicitly non-singular. This is one way to generalize a previous non-singular treatment of the two body problem.[13] Since only the Z^E are physically observable in three particle systems, this two step process has the advantage that the solution for Z^I need only be good enough to guarantee the accuracy of the quadrature which occurs in the equations for Z^E. This is obviously a less stringent requirement than having to solve for the functional dependence on p.

Since these equations for Z are unique, there is no guarantee that they will agree with experiment. According to elementary particle theory, there will be additional effects generated by mesonic degrees of freedom at short distance. In conventional non-relativistic theories these are replaced by two body off shell effects arising from some assumed potential model and in some cases by three body forces. Unfortunately there is no consensus as to how to separate these effects theoretically, and as has been pointed out[4] they are impossible to separate using only the two and three body observables themselves. However, now that we have a unique reference theory based only on two particle observables, we do have a way of measuring the combined mesonic effects. One way to make this explicit is to introduce into the model a direct "zero range three particle" scattering amplitude. If this itself is unitary, like the two particle amplitudes, the equations remain unitary and provide via the parameters in this added amplitude a way to parameterize the discrepancy between the unique theory and experiment. Such a system need be inverted only once to obtain fitting formulae which can be used for data analysis, rather than requiring the solution of an integral equation each time a parameter is varied. Brayshaw[14] has demonstrated the practicality of this approach to data fitting in various relativistic situations. So far as we can see, our zero range equations can be extended to the relativistic case and form a special case of a general separable model given by Brayshaw.[14] We have also shown that minimal four particle equations can be obtained in a similar way, but will not pursue either of these applications of our approach here.

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Note Added, January 1980: The restriction imposed by Eq. (7) drastically limits the field of application of this theory. In particular, since all the quantities appearing in that equation are observable, if the amplitude has a "left hand cut" this equation (a partial wave dispersion relation) can be used to calculate the contribution from this cut in the physical region. For nucleon-nucleon scattering this contribution is substantial, as has been shown many times, for example by D. V. Bugg.[15] Another way to see this is to start from the Low equation to construct the fully off-shell t-matrix.[16] If this is done, and the on-shell amplitude has a left hand cut, it is easy to show that the zero range limit cannot be taken. If there is no left hand, the limit can be taken, but the solution of the Low equation implied is of the Castillejo, Dalitz-Dyson type in which scattering persists even though the "interaction" goes to zero. All of this will be discussed in detail in a longer paper now in preparation. In spite of these dismal conclusions, which make this zero range theory of little direct use for nuclear physics, the elementary particle extension using relativistic kinematics looks promising.

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