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BRIGHTNESS OF SYNCHROTRON RADIATION FROM ELECTRON STORAGE RINGS*

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ABSTRACT

Parameters of an electron storage ring which are relevant to brightness are defined. For the case of a FODO lattice scaling laws and maximum beam brightness are calculated.

I. Introduction

Electron storage rings are used more and more frequently as a source of light in a wide range of wavelengths for research in solid state physics, microbiology, technology and other fields. For many experiments it is desirable to have the light come from a point-like source or to have the brightness of an extended source as high as possible.

Electrons circulating in a storage ring emit synchrotron light photons almost tangentially to the particle's trajectory at the point of emission with a root mean square angle of $1/\gamma$, where $\gamma \cdot mc^2$ is the total electron energy.

Each electron as it moves around the ring oscillates transversely about an equilibrium orbit which usually runs through the middle of the

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magnets. The amplitude of these oscillations is determined by the quantized emission of synchrotron radiation photons, a damping mechanism due to the accelerating fields which restore the energy lost to radiation and the particular arrangement of focussing and bending magnets. As a result of these processes we get a stationary Gaussian distribution of electrons in the transverse plane.

In this paper the physics which leads to a certain electron density distribution is described and scaling laws as well as limitations to the beam brightness will be derived.

II. Definition of Relevant Storage Ring Parameters

The magnet lattice of a storage ring is generally composed of a set of evenly spaced bending magnets on a circle. Between the bending magnets the magnetic focussing elements - quadrupoles - are located. The quadrupole fields are able to change the slope of a trajectory proportional to the distance of that trajectory from the center of the quadrupole. The equation for the particle trajectory in the horizontal plane then can be written as:[1]

$$\frac{d^2x_\beta}{ds^2} = -k(s) \cdot x_\beta \tag{1}$$

where x_β is the distance of the particle from the ideal orbit going through the centers of the magnets and s the coordinate along the trajectory. k(s) is the strength of the focussing elements being a positive or negative number where there are quadrupoles and zero elsewhere. The strength multiplied by the length of the quadrupole is just the inverse of the focal length of the quadrupole $k \cdot \ell = 1/f$. Parameters of the bending magnets do not appear in Eq. (1) since we use a coordinate system (x,y,s) which moves with the particles along the ideal orbit. Since k(s) is a periodic function around the storage ring, the solution of Eq. (1) is:

$$x_\beta(s) = a \cdot \sqrt{\beta(s)} \cdot \cos(\psi(s) + \delta) \qquad (2)$$

with a an amplitude factor and δ a phase constant. $\beta(s)$ is the so-called betatron function and is the one and only periodic solution of the differential equation $d^2w/ds^2 + k(s) \cdot w - 1/w^3 = 0$ ($w = \sqrt{\beta(s)}$) which can be derived by inserting Eq. (2) into Eq. (1). The betatron phase $\psi(s)$ is defined by $\psi(s) = \int_0^s d\sigma/\beta(\sigma)$.

Equation (2) describes the trajectory of a single particle with a phase δ at $s = 0$. In a particle beam each particle has a different phase constant δ and if we follow all particles with the same amplitude factor a around the ring we find for the envelope E(s) of all these particles:

$$E(s) = \pm a\sqrt{\beta(s)} \qquad (3)$$

The total beam width then is just $2 \cdot |E(s)|$.

In the remainder of this paper we are mainly concerned with determining the amplitude factor a. If we take the derivative of Eq. (2) with respect to s and eliminate from both equations the phase $\psi(s) + \delta$ we get:

$$a^2 = \beta x_\beta'^2 + 2\alpha x_\beta x_\beta' + \gamma x_\beta^2 \qquad (4)$$

with $\alpha = -\beta'/2$ and $\gamma = (1 + \alpha^2)/\beta$. Eq. (4) tells us that a particle with an amplitude factor a moves on an ellipse in the (x,x') phase space. The significance of the phase ellipse is that all particles with amplitude factors $a < a_{max}$ move on similar ellipses within the ellipse defined by a_{max}. The area enclosed by the ellipse a_{max} is $\pi a_{max}^2 = \pi \cdot \varepsilon$ where ε is defined as the emittance of the beam. For a Gaussian density distribution we define the beam emittance ε for those particles which are within an envelope of $\sigma_x^2 = \langle x_\beta^2 \rangle = \frac{1}{2} \beta a^2 = \beta\varepsilon$ (see Eq. (2)).

There is another quantity – the η-function – we need to know before we can calculate the beam emittance. Particles with the right energy E_0 perform betatron oscillations about the ideal orbit. If, however, the particle has a different energy $E = E_0 + \Delta E$ it performs betatron oscillations about a different orbit which is offset from the ideal orbit by $\eta(s) \cdot \Delta E/E_0$. The η-function is the one and only periodic solution of the equation of motion:[1]

$$\frac{d^2}{ds^2} \eta + k(s) \cdot \eta = -\frac{1}{\rho} \qquad (5)$$

with $1/\rho$ the curvature of the bending magnets.

III. Beam Emittance in Electron Storage Rings[2]

While the particle moves around the ring performing betatron oscillations there comes the moment when it emits a photon of energy δE. From that point on the particle oscillates about a new orbit which is offset from its previous orbit by $-\eta(s) \cdot \delta E/E_0$. Since the particle cannot make a jump in space we have for the variation of its betatron oscillation amplitude $\delta x = 0 = \delta x_\beta - \eta(s) \cdot \delta E/E_0$ or $\delta x_\beta = \eta(s) \cdot \delta E/E_0$. Analogously, we have for the variation of the slope $\delta x'_\beta = \eta'(s) \cdot \delta E/E_0$.

The sudden variation of the betatron amplitude also changes the amplitude factor a. From Eq. (4) we have for the variation of a

$$\delta a^2 = \beta \cdot \delta(x_\beta'^2) + 2\alpha \cdot \delta(x_\beta \cdot x_\beta') + \gamma \cdot \delta(x_\beta^2)$$

with $x_\beta = x_{0\beta} + \delta x_\beta$ and $x_\beta' = x_{0\beta}' + \delta x_\beta'$, we get

$$\delta\langle a^2\rangle = \left(\frac{\delta E}{E_0}\right)^2 \cdot \left\{\beta\eta'^2 + 2\alpha\,\eta\,\eta' + \gamma\eta^2\right\} = \left(\frac{\delta E}{E_0}\right)^2 \mathcal{H}(s) \qquad (6)$$

Here we have averaged over all possible amplitudes a particle can have when it emits a photon which leads to the disappearance of all terms linear in $x_{0\beta}$ or $x_{0\beta}'$.

Equation (6) gives us the increase of the amplitude factor a due to the emission of one photon of energy δE. With $\mathcal{N}$ the number of photons emitted per second with an average energy $\langle\delta E^2\rangle$ we arrive at an increase of the amplitude factor a per turn

$$\frac{d\langle a^2\rangle}{dt} = \frac{1}{cT_0E_0^2}\oint\left\{\mathcal{N}\langle\delta E^2\rangle\mathcal{H}\right\} ds = \frac{1}{E_0}\langle\mathcal{N}\langle\delta E^2\rangle\mathcal{H}\rangle_s \qquad (7)$$

where T_0 is the revolution time, c the speed of light, $\langle\delta E^2\rangle$ the average of the square of the photon energy and $\langle\mathcal{N}\langle\delta E^2\rangle\mathcal{H}\rangle_s$ the average of the enclosed function around the whole ring. Due to the emission of synchrotron radiation we have a continuous increase in the beam emittance according to Eq. (7). This increase is counteracted by the particular way storage rings restore the lost energy which leads to a damping of the betatron oscillations according to

$$\frac{d\langle a^2\rangle}{dt} = -\frac{2}{\tau}\langle a^2\rangle \qquad (8)$$

where τ is the damping time for transverse betatron oscillations.

Equations (7) and (8) lead to an equilibrium state defined by

$$\frac{\sigma_x^2}{\beta} = \varepsilon_x = \frac{\tau}{4E_0^2} \langle \mathcal{N} \langle \delta E^2 \rangle \mathcal{H} \rangle_s \tag{9}$$

where we have used $\varepsilon_x = \langle a^2 \rangle /2 = \sigma_x^2/\beta$. With the well-known relation $\mathcal{N} \langle \delta E^2 \rangle = 55 \varepsilon_c P_\gamma /(24\sqrt{3})$ ($\varepsilon_c = 3\hbar c \gamma^3/(2|\rho|)$ the critical energy of the synchrotron radiation and $P_\gamma = 2 r_e c E_0^4/(3(mc^2)^3 \rho^2)$ the instantaneous radiation power for a single electron) and $\tau = 2E_0/\langle P_\gamma \rangle_s$, we get:

$$\frac{\sigma_x^2}{\beta} = \varepsilon_x = \frac{55}{32\sqrt{3}} \frac{\hbar c}{mc^2} \gamma_0^2 \frac{\langle |1/\rho^3| \mathcal{H} \rangle_s}{\langle 1/\rho^2 \rangle_s} \tag{10}$$

So far we have calculated only the horizontal beam size. In the vertical plane there is no quantum fluctuation in a plain ring with no vertical bending magnets. However, there is still damping leading to a zero vertical emittance. The vertical beam emittance in a storage ring, therefore, is determined by other effects, such as the emission of photons at an average angle $\langle \theta^2 \rangle = 1/\gamma^2$ to the direction of the particle's trajectory. This effect leads to a vertical emittance of $\varepsilon_y(\text{rad m}) = 1.9 \times 10^{-13}\ \bar{\beta}_y \langle |1/\rho^3| \rangle_s / \langle 1/\rho^2 \rangle$ which is very small in practical storage rings. Another effect is the coupling of horizontal betatron oscillations into the vertical plane due to rotational misalignments of quadrupoles. In a well-aligned storage ring this results in a vertical emittance of

$$\varepsilon_y \approx 0.01 \cdot \varepsilon_x \tag{11}$$

and is usually the most dominant effect in determining vertical beam emittance.

IV. Scaling Laws for Magnet Lattice Parameters

In order to calculate the beam emittance according to Eq. (10) we need to know more about the function $\mathscr{H}(s)$. Generally this function has to be computed and averaged by a computer. With some simplifying assumptions on the magnet lattice, however, we can derive algebraic expressions which are good approximations for design purposes. We assume a simple FODO-lattice which is a series of focussing (F) and defocussing (D) quadrupoles equally spaced and the bending magnets (O) placed between them. We further assume infinitely short quadrupoles of focal length f and all the space between the quadrupoles filled with bending magnets (see Fig. 1).

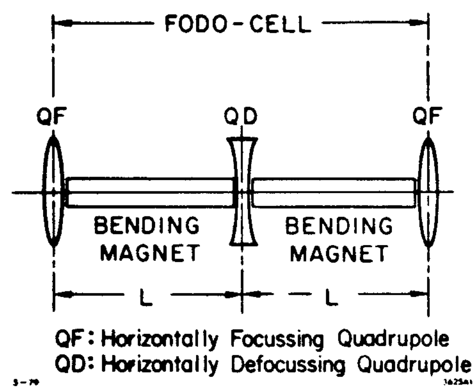

Fig. 1. FODO-CELL (schematic).

The values of the betatron and η-function in the quadrupoles are listed in Table I. We use the parameter $\alpha = L/f$ and $1/f = k \cdot \ell_q$ with ℓ_q the length of half a quadrupole.[3] If θ is the bending angle of one bending magnet ($\theta = L/\rho$) we get[4]

$$\langle \mathscr{H} \rangle = \frac{\rho\,\alpha^4}{\sqrt{\alpha^2-1}\,\sin\theta}\left\{16\sin^4\frac{\theta}{2} - \frac{2\sin^2\frac{\theta}{2}}{\alpha^2}\left(5-2\cos\theta-3\,\frac{\sin\theta}{\theta}\right) + \frac{1}{\alpha^4}\left(2+\cos\theta-3\,\frac{\sin\theta}{\theta}\right)\right\} \qquad (12)$$

TABLE I

Horizontally Focussing		Defocussing Quadrupole
focal length (m)	f	-f
betatron function	$\beta = L \frac{\alpha(\alpha+1)}{\sqrt{\alpha^2-1}}$	$\beta = L \frac{\alpha(\alpha-1)}{\sqrt{\alpha^2-1}}$
η-function	$\eta = \frac{L^2}{2\rho}\alpha(2\alpha+1)$	$\eta = \frac{L^2}{2\rho}\alpha(2\alpha-1)$

Up to large bending angles ($\theta \approx 50$ to 60°) we can replace $\sin\theta$ by θ and using the betatron phase advance per half cell $\sin\psi = 1/\alpha$ we get

$$\langle \mathcal{H} \rangle = \rho\theta^3 \frac{2(1 - 3/4 \sin^3\psi + 1/60 \sin^4\psi)}{\sin^2\psi \cdot \sin 2\psi} \tag{13}$$

The quantity $\langle \mathcal{H} \rangle/\rho\theta^3$ is shown in Fig. 2 as a function of ψ. We observe that the beam emittance reaches a minimum for $\psi \approx 65^\circ$.

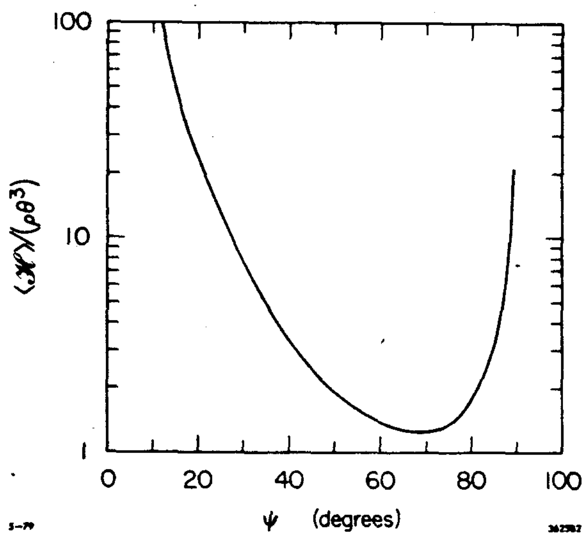

Fig. 2. Beam emittance as a function of the betatron phase per half cell.

V. Beam Brightness

The beam brightness is proportional to the inverse of the beam cross section at the source point of the synchrotron radiation.

The beam width is a composition of the beam width due to betatron motion only and wherever $\eta(s) \neq 0$ due to the energy spread in the beam. Since both the particle distribution in the betatron phase space and the energy distribution are Gaussian we have for the total beam width

$$\sigma_x^2 = \sigma_\beta^2 + \eta^2 \left(\frac{\sigma_\varepsilon}{E}\right)^2 \tag{14}$$

where

$$\left(\frac{\sigma_\varepsilon}{E}\right)^2 = 1.92 \times 10^{-13}\ \gamma_0^2 \frac{\langle |1/\rho^3| \rangle_s}{\langle 1/\rho^2 \rangle_s} \tag{15}$$

In the FODO lattice assumed we always have $\eta^2(\sigma_\varepsilon/E)^2/\sigma_\beta^2 \lesssim 0.5$. The total beam width varys somewhat depending on whether the source of the synchrotron radiation is close to the focussing or defocussing quadrupole (see Table I). In the vertical plane we have assumed the emittance to be about 1% of the horizontal emittance. There is no contribution to the beam height due to energy spread since for a plane storage ring we have $\eta_y \equiv 0$. In the vertical plane, however, the meaning of focussing or defocussing is just reversed with respect to the horizontal plane which is important when we determine the vertical betatron function at the source point. In Table II the beam sizes are shown for the two extreme source points at the horizontally focussing and defocussing quadrupoles.

TABLE II

Horizontally	Focussing	Defocussing Quadrupole
beam width σ_x	$\left\{ L \frac{\alpha(\alpha+1)}{\sqrt{\alpha^2-1}} \varepsilon_x + \frac{L^4}{4\rho^2} \alpha^2 (2\alpha+1)^2 \left(\frac{\sigma_\varepsilon}{E}\right)^2 \right\}^{\frac{1}{2}}$	$\left\{ L \frac{\alpha(\alpha-1)}{\sqrt{\alpha^2-1}} \varepsilon_x + \frac{L^4}{4\rho^2} \alpha^2 (2\alpha-1) \left(\frac{\sigma_\varepsilon}{E}\right)^2 \right\}^{\frac{1}{2}}$
beam height σ_x	$\left\{ L \frac{\alpha(\alpha-1)}{\sqrt{\alpha^2-1}} \frac{\varepsilon_x}{100} \right\}^{\frac{1}{2}}$	$\left\{ L \frac{\alpha(\alpha+1)}{\sqrt{\alpha^2-1}} \frac{\varepsilon_x}{100} \right\}^{\frac{1}{2}}$

We now have all the quantities to calculate the beam cross section or the beam brightness. Using Eqs. (10), (11), (13), (14), (15) and Table II we get at the horizontally focussing quadrupole

$$(\sigma_x \cdot \sigma_y)_F = C_q \cdot \frac{1}{10} \gamma_0^2 \frac{L^4}{\rho^3} F_F(\alpha) \qquad (16)$$

and at the horizontally defocussing quadrupole

$$(\sigma_x \cdot \sigma_y)_D = C_q \cdot \frac{1}{10} \gamma_0^2 \frac{L^4}{\rho^3} F_D(\alpha) \qquad (17)$$

where $C_q = 55 \cdot \hbar c/(24\sqrt{3}\, mc^2) = 3.84 \times 10^{-13}$ m) and the functions $F_F(\alpha)$ and $F_D(\alpha)$ are plotted versus the half-cell phase advance ψ in Fig. 3. For any source point between the two quadrupoles the beam brightness can be calculated by interpolation.

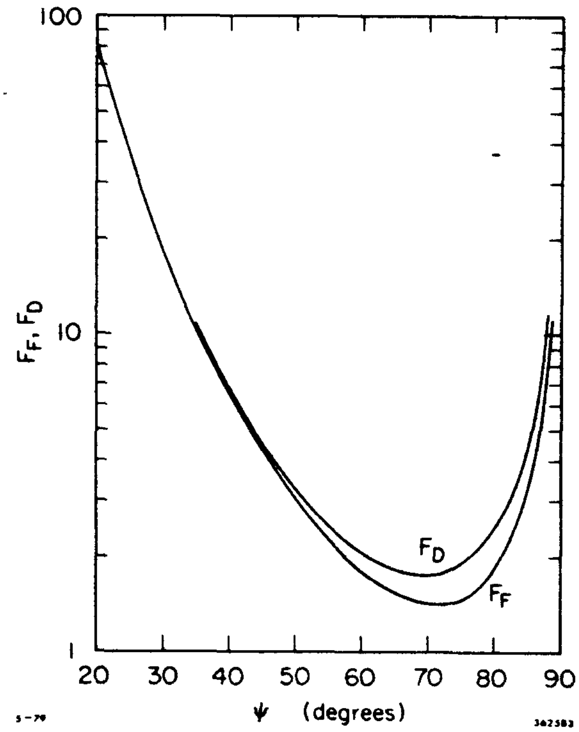

Fig. 3. Beam brightness as a function of the betatron phase per half cell.

VI. Conclusion

In a review of the influence of storage ring parameters the beam cross sections as a function of the betatron phase advance in a FODO-cell lattice have been calculated. Some simplifying assumptions have been made to make the algebra manageable, however, the approximation is valid for a very wide variety of lattice structures and is, therefore, a useful tool in designing storage ring lattices and calculating expected beam brightness. From the scaling it is found that the maximum beam brightness is achieved for a focussing lattice with a phase advance per half cell of about 70° with a beam cross section of

$$\sigma_x \cdot \sigma_y \ (\mathrm{mm}^2) \approx 0.24 \cdot E^2(\mathrm{GeV}^2) \cdot \frac{L^4(\mathrm{m}^4)}{\rho^3(\mathrm{m}^3)} \qquad (18)$$

REFERENCES

(1) E. D. Courant and H. S. Snyder: Annals of Phys. 3 (1958) 1-48.

(2) M. Sands in Physics with Intersecting Storage Rings, ed., B. Touschek, Academic Press, New York, 1971.

(3) H. Wiedemann: PEP-Note 39, Stanford Linear Accelerator Center, 1973 (unpublished).

(4) R. Helm and H. Wiedemann: PEP-Note 303, Stanford Linear Accelerator Center, 1979 (unpublished).