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CONFINEMENT IN COMPACT QED FOR LOW COUPLINGS*

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ABSTRACT

A comparison is made between several models of lattice QED, in 2+1 dimensions, which have been shown to exhibit confinement for small couplings. We also compare two different approaches which have been employed to study this phenomenon: the variational and path integral methods. The reasons for differences in the results are explored. The importance of compactness for low-coupling confinement is demonstrated.

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1. Introduction

There have been several approaches to the problem of confinement in 2+1 dimensional compact QED, for small coupling constant. All approaches yield a static potential energy $E(D) \simeq \gamma D$ between a $q\bar{q}$ pair a distance D apart, where γ vanishes nonanalytically with the coupling. This means there is confinement (but there are differences in the results). Since the approaches, and the models themselves differ widely, we think it is worthwhile to try answering the following questions: How are the confinement mechanisms related? How much of the difference between the various results is caused by making different approximations in each case? There is also the question of periodicity. In one approach,[3] heavy use is made of the periodicity of the Hamiltonian as a function of magnetic field, whereas in the path integral approach of Ref. 1, the apparent cause of confinement is the presence of multi-pseudo-particle solutions, and these exist for non-periodic field potentials as well (those possessing degenerate minima). One would like to know whether periodicity is really necessary for confinement.

What follows is an investigation of these points. We will begin by describing the main features of the approaches we will be referring to, but first a word about notation: our convention for electric and magnetic fields vary in different parts of the paper. To prevent confusion we will give their definitions in terms of the vector potentials A_i, whenever necessary. A_i are always normalized to satisfy canonical commutation relations with $\dot{A}_i$. We will always be working in the temporal $A_0 = 0$ gauge. Other conventions used are:

1) The lattic indices i,j run over 1,2.

2) Greek indices μ, ν, etc. run over 1,2,3.

3) The 3 direction is the time direction (real or imaginary).

4) Summation over like indices is sometimes implicitly used.

5) Δ_μ, Δ_i are lattice difference operators.

6) $\hat{\mu}$, $\hat{i}$ are unit vectors in the directions μ, i, respectively.

1.a. The path-integral method

This approach was implemented in two different ways which converge at a certain point.[1,2] Polyakov used a Georgi-Glashow QED, which we call model I. The Euclidean action is

$$\int d^3x \left[\frac{1}{4e^2} \vec{F}^2_{\mu\nu} + \frac{1}{2} (D_\mu \vec{\phi})^2 + \frac{1}{4} \lambda (\vec{\phi}^2 - \eta^2)^2 \right]$$

$$\vec{F}_{\mu\nu} = e(\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu) + e^2 \vec{A}_\mu \times \vec{A}_\nu \qquad (1.a.1)$$

where $\vec{\phi}$ is the isovector Higgs field, and $\vec{F}_{\mu\nu}$ are the non-Abelian gauge fields for the SU(2) gauge group. $F^a_{\mu\nu}$ are derived from vector potentials A^a_μ, where a is the isospin index. η, λ are constants; e is the gauge coupling, and has dimension $(\text{length})^{-\frac{1}{2}}$. D_μ are covariant derivatives.

Model I contains a pseudo-particle solution, the 't Hooft-Polyakov monopole. By a suitable gauge transformation, $A^{1,2}_\mu$ are gauged away far from a monopole, and one is left with the vector potential $A_\mu = A^3_\mu$. Only a U(1) subgroup of the original SU(2) symmetry remains. Polyakov computes the correlation function

$$F(C) = \langle \exp ie \oint_C A_\mu \, dx_\mu \rangle$$

for small e. C is the Wilson loop. He considers only the multi-monopole contributions to the path-integral. F(C) is thus expressed as a correlation function in a magnetic monopole gas, in the dilute gas approximation. The monopoles have a long-range Coulomb interaction, but a screening occurs that makes the Green's function of the gas short range. In fact, the $k^2 = 0$ pole in the correlation function

$$\int e^{ik\cdot x} \langle T(B(0)\, B(x)) \rangle d^3x$$

where B is the magnetic field, disappears when an infinite set of Feynman graphs is summed. Instead of the massless vector Boson A_μ, one gets a pole corresponding to a massive scalar field, ϕ. F(C) is expressed as a path-integral over this field:

$$F(C) = \int [d\phi] \exp\left\{-\frac{1}{2}\, \pi e^2 \int d^3x [(\nabla(\phi - \eta))^2 - 2M^2 \cos\phi]\right\}$$

where

$$\eta(x) = \frac{1}{2} \int_S d\vec{\sigma} \cdot \frac{\vec{x} - \vec{y}}{|\vec{x} - \vec{y}|^3} \qquad (1.a.2)$$

S is the area bounded by the contour C. It is unambiguous if one chooses a planar C.

This path integral is computed using a stationary point. The result is an area-law decrease,

$$F(C) \propto e^{-\gamma A} \qquad (1.a.3)$$

where A is the area of S and γ is a constant.[1]

The second model in which the path integral approach was used is compact QED on a spacetime lattice.[2] The Euclidean action is

$$\beta \sum_{r,\mu,\nu} (1 - \cos\theta_{\mu\nu}(r)) \qquad (1.a.4)$$

where $\theta_{\mu\nu}$ is a dimensionless multiple of the electromagnetic field tensor.

We call this model II. The $\beta \to \infty$ limit is considered. Using a Villain approximation, a monopole partition sum is derived, and the results are a lattice version of those in Ref. 1. The discreteness of spacetime in this model eliminates short distance problems; in particular, the self-action of a monopole is computable. But the long distance behavior is not affected by discreteness, so (1.a.3) is still true. β corresponds to $1/e^2 a$, where e is the coupling in Ref. 1 and a is the lattice spacing. In both models I and II the static energy of a quark-antiquark pair at large separation Da is read off (1.a.3) and is

$$E(D) \simeq \gamma D$$

where γ vanishes non-analytically with the coupling. The treatments in Refs. 1 and 2 are valid for small couplings e; we further know compact QED confines for large e.[4] Hence these models cannot have a phase transition.

1.b. <u>The variational approach</u>

This approach to the problem is a Hamiltonian formulation on a two-dimensional lattice,[3] with the following compact QED Hamiltonian:

$$H = \frac{1}{2}\left\{ g^2 \sum_{\vec{p},i} E^2_{\vec{p},i} + \frac{2}{g^2} \sum_{\vec{p}} (1 - \cos B_{\vec{p}}) \right\} \tag{1.b.1}$$

where $E_{\vec{p},i}$ is the <u>electric</u> field on a link $\vec{p} \to \vec{p} + \hat{i}$, $B_{\vec{p}}$ is the magnetic field on a plaquette, and we have set the lattice spacing, a, to be 1 for convenience. g corresponds to e in Ref. 1. In terms of the vector potentials we define

$$E_{\vec{p},i} = \frac{1}{g} \partial_3 A_{\vec{p},i}$$

$$B_{\vec{p}} = g\, \varepsilon_{ij}\, \Delta_i\, A_{\vec{p},j} \tag{1.b.1'}$$

We name this model III.

The separation energy E(D) is computed variationally in a special sector of Hilbert space. Let $|\{B_{\vec{p}}\}\rangle$ be a state with a well-defined magnetic field configuration, $B_{\vec{p}}$ at site $\vec{p}$. Working in this basis, the wave function is

$$\chi(\{B_{\vec{p}}\}) = \langle \{ B_{\vec{p}} \} | \psi \rangle \tag{1.b.2}$$

where $|\psi\rangle$ is that part of the $q\bar{q}$ state belonging to the photon sector. The Hamiltonian is periodic in each $B_{\vec{p}}$ with a period 2π. An operator $L_{\vec{p}}$, conjugate to $B_{\vec{p}}$, is defined as follows:

$$E_{\vec{p},i} = \varepsilon_{ij} \Delta_j L_{\vec{p}} + E^{\parallel}_{\vec{p},i}$$

$E^{\parallel}_{\vec{p},i}$ is the Coulomb static field and we choose it to be that generated by a static $q\bar{q}$ pair a distance D apart. Δ_j are lattice difference operators. The above-mentioned periodicity then means that

$$\left[\exp\,(2\pi\, i\, L_{\vec{p}})\,,\, H\right] = 0$$

Therefore $G_{\vec{p}} = \exp\,(2\pi\, i\, L_{\vec{p}})$ are good quantum numbers, and so are $L_{\vec{p}}$ up to integral fluctuations. Choosing then a sector in which

$$G_{\vec{p}} = \exp\,(2\pi\, i\, \varepsilon_{\vec{p}})\,,\; L_{\vec{p}} = \varepsilon_{\vec{p}} + \text{integer} \tag{1.b.3}$$

We can label the trial wave function as

$$\chi\,(\{\, B_{\vec{p}}\,;\, \varepsilon_{\vec{p}}\,\}) \qquad .$$

This is a Bloch-type wave function, with "wave numbers" $\varepsilon_{\vec{p}}$. The natural choice for $\varepsilon_{\vec{p}}$ is that which will screen the Coulomb field and squeeze the electric flux links into a string, which runs in a straight line between the quark and antiquark. Then the question of confinement translates itself into the question: will the string be stable? As is well-known this approach to confinement is equivalent to using the Wilson loop. Making this choice, one must now determine the shape of χ inside a single potential well. χ is chosen to be a sume of Gaussians.[3]

$$\chi_{\mathrm{trial}}(\{B_{\vec{p}};\,\varepsilon_{\vec{p}}\}) = \prod_{\vec{p}} \left(\sum_{n_{\vec{p}}=-\infty}^{\infty} \right) e^{2\pi i \sum_{\vec{p}} n_{\vec{p}} \varepsilon_{\vec{p}}}$$

$$\times\ e^{-\frac{1}{2g^2} \sum_{\vec{p}\vec{p}'} (B_{\vec{p}} - 2\pi n_{\vec{p}})\, \Delta_{\vec{p}\vec{p}'}\, (B_{\vec{p}'} - 2\pi n_{\vec{p}'})\, \delta\left(\sum_{\vec{p}} B_{\vec{p}'}\right)} \qquad (1.b.4)$$

and $\Delta_{\vec{p}\vec{p}'}$ is a Green's function, determined variationally to be

$$\Delta_{\vec{p}\vec{p}'} = \frac{1}{V} \sum_{\vec{k}\neq 0} e^{i\vec{k}\cdot(\vec{p}'-\vec{p})}\,(4 - 2\cos k_x - 2\cos k_y)^{-\frac{1}{2}} \qquad (1.b.5)$$

up to corrections of orders g^2 and $e^{-\mathrm{const}/g^2}$. $\vec{k}$ is the discrete lattice momentum. The variational computation then gives for E(D) an expression, derived from a partition sum

$$Z(\{\varepsilon_{\vec{p}}\}) = \prod_{\vec{p}} \left(\sum_{N_{\vec{p}}=-\infty}^{\infty} \right) \exp\left[-\frac{\pi^2}{g^2} \sum_{\vec{p}\vec{p}'} N_{\vec{p}}\, \Delta_{\vec{p}\vec{p}'}\, N_{\vec{p}'} + 2\pi i \sum_{\vec{p}} N_{\vec{p}}\, \varepsilon_{\vec{p}} \right] \delta\left(\sum_{\vec{p}} N_{\vec{p}} \right)$$

$$(1.b.6)$$

This is a partition sum of a 2-dimensional Coulomb gas on the lattice, interacting with an imaginary external field. The charges of the gas have the interpretation of being tunnelings between the different B-vacua, i.e., they appear in the sum as a result of overlap integrals between different terms in (1.b.4). If one wants to push the tunneling interpretation further, one can say that the magnetic charges are electric vortices; a tunneling means a change of B with time, which causes an electric field circulation, by Faraday's law. In Sections 2 and 3, we will examine the relation between these tunnelings and the monopoles of the other approach.

1.c.

In what follows, we will compare the various aspects of the different procedures - the "variational" (or "tunneling") approach and the "path integral" (or "monopole") approach. The plan of the paper is as follows: in Section 2 we discuss classical monopoles in a certain continuum model, in order to gain a better understanding of the relation between model I and the other two models - in which no exact monopole solutions are known. In Section 3 the mechanisms for screening and confinement are explained and compared in the two approaches. In Section 4 we explore the importance of compactness for screening and confinement. In Section 5, a numerical comparison is made between the results of Refs. 2 and 3, and an attempt is made to explain the different long-distance behavior of the Green's functions. In Section 6, the treatment of Ref. 2 (namely a spacetime discretization) is applied to model III, and comparisons are made with both Refs. 2 and 3. In Section 7 we present concluding remarks.

2. Monopoles in Interacting QED

To better understand the connection between the approaches, and also between models I and II, we will demonstrate how classical monopole solutions arise in certain interacting QED models. Models II and III are such models. Consider a generalization of model III for Euclidean time:

$$\text{Action} \;=\; \int dt \, \frac{1}{2g^2} \sum_{\vec{p}i} B^2_{\vec{p}i} + \frac{1}{g^2}\int dt \sum_{\vec{p}} V(B_{\vec{p}}) \qquad (2.1)$$

Again we have set $a = 1$ for convenience, so the problem is now devoid of dimensional quantities. The convention for the fields is now as follows:

$$B_{\vec{p},i} \;=\; g\,\varepsilon_{ij}\,\partial_3\,A_{\vec{p},j}$$

$$B_{\vec{p}} \;=\; B_{\vec{p},3} \;=\; g\,\varepsilon_{ij}\,\Delta_i\,A_{\vec{p},j} \qquad (2.1)'$$

Note we have passed from a 2+1 dimensional notation, in which there are two electric-field components and one magnetic component, to a 3 dimensional notation where there are three magnetic field components. We choose a field potential V having two degenerate minima:

$$V(B) \;=\; \frac{1}{2}B^2 - 2\pi(B-\pi)\,\theta(B-\pi) \qquad (2.2)$$

(see Fig. 1). This is a piecewise quadratic function. To find a classical solution of the equations of motion, we should solve a set of difference-differential equations. However, in order to make our point as simply as possible, we will replace the spatial lattice by a continuum for the remainder of this section. The action is now,

$$\text{Action} \;=\; \frac{1}{2g^2}\int d^3x \sum_i (B_i(x))^2 + \frac{1}{g^2}\int d^3x\, V(B) \qquad (2.3)$$

where

$$\partial_\mu B_\mu \;=\; 0$$

The last follows from (2.1)'. The equations of motion obtained from this action are

$$\partial_3^2 B + \sum_i \partial_i^2 V'(B) = 0 \tag{2.4}$$

$$\sum_{ij} \varepsilon_{ij} \partial_i B_j = 0 \quad , \quad \partial_\mu B_\mu = 0 \tag{2.5}$$

We seek a monopole solution. From (2.2) and (2.4)

$$\partial_3^2 B + \sum_i \partial_i^2 (B - 2\pi\theta(B-\pi)) = 0$$

$$\vec{\nabla}^2 B = 2\pi(\partial_1^2 + \partial_2^2)\,\theta(B-\pi) \tag{2.6}$$

Imagine a configuration, $B^{(0)}$, in which $B = 2\pi$ in an infinitely long tube, becomes $B \geq \pi$ on some surface K at the end of the tube, and is everywhere $< \pi$ outside this tube (Fig. 2). Obviously, the R.H.S. in (2.6) is nonzero only on K and T, and singular there. It is equivalent to an infinite current-solenoid terminating near the point P, which will generate another configuration, $B^{(1)}$, similar to the first. It will have the same T but a different K. Thus we can find an approximate monopole solution to (2.7) by iteration, which presumably will converge. From the first iteration $B^{(0)}$ onward, B behaves like a monopole field far from the tube, and from (2.5), all three components will behave like $\vec{r}/r^3$. We have used a singular field potential; for a potential V(B) that varies smoothly near $B = \pi$, like a cosine potential, the current solenoid may develop a thickness.

Note that the equation (2.6) has no length scale, so the width of the monopole tube is arbitrary. The monopole point P is also arbitrary, but the tube must be along the $\hat{3}$ direction.

One can presumably find such monopole solutions for any potential V(B) possessing degenerate minima. This condition is satisfied in particular for periodic potentials, but periodicity is not required. This leads to the question raised in Section 1, of whether periodicity (compactness) is necessary for confinement. We will investigate this point in Section 4.

3. Screening and Confinement Mechanisms

In both approaches confinement is achieved by screening. The monopole picture in 2+1 dimensions is as follows:

The Wilson quark loop is an (imaginary) current loop C, producing a magnetic field. This field is the same as that of a narrow magnetic dipole sheet on the area S bounded by C, except on S itself. On the lattice, this sheet is one spacing thick. The energy of the loop increases only as the loop perimeter. However, the loop is immersed in a magnetic monopole gas, which interacts directly with the dipole sheet, thus adding a term proportional to the area of S to the action. The gas becomes polarized, and monopoles of opposite polarity accumulate on the two sides of the sheet, screening both the magnetic field away from the sheet and the monopole-monopole interaction (Fig. 3). This gives the correlation function a finite range, whose inverse is interpreted as a mass of a scalar field. The mass vanishes nonanalytically with g. This effect causes the action to increase by an amount proportional to S. Thus

$$\langle \exp \left(ie \oint_C A_\mu \, dx_\mu \right) \rangle$$

decreases according to an area law.

This picture can be projected into a static one, in the 2 space dimensions. Here the current loop becomes a quark-antiquark pair a distance D apart, creating a Coulomb electric field. The electric field in this picture is simply related to the loop magnetic field by

$$E_i = \epsilon_{ij} B_j \tag{3.1}$$

where $B_3 = B$ remains magnetic. Unlike in Section 2, E now includes a longitudinal part. The Coulomb field is purely longitudinal; a monopole adds an electric vortex in this picture. Vortices of opposite circulations accumulate on both sides of the $q\bar{q}$ line, focusing the electric flux to a narrow tube, which causes confinement (Fig. 4). This same spatial picture of confinement in the monopole approach can be seen in the mechanism of Ref. 3. There one has vacuum tunnelings rather than monopoles, but these are known to be related. Specifically, we saw such a relation in Section 2. The sense in which electric vertices are present in Ref. 3 is the following one: if we isolate a single-gas-particle term in the partition sum (1.b.6), i.e., a configuration $N_{\vec{q}} = \delta_{\vec{p}\vec{q}}$, its contribution to $\langle L_{\vec{q}} \rangle$ is $i \frac{\pi}{g^2} \Delta_{\vec{q}\vec{p}}$. This shows that in the Hamiltonian approach, too, the "gas particles" are associated with electrical vertices. The i results from the fact that in the Hamiltonian formulaism one has real, rather than imaginary time. In both approaches, screening makes the correlation function of the gas well behaved at large spatial separations. But in the monopole approach the screened propagator decreases exponentially with distance (it developes a mass), whereas in Ref. 3 the decrease is a power law. In Section 5 we will discuss this discrepancy, and show how it might be an artifact of certain unjustified approximations made in Ref. 3.

4. Compactness

4.a. The case of a non-periodic potential

In all three models I, II, and III the action is compact in some way. In model I it is compact through the non-Abelian group in which U(1) is embedded; in model II, it is periodic in all three non-vanishing components of $\theta_{\mu\nu}$. In model III, the action is periodic only in B. If the action were not compact, the theory would not necessarily confine for large g. What happens to the small-g arguments for confinement in such a case? If V(B) is non-periodic, $\{\varepsilon_{\vec{p}}\}$ are no longer well defined quantum numbers, since $G_{\vec{p}}$ (defined in 1.b) no longer commute with the Hamiltonian. To put it another way, the fluctuations of $L_{\vec{p}}$ around $\varepsilon_{\vec{p}}$ are no longer discrete. Therefore, even if V(B) has a (finite) number of degenerate minima, (1.b.4) is no longer a reasonable trial wave function. Since no set $\{\varepsilon_{\vec{p}}\}$ is favored now, one should take the one leading to lowest energy, i.e., the vacuum assignment, and confinement is lost. Formulating this in the Wilson loop language, where the loop C is the rectangle shown in Fig. 5, one has for Euclidean time of duration T,

$$F(C) = \langle 0 | e^{-ig\sum_{\vec{p}=0}^{(D-1)\hat{1}} A_1(\vec{p})} e^{-HT} e^{ig\sum_{\vec{p}=0}^{(D-1)\hat{1}} A_1(\vec{p})} | 0 \rangle$$

and as $T \to \infty$, $F(C) \to 1$ unless $\langle 0 | e^{ig\sum_{\vec{p}=0}^{(D-1)\hat{1}} A_1(\vec{p})} | 0 \rangle = 0$. The latter is true for any quark positions only if the vacuum wave function is periodic, which in turn means V(B) must be periodic. Thus there is no area-law and no confinement for non-periodic V(B). But consider a potential with a finite number > 1 of degenerate minima. There are still tunneling and monopoles in the theory, and one can argue that in a path integral calculation of F(C) these will cause screening and confinement, for $g \ll 1$, as describe in Section 3. We will now demonstrate, for a particular double-well potential, how the screening mechanism breaks down, thus affirming that for small g, too, periodicity is needed for confinement as can be deduced from the general argument above.*

4.b. An example

Let us choose a particular non-periodic potential, and demonstrate how screening breaks down, even though V(B) has two degenerate minima and therefore tunneling and monopoles.

The breakdown will appear in a different way than in 4.a. Working on a space-time lattice, one cannot have two tunnelings of the same signature, at the same spatial point and two consecutive times. This is because the potential we have chosen has only two minima. One does obtain a monopole gas, as in Ref. 2, but the monopole charges m(r) do not vary independently, and we show that ruins screening.

We employ here the following convention for the fields:

$$E_i = \partial_3 A_i$$

$$B = \varepsilon_{ij} \Delta_i A_j \tag{4.b.1}$$

The potential we choose is (Fig. 6)

$$V(B) = -a^{-3} \ln\left(e^{-\frac{1}{2}a^3 B^2} + e^{-\frac{1}{2}a^3(B-B_0)^2}\right), \quad a = \text{lattice spacing} \tag{4.b.2}$$

* Note however that just because screening does not appear in the usual way does not prove lack of confinement.

for large B_0. It has two minima, and is non-compact. It corresponds to a free photon in two limits: $B \to 0$ and $B \to \infty$. For small B, V(B) can be expanded around 1/2 B^2 and gives, for example, a 3-photon coupling proportional to

$$B_0^3 \, e^{-\frac{1}{2} B_0^2 a^3}$$

Thus this potential has no strong coupling limit. But we choose it because a monopole gas can be derived exactly from the Path-integral. In that monopole gas, B_0, corresponds to 1/g in Ref. 2.

Setting $a = 1$ for convenience, the correlation function is

$$F(C) = \langle e^{ie \sum_{\mu,r} A_\mu(r) J_\mu(r)} \rangle \propto \int [dA] \exp\left\{ie \sum_{r\mu} A_\mu(r) J_\mu(r) - \frac{1}{2} \sum_{r,i} E_i^2(r) - \sum_r V(B(r))\right\} \tag{4.b.3}$$

where r is the lattice site, and J_μ is[2]

$$J_\mu(r) = \begin{cases} 1 & \text{if the link } r \to r+\hat{\mu} \text{ is on C} \\ -1 & \text{if the link } r+\hat{\mu} \to r \text{ is on C} \\ 0 & \text{otherwise} \quad . \end{cases}$$

For the potential (4.b.2),

$$e^{-\sum_r V(B(r))} = \left(\prod_r \sum_{n_r=0}^{1}\right) \prod_r e^{-\frac{1}{2}(B(r) - n_r)^2}$$

Therefore the integrand in (4.b.3) is Gaussian, and the path integration can be performed exactly. The result is, up to perimenter-law factors,

$$F(C) \propto \left(\prod_r \sum_{n_r=0}^{1} \sum_{N_r=-1}^{1}\right) \exp\left[\sum_{rr'} \frac{B_0^2}{2} N_r \left(\frac{1}{\Delta^2}\right)_{rr'} N_{r'}\right.$$

$$\left. - i(eB_0) \sum_r N_r \frac{\Delta_2}{\Delta^2} (\delta_{y,0}\ \theta_S(x,t))\right] \cdot \prod_r \delta_{N_r\,,\, n_{r+\hat{3}} - n_r} \qquad (4.b.4)$$

If we choose the curve C as in Fig. 6. $\theta_S(x,t)$ is 1 on S and 0 outside. The interpretation of (4.b.4) is a (dilute) monopole gas, where N_r are the monopole charges. They arise from tunneling between vacua which have $B(r) = B_0 n_r$, $n_r = 0$ or 1. This implies a constraint on the monopole charges, which is the Kronecker delta. It is this constraint that ruins the screening, as we will show. The external source coupled to N_r in (4.b.4) is the same as in Refs. 2 and 3. Defining

$$\eta_r = \frac{\Delta_2}{\Delta^2} (\delta_{y,0}\ \theta_S(x,t))\ eB_0 \qquad (4.b.5)$$

We get by the usual resummation techniques,

$$F(C) \propto \left(\prod_r \sum_{n_r} \sum_{N_r} \int_{-\infty}^{\infty} d\phi_r\right) \prod_r \delta_{N_r\,,\, n_{r+\hat{3}} - n_r}$$

$$\times \exp\left\{i \sum_r (\phi_r - \eta_r) N_r\right\} \exp\left\{-\frac{1}{2B_0^2} \sum_{r,\mu} (\Delta_\mu \phi_r)^2\right\} \qquad (4.b.6)$$

Without the constraint, this would give $(\cos \phi_r)$ factors, whose normal ordering in a diagram expansion give the Greens function a mass, similar to the mechanism in Ref. 3. However, due to the constraint,

$$\prod_r \left(\sum_{n_r} \sum_{N_r}\right) \prod_r \delta_{N_r\,,\,n_{r+\hat{3}}-n_r} \exp\left\{i\sum_r (\phi_r - \eta_r) N_r\right\}$$

$$= \prod_r \sum_{n_r} \exp\left\{i\sum_r (\phi_r - \eta_r)(n_{r+3} - n_r)\right\}$$

$$= \prod_r \sum_{n_r} \exp\left\{-i\sum_r n_r (\phi_r - \phi_{r-\hat{3}} + \eta_{r-\hat{3}} - \eta_r)\right\}$$

$$= \prod_r \left\{1 + \exp\left[i(\phi_{r-\hat{3}} - \phi_r + \eta_r - \eta_{r-\hat{3}})\right]\right\}$$

This introduces a derivative coupling of the ϕ_r field, and the self-energy will vanish as momentum $\rightarrow 0$. Thus, screening breaks down.

5. A Detailed Comparison: Models II and III

Models II and III differ in two respects: time is continuous in III while discrete in II, and the action in III is periodic only in B, not in the electric fields. In addition, the approximations used in Refs. 2 and 3 are different, although similar in nature. The strength γ of the linear confining potential comes out different in the two papers; in Section 6 we will show this is mainly an artifact of the different treatments of time. In 5.a we will compute the numerical coefficients in γ, based on Refs. 2 and 3.

A major qualitative difference in the results concerns the nature of screening: in Refs. 1 and 2, the screened Green's function of the monopole gas drops exponentially at large distance, whereas in Ref. 3 it drops as a power. In 5.b we will demonstrate how the approximations made in Ref. 3 may be responsible for this.

5.a. Numerical comparison for the linear potential

In both approaches, the separation energy for a $q\bar{q}$ pair a distance Da apart (a is the spatial lattice spacing) is

$$E(D) \simeq \gamma D \tag{5.a.1}$$

behaves for small g as

$$\gamma \simeq \delta \exp\left(-d \frac{1}{g^2 a}\right)(g^2 a)^c \, a^{-2} \tag{5.a.2}$$

where c,d,δ are numbers. We will compare them for the two approaches. From Ref. 2 we found*

$$\gamma = \frac{2g^2 M}{\pi^2} = \frac{4g\sqrt{2}}{\pi a^{3/2}} \exp\left(-\pi^2 v_0/g^2 a\right) \tag{5.a.3}$$

where M is the screening mass, and v_0 is quoted as being

$$v_0 \simeq 0.253 \quad .$$

Thus,

$$c_2 = \frac{1}{2}$$

$$d_2 = 2.50$$

$$\delta_2 = 1.80$$

where the subscript refers to Ref. 2. In Ref. 3,

$$\gamma \simeq \frac{\pi^2 - 4}{\pi} a^{-2} \exp\left(-\frac{\pi^2 J}{g^2 a}\right)$$

where

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* There are some wrong factors of π and 2 in Ref. 2. In Ref. 1, the formula (V.24) for γ is nuclear; we have obtained there $\gamma = 8\pi e^2 M$. Banks *et al.*[2] rely on Polyakov for γ.

$$J = \lim_{L\to\infty} \frac{1}{4L^2} \sum_{n_1=-L}^{L} \sum_{n_2=-L}^{L} \left(4 - 2\cos\frac{\pi n_1}{L} - 2\cos\frac{\pi n_2}{L}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} dk_1\, dk_2\, (4 - 2\cos k_1 - 2\cos k_2)^{-\frac{1}{2}} \simeq 0.64$$

Thus,

$$c_3 = 0$$

$$d_3 = \pi^2 J \simeq 6.33$$

$$\delta_3 = 1.87$$

5.b. The long-distance behavior of the Green's function

The Green's function of the gas has different long-distance behavior in Refs. 2 and 3. In Ref. 3, $\Delta_{\vec{p}\vec{p}'}$ is that given by (1.b.5) and normal-ordering the diagram expansion modifies it to

$$\Delta^{\mu^2}_{\vec{p}\vec{p}'} = \frac{1}{V} \sum_{\vec{k}} e^{i\vec{k}\cdot(\vec{p}'-\vec{p})} \left((4 - 2\cos k_x - 2\cos k_y)^{\frac{1}{2}} + \mu^2/g^2\right)^{-1} \qquad (5.b.1)$$

where $\mu^2 = e^{-\pi^2 \Delta_0/g^2}$. The latter is actually an equation for μ^2. $\Delta^{\mu^2}_{pp'}$ behaves as $1/|\vec{p}'-\vec{p}|^3$ at large distances, which is enough to eliminate volume divergences. However, this behavior differs from an exponential falloff. One would expect such a falloff when the 3 dimensional monopole gas of Ref. 2 is projected on the two spatial dimensions, as described in Section 3.

The trial wave function χ is chosen to have a Gaussian form in Ref. 3. Actually, for a smooth field potential V(B), a Gaussian would only be accurate near the bottom of the potential wells. But it would be difficult to compute the integrals with a non-Gaussian wave function. However, an approximation is made in Ref. 3 even in deriving the $\Delta_{\vec{p}\vec{p}'}$ that extremizes $\langle\chi|H|\chi\rangle$.

We have found that changing this approximation can make $\Delta_{\vec{p}\vec{p}'}$ well behaved to begin with at large distances. We suggest that such effects may combine to make the photon propogator drop exponentially, as in Ref. 2, but cannot prove it.

Defining as in Ref. 3

$$\Delta_{\vec{p}\vec{p}'} = \frac{1}{V}\sum_{\vec{k}} e^{i\vec{k}\cdot(\vec{p}'-\vec{p})}\gamma_{\vec{k}}$$

We start by assuming that γ_0 is finite, and seek to show that this leads to self-consistent results. Thus, there is no need to freeze out the degree of freedom $\sum_{\vec{p}} B_{\vec{p}}$ as done in (1.b.4), so we remove the δ-function from χ_{trial}. Also, since $\Delta_{\vec{p}\vec{p}'}$ is well-behaved at large distance, we can use the cluster expansion. This consists of expanding the exponential factor in (1.b.6) around $\exp\left[-\frac{\pi^2}{g^2}\Delta_0\sum_{\vec{p}} N_{\vec{p}}^2\right]$, where $\Delta_0 = \Delta_{\vec{p}\vec{p}'}$. For no external sources $\varepsilon_{\vec{p}} = 0$, one must minimize $\langle\chi|H|\chi\rangle = E(\{\gamma_{\vec{k}}\})$ as a function of the $\gamma_{\vec{k}}$ - s. In the first approximation to the cluster expansion, this gives the equation

$$\frac{1}{4}\gamma_0^{-2} + \frac{4\pi^2}{g^2}e^{-\pi^2\Delta_0/g^2} = 0$$

at $\vec{k}^2 = 0$, which has no solution. In the second approximation the equations becomes

$$\frac{1}{4}\gamma_0^{-2} + \frac{4\pi^2}{g^2}e^{-\pi^2\Delta_0/g^2} = \frac{32\pi^4}{g^4}e^{-2\pi^2\Delta_0/g^2}\gamma_0 \qquad (5.b.2)$$

which admits the solution

$$\gamma_0 = \frac{g^2}{8\pi^2} e^{\pi^2\Delta_0/g^2} + O(1)$$

Had we neglected all $e^{-\mathrm{const}/g^2}$ terms in (5.b.2) we would obtain $\gamma_0^{-2} = 0$ as in Ref. 3; it is this approximation that we have changed. Furthermore, for $\vec{k}^2$ small compared to γ_0^{-1} we can expand

$$\gamma_{\vec{k}} = \frac{g^2}{8\pi^2} e^{\pi^2\Delta_0/g^2} + \frac{g^4}{1024\pi^4} e^{2\pi^2\Delta_0/g^2} \vec{k}^2 + O(1)$$

and at finite $\vec{k}$, we obtain as in Ref. 3

$$\gamma_{\vec{k}} = (4 - 2\cos k_x - 2\cos k_y)^{-\frac{1}{2}} + O(g^2) + O(e^{-\pi^2\Delta_0/g^2})$$

Inclusion of higher-order terms in the cluster expansion will change (5.b.2), but to all orders we still have, at least formally,

$$\gamma_0 \sim e^{\pi^2\Delta_0/g^2} .$$

6. A modified Comparison

We will now refine the comparison between the path-integral and variational approaches by making the models, used as inputs for the two procedures, identical. That is, we will apply a path-integral method to model III. As a first step, we will put this model on a lattice in time as well as space, but with lattice spacing b in time and a in space. For b finite, we obtain results similar to those of Ref. 2; for $b = a$ they coincide. This is done by employing approximations similar to those used in Ref. 2. We then take the continuum limit $b \to 0$. There our results are less accurate, since the approximation that all monopole charges are ±1 breaks down. However, we are able to argue that the crucial numerical coefficient d, governing the non-analytic behavior of confinement near $g=0$, is the same as obtained in the variational treatment of Ref. 3.

Model III for continuous time has the following action:

$$\int dt\ a^2 \sum_{\vec{p}} \left[\frac{1}{2} \sum_{i=1}^{2} (E_i(\vec{p},t))^2 + \frac{1}{g^2 a^4} (1 - \cos(ga^2 B(\vec{p},t))) \right] \quad (6.1)$$

where g is the coupling constant, a is the spatial lattice spacing, and $\vec{p}$ denotes a lattice site. g has dimension $(\text{length})^{-\frac{1}{2}}$.

We now make time discrete, with spacing b, and make space-time Euclidean. The action now becomes

$$S = ba^2 \sum_{r} \left[\frac{1}{2} \sum_{i} (E_i(r))^2 + \frac{1}{g^2 a^4} (1 - \cos(ga^2 B(r))) \right] \quad (6.2)$$

where r now denotes a three-dimensional lattice site. In terms of the vector potentials the electric and magnetic fields are

$$B = \frac{1}{a} (\Delta_1 A_2 - \Delta_2 A_1)$$

$$E_i = \varepsilon_{ij} \left(\frac{1}{a} \Delta_j A_3 - \frac{1}{b} \Delta_3 A_j \right) \quad (6.3)$$

Δ_μ are the difference operators in the three directions. We will rescale the A_μ,

$$\theta_i = ga\ A_i$$

$$\theta_3 = gb\ A_3$$

and define

$$\theta_{\mu\nu} = \Delta_\mu \theta_\nu - \Delta_\nu \theta_\mu \quad . \quad (6.4)$$

So that

$$S = \frac{b}{g^2a^2}\sum_r (1-\cos\theta_{12}(r)) + \frac{1}{2g^2b}\sum_{r,i}(\theta_{3i}(r))^2 \qquad (6.5)$$

and our aim is to compute

$$Z = \prod_r \int [d\theta_\mu(r)] \exp\left\{-S - i\sum_{r,\mu} j_\mu(r)\ \theta_\mu(r)\right\} \qquad (6.6)$$

where j_μ is the external source associated with the Wilson loop. To find j for a given contour C, we note that (6.4) gives

$$\sum_{r,\mu} j_\mu(r)\ \theta_\mu(r) = g\sum_{r,i} aA_i(r)\, j_i(r) + g\sum_r bA_3(r)\, j_3(r)$$

And we therefore choose,

$$j_\mu(r) = \begin{cases} 1 & \text{if the link } r \to r+\hat{\mu} \text{ is on C} \\ -1 & \text{if the link } r+\hat{\mu} \to r \text{ is on C} \\ 0 & \text{otherwise} \quad . \end{cases} \qquad (6.7)$$

and this current is conserved: $\Delta_\mu\ j_\mu(r) = 0$.

What is the range of integration over $\theta_\mu(r)$ in (6.1)? The action is periodical in either θ_3 nor θ_1, so we let $\theta_\mu(r)$ vary over the entire real axis.

Separating θ_i into longitudinal and transverse parts,

$$\theta_i(r) = \Delta_i\lambda(r) - \varepsilon_{ij}\ \Delta_j \frac{1}{\Delta_1^2 + \Delta_2^2}\,\theta(r)$$

$$\therefore \quad \theta_{12}(r) = \theta(r)$$

We get the following exponent in (6.6):

$$S + i \sum_{r,\mu} j_\mu \theta_\mu = \frac{b}{g^2 a^2} \sum_r (1 - \cos\theta(r))$$

$$+ \frac{1}{2g^2 b} \sum_r \left(\lambda \Delta_3^2 \Delta_i^2 \lambda + \theta \frac{\Delta_3^2}{\Delta_i^2} \theta - \theta_3 \Delta_i^2 \theta_3 - 2(\Delta_3\theta_3)\Delta_i^2 \lambda \right)$$

$$+ i \sum_r j_3(\theta_3 - \Delta_3 \lambda) - i \sum_r j_i \, \epsilon_{ij} \frac{1}{\Delta_k^2} \Delta_j \theta \qquad (6.8)$$

with $\Delta_i^2 = \Delta_i \Delta_i = \Delta_1^2 + \Delta_2^2$. This expression is quadratic in θ_3 and λ, so integration over these variables can be easily perofrmed, leaving only a path integral over the magnetic field $\theta(r)$. The result is

$$Z \propto \prod_r \int_{-\infty}^{\infty} d\theta(r) \exp \left\{ - \frac{b}{g^2 a^2} \sum_r (1 - \cos\theta(r)) \right.$$

$$\left. + \frac{1}{2g^2 b} \sum_r (\Delta_3 \theta) \frac{1}{\Delta_i^2} (\Delta_3 \theta) + i \sum_r (\Delta_1 j_2 - \Delta_2 j_1) \frac{1}{\Delta_i^2} \theta \right\} \quad (6.9)$$

The proportionality is up to a factor resulting from the self-interaction of the current loop j. Such factors contribute only a perimeter law and will be subsequently ignored. To compute Z we replace the cosine potential by its Villain version,*

$$\exp\left[-\beta(1 - \cos\theta)\right] \rightarrow \sum_{\ell=-\infty}^{\infty} e^{i\ell\theta} \frac{1}{\sqrt{2\pi\beta}} e^{-\ell^2/2\beta} \qquad (6.10)$$

and perform the θ integration, which is now Gaussian:

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* This step is comparable to the approximation of choosing the trial wave function used in Ref. 3.

$$Z \propto \prod_r \sum_{\ell(r)=-\infty} \exp\left\{-\frac{g^2 a}{2}\sum_r \ell(r)\left[\frac{a}{b}+\frac{b}{a}\,\frac{\Delta_i^2}{\Delta_3^2}\right]\ell(r)\right.$$

$$\left. - g^2 b \sum_r \ell(r)\,\frac{1}{\Delta_3^2}\,(\Delta_1 j_2 - \Delta_2 j_1)\right\} \qquad (6.11)$$

Substituting $\ell(r) = \Delta_3 L(r)$ and using the Poisson resummation technique gives

$$Z \propto \prod_r \sum_{m(r)=-\infty} \exp\left\{\frac{2\pi^2}{ba^2g^2}\sum_r m(r)\,\frac{1}{\partial_3^2 + \frac{1}{a^2}\Delta_i^2}\,m(r)\right.$$

$$\left. + \frac{2\pi i}{a^2}\sum_r m(r)\,\frac{1}{\left(\partial_3^2 + \frac{1}{a}\Delta_1^2\right)}\,\frac{1}{\Delta_3}\,(\Delta_1 j_2 - \Delta_2 j_1)\right\} \qquad (6.12)$$

where $\partial_3 = \frac{1}{b}\Delta_3$. This is, once again, the partition sum for a monopole gas, interacting with a current loop. We now consider two cases.

I. $b=a$. In this case (6.12) coincides precisely with (A.2) of Ref. 2. This is interesting, since model III is not symmetric in space and time, and yet yields a symmetrical monopole gas.

II. $b \ll a$. Let us characterize a configuration of $\{m(r)\}$ as

$$m(r) = \sum_k M_{xy}(k)\,\delta_{t,t_{xy}(k)} \qquad (6.13)$$

where $r = (x,y,t)$, and $\{M\}$ are restricted to be integers $\neq 0$. Now a configuration is characterized by a set $\{M_{xy}(k)\}$ of monopole strengths, and a set $\{t_{xy}(k)\}$ giving their times for every spatial site.

In the continuous time limit, $b \ll a$, ∂_3 becomes a true derivative, and $1/\left(\partial_3^2 + \frac{1}{a^2}\Delta_i^2\right)$ becomes a mixed matrix-integral operator. The external source

$$\eta_{xy}(t) = \frac{1}{\Delta_3}(\Delta_1 j_2 - \Delta_2 j_1) \tag{6.14}$$

is a finite, discontinuous function of t for a given xy. Finally, we can replace

$$b\sum_r \to \int dt \sum_{xy}$$

$$\frac{1}{b}\,\delta_{t,t_{xy}(k)} \to \delta(t - t_{xy}(k))$$

And all the above gives:

$$Z \propto \pi \sum_{\{t_{xy}(k)\}} \sum_{\{M_{xy}(k)\}} \exp\left\{ \frac{2\pi^2}{g^2a^2} \sum_{xyk,x'y'k'} M_{xy}(k)\, B_{xyk,x'y'k'}\, M_{x'y'}(k') \right.$$
$$\left. + \frac{2\pi i}{a^2} \sum_{xyk} M_{xy}(k) \left(\frac{1}{\partial_t^2 + \frac{1}{a^2}\Delta_i^2}\, \eta_{xy}(t) \right)_{t=t_{xy}(k)} \right\} \tag{6.15}$$

$$B_{xyk,x_0'y_0'k'} = \left(\frac{1}{\partial_t^2 + \frac{1}{a^2}\Delta_i^2}\, \delta(t - t_{x_0'y_0'}(k'))\, \delta_{x,x_0'}\, \delta_{y,y_0'} \right)_{t=t_{xy}(k)}$$

The summation over $\{t_{xy}(k)\}$ includes summation over the number of monopoles at the point xy. The numbers $B_{xyk,x'y'k'}$ are finite, and proportional to a. According to the procedure employed in Refs. 1,2 and 3 one should at this point neglect all monopoles with charges not ±1, since a monopole with a higher charge gets extra powers of exp $(-1/g^2a)$. But since b << a, these configurations are no longer suppressed, since two big and opposite charges can come close to within a few b on the time axis and have a small action. We do not know how to compute Z in the limit $b \to 0$, but we now make the following assumption: along the time axis, monopoles will cluster into segments. These segments are of length ~ a, since this is the scale of the Green's function B. For two such segments, separated by a time interval $\gtrsim$ a, the interaction B between them is less important than their self-action, so we can apply the approximation $M_{xy}(k) = \pm 1$ to the total charge of a segment. According to this mechanism, (6.15) effectively becomes the gas of Ref. 2, where "segments" replace monopoles. The main difference is in $D_0 = B_{xyk,xyk}$, which is derived from a continuous-time, rather than a discrete-time, Green's function. The d coefficient, defined in Section 5, is therefore

$$d = \frac{2\pi^2}{a} D_0 = \frac{2\pi^2}{a} \left(\frac{1}{\partial_t^2 + \frac{1}{a^2} \Delta_i^2} \delta(t)\, \delta_{x,0}\, \delta_{y,0} \right)_{t=x=y=0} \tag{6.16}$$

The less important coefficients, c and δ, presumably depend on the details of the clustering phenomenon. Going to momentum base, (6.16) gives

$$d = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega \int_{-\pi/a}^{\pi/a} \int_{-\pi/a}^{\pi/a} dk_1\, dk_2 \left[\omega^2 + \frac{1}{a^2} (4 - 2\cos(k_1 a) - 2\cos(k_2 a)) \right]^{-1}$$

$$= \frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} dk_1\, dk_2\, (\omega^2 + 4 - 2\cos k_1 - 2\cos k_2)^{-1}$$

$$= \frac{1}{4} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (4 - 2\cos k_1 - 2\cos k_2)^{-\frac{1}{2}}\, dk_1\, dk_2 \tag{6.17}$$

which is exactly d_3, the result in Ref. 3, as written in Section 5.

7. Conclusions

We have compared the monopole and variational approaches to the problem of confinement in compact QED in 2+1 dimensions. We saw that the mechanisms for screening and confinement are qualitatively the same, and that the crucial coefficient d comes out the same if one takes care to treat time in the same way in the two approaches. The main remaining discrepancy is the nature of screening, which is only a power-law in Ref. 3 but is exponential in Refs. 1 and 2. However, we indicated that this might be an artifact of certain approximations made in Ref. 3. We have also demonstrated that the confinement mechanism breaks down for nonperiodic potentials.

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FIGURE CAPTIONS

1. A piecewise-quadratic field potential.

2. A monopole. B becomes π on surfaces T, k.

3. The Wilson current loop, immersed in a monopole gas.

4. A quark-antiquark pair. The lines are Coulomb field lines, the circles denote electric vortices.

5. The Wilson Contour.

6. A double-well field potential which does not confine.

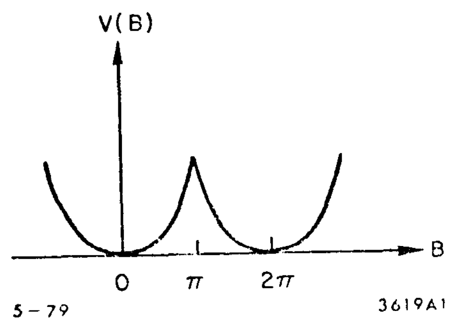

Fig. 1

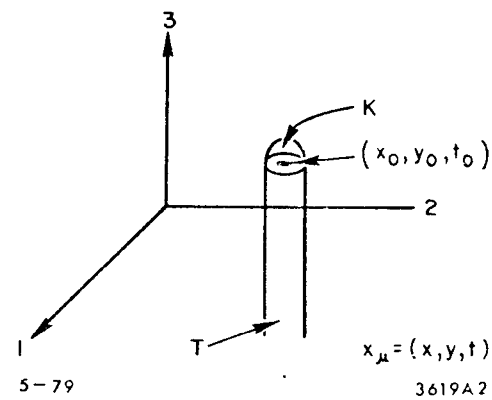

Fig. 2

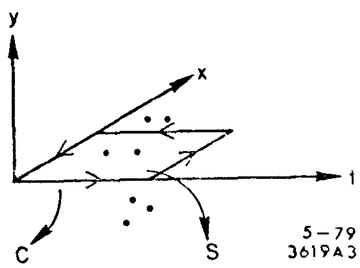

Fig. 3

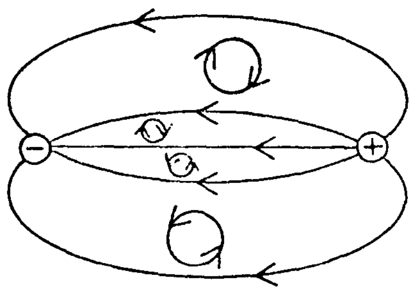

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Fig. 4

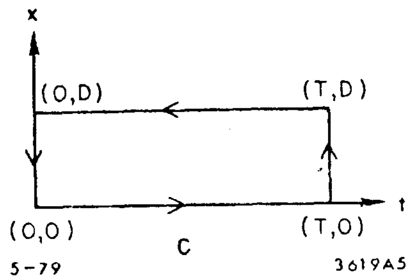

Fig. 5

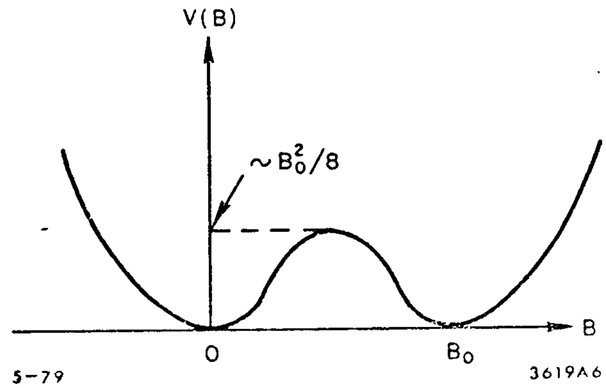

Fig. 6