

SLAC-PUB-2328  
May 1979  
(T)

VANISHING GRAPHS, PLANARITY AND REGGEIZATION\*

Giovanni M. Cicuta<sup>†</sup>

Stanford Linear Accelerator Center  
Stanford University, Stanford, California 94305

ABSTRACT

An infinite class of non-planar skeleton graphs is found to vanish in any non-abelian gauge theory. Thus, the dominance of planar graphs is enhanced, particularly in processes where some momenta are very large.

(Submitted to Phys. Rev. Lett.)

---

\* This work was supported by the Department of Energy under contract number DE-AC03-76SF00515.

<sup>†</sup> Permanent address: Istituto Nazionale Fisica Nucleare,  
Via Celoria 16, 20133 Milano, ITALY.

Non-abelian gauge theories have surprising properties which enhance the interest of the topological expansion. The techniques used in the present perturbative computations are known, yet the interplay of the group properties with the high energy limit of the space-time factors seems very interesting. I shall then list the main results and sketch the derivations.

Every Feynman graph in a non-abelian gauge theory is conveniently written as a product of a group weight factor  $W_G$  times a space-time factor  $M_G$ . Diagrammatic methods which efficiently compute  $W_G$  were described in Ref. 1. The first results (A) and (B) in this Letter easily follow from that paper.

(A) In every non-abelian gauge theory there exist an infinite class of skeleton graphs with vanishing group weight factor.

The lowest order vanishing graph in Fig. 1 is at order  $g^5$ .  $W_G$  vanishes since it results from contracting an antisymmetric tensor  $C_{aji}$  with a tensor  $T_{ijbc}$  symmetric in the exchange  $i \leftrightarrow j$ . A more general vanishing graph is in Fig. 2. Again one checks that the tensor  $T_{ijbc}$  is symmetric in  $i \leftrightarrow j$ , provided the two ladders have the same number of rungs.<sup>2</sup> Next one may notice that in a pure gauge theory (no fermions and no Higgs) the group weight factor of every 2-point function is proportional to  $\delta_{ab}$  and for a 3-point function to  $C_{ijk}$ , then each graph in Fig. 1 or 2 may be understood as a skeleton graph, where every trigluon vertex is replaced by an arbitrary (planar or otherwise) 3-point function and every gluon line may be replaced by arbitrary 2-point function. By use of diagrammatic methods one sees that properly (A) holds in any non-abelian theory with a compact Lie group. All vanishing graphs are

nonplanar.<sup>3</sup> They are many, yet negligible when compared to nonvanishing ones,<sup>4</sup> still they seem to have intriguing consequences, described in property (C).

(B) In the non-abelian SU(N) theory, without fermions, the group weight factor of a graph is a polynomial in  $N^2$ , i.e., the topological expansion is a power series in  $(1/N^2)$  rather than  $(1/N)$ .<sup>5</sup>

Indeed to compute the group factor  $W_G$  one performs two steps:<sup>1</sup>

(a) reexpress all three gluon vertices in terms of the defining representation

$$i C_{ijk} = 2 \text{Tr}(T_i T_j T_k - T_k T_j T_i) \quad (1)$$

(b) replace internal gluon lines with gluon projection operators:

$$2 (T_i)_b^a (T_i)_d^c = \delta_d^a \delta_b^c - \frac{1}{N} \delta_b^a \delta_d^c \quad (2)$$

However, because of the trilinear nature of the coupling<sup>6</sup> the singlet term  $-\frac{1}{N} \delta_b^a \delta_d^c$  of step (b) is seen to cancel and it can be ignored. That is, one may use, for the internal gluon lines, the simpler replacement, proper for the U(N) theory,

$$2 (T_i)_b^a (T_i)_d^c = \delta_d^a \delta_b^c \quad (3)$$

For 2-point functions and 3-point functions, where there is just one basic tensor (respectively  $\delta_{ab}$  and  $f_{abc}$ , the group weight  $W_G$  of the generic Feynman graph is

$$W_G = \delta_{ab} (Ng^2)^s \sum_{P=0}^{[s/2]} c_P (N^2)^{-P} \quad \text{at order } g^{2s} \quad (4)$$

$$W_G = f_{abc} g (Ng^2)^s \sum_{P=0}^{[s/2]} c_P (N^2)^{-P} \quad \text{at order } g^{2s+1} \quad (5)$$

where the leading coefficient  $c_0$  is different from zero if and only if the graph is planar.<sup>7</sup>

For the 4-point function one has six basic tensors,<sup>8</sup> three of which (A, B, C) have one boundary<sup>5</sup> and three (D, E, F) have two boundaries.

At order  $g^{2s+2}$  one finds

$$W_G = g^2 (g^2 N)^s \left\{ A \sum_0^{[s/2]} a_P (N^2)^{-P} + B \sum_0^{[s/2]} b_P (N^2)^{-P} + C \sum_0^{[s/2]} c_P (N^2)^{-P} + \frac{1}{N} \left[ D \sum_0^{[s/2]} d_P (N^2)^{-P} + E \sum_0^{[s/2]} e_P (N^2)^{-P} + F \sum_0^{[s/2]} f_P (N^2)^{-P} \right] \right\} \quad (6)$$

Higher n-point functions have weights  $W_G$  expressed in the same form after one has taken care of the N factors associated with the number of boundaries of the basic tensors.

One may remark that the properties (A) and (B) hold for every  $N \geq 2$ , every value of the coupling constant  $g$  and every dimensionality (complex too) of the space-time dimensions. They also hold for spontaneously broken theories, provided the local gauge group still survives as a global symmetry.

(C) There are kinematical (asymptotic, leading log) regions of the Lorentz invariants where the large N expansion is exact (graphs with non-dominant weight are not leading log dominant).

The property (C) will here be shown by quoting some results of a new study<sup>9</sup> of reggeization in nonabelian gauge theory. The high energy behavior (large  $s$ , fixed  $t$ ) of the elastic scattering amplitude has been computed in a pure gauge (without fermions or Higgs) SU(N) theory. The main differences with previous investigations<sup>10</sup> are: (a) dimensional regularization is used, instead of the Higgs bosons; (b) an improved treatment of the large energy limit of Feynman integrals, by which the

numerators of relevant Feynman integrals are decomposed into sums of terms, each of which may be associated to a contracted scalar Feynman integral. The asymptotic behavior of the latter is then computed by counting the number and length of the shortest  $t$  paths.<sup>11</sup> The main results of previous studies<sup>10</sup> are reproduced: in the  $t$ -channel with the quantum numbers of the gluon (adjoint representation  $t$ -channel) there is just one Regge pole, the reggeized gluon, with the trajectory:<sup>12</sup>

$$\alpha(t) = 1 + g^2 \frac{N}{2} t K_{d-2}(t) + O\left((g^2 N)^2\right) \quad (7)$$

while in the Pomeron channel the perturbative results are consistent with the Froissart violating fixed cut previously found.<sup>10</sup> Yet the improved treatment of the space-time factor<sup>9,13</sup> shows a different mechanism leading to these results. The set of leading (i.e., leading  $\log s$ ) planar graphs is divided into two sets, the strictly planar graphs<sup>14</sup> and the set of graphs obtained by the former after the exchange  $s \leftrightarrow u$ . At order  $g^{2n+2}$  the leading  $\log s$  contribution of the first set in the  $t$ -channel of the adjoint representation is

$$C = g^2 T_1 s(t)^{n-1} \frac{[g^2 K_{d-2}(t) \log s]^n}{n!} \quad (8)$$

while the contribution of the second set ( $s \leftrightarrow u$  interchanged) is

$$\tilde{C} = g^2 T_2 (-s)t^{n-1} \frac{[g^2 K_{d-2}(t) \log s]^n}{n!} \quad (9)$$

where the group weight tensors  $T_1$  and  $T_2$  have the property  $T_1 - T_2 = (N/2)^n T_{Ad}$  where  $T_{Ad}$  is the projection operator of the adjoint representation.

An example is given in Fig. 3, where use is made of Jacobi identity and

triangle contraction. The relevant (leading log s) nonplanar graphs are also divided into two sets such that the second set may be obtained by the first set after  $s \leftrightarrow u$  permutation. Each graph of the first set gives the (leading log s) contribution:

$$D = (T_P + \frac{1}{N} T_3 + T_4) g^4 (g^2 N)^{P-1} s f(t) (\log s)^P \quad (10)$$

where  $T_P$  is the projection operator for the pomeron channel,  $T_3$  is a tensor contributing to the adjoint channel,  $T_4$  is a tensor contributing to other channels, furthermore  $T_P$ ,  $T_3$  and  $T_4$  are symmetric under the interchange  $s \leftrightarrow u$ . Then after summing the contribution of the second set of nonplanar graphs (see an example of the cancellation in Fig. 4, the last term vanishes as it contains as a subgraph the Fig. 1) the contribution of nonplanar graphs is of the order  $g^4 s f(t) (Ng^2 \log s)^{P-1}$  in the pomeron channel and next to the leading logs in the adjoint representation t-channel. In other words, to prove reggeization of the vector mesons, one does not need the nonplanar graphs. Furthermore, because the non-leading N terms in planar graphs cancel (Fig. 3) one would obtain the correct results, in the leading logs approximation, by computing only the leading N contribution of the weight factors  $W_G$ , rather than their complete value.

#### ACKNOWLEDGMENTS

I would like to thank G. Marchesini for being able to involve me into this computation, E. Witten for explaining the topological expansion, P. Butera and M. Enriotti for many discussions and R. Blankenbecler for the kind hospitality of the SLAC theory group.

This work was supported by the Department of Energy under contract number DE-AC03-76SF00515.

REFERENCES AND FOOTNOTES

1. P. Cvitanovic, Phys. Rev. D14, 1537 (1976). Many references to previous work on diagrammatic methods are given.
2. It was already found in Ref. 1 that the group weight factor  $W_G$  vanishes for the graph in Fig. 1, which is topologically the complete bipartite  $K_{3,3}$  (after adding the external vertex<sup>3</sup>), and the following one, at order  $g^9$  which is topologically the Petersen graph, and is obtained in Fig. 2 by deleting all the rungs of the two ladders and one side of each ladder. Also some generalizations were noticed, which were obtained by replacing a gluon line with a fermion line or by including such graphs as subgraphs in other larger graphs.
3. Planarity of Feynman graphs is related to the definition of planarity in graph theory. See N. Nakanishi, Graph Theory and Feynman Integrals, Gordon and Breach Pub., 1971, §4.
4. The class of vanishing skeletons is discussed in P. Butera, G. Cicuta, M. Enriotti, SLAC-PUB-2376 (1979). At large order  $n$  in perturbation theory, the number of vanishing skeletons is shown to be roughly  $\left(\frac{n}{2}\right)!$
5. G. 't Hooft, Nucl. Phys. B72, 461 (1974). G. Veneziano, Phys. Lett. 52B, 220 (1974) and Nucl. Phys. B74, 365 (1974). M. Ciafaloni, G. Marchesini, G. Veneziano, Nucl. Phys. B98, 472 (1975), and *ibid.* B98, 493 (1975). Recent works that include many references to further work are G. Chew and C. Rosenzweig, Physics Reports C41, 263 (1978). E. Witten, Harvard preprint HUTP 79/A007 (1979). Property (B) was also stated in G. P. Canning Phys. Rev. D12, 2505 (1975) where graphs without color external sources are discussed. The consequences of Eq. (6) in channels of definite quantum numbers are discussed in Ref. 4.

6. It is well-known that the four-gluon couplings may always be replaced by a sum of three-gluon couplings.
7. All sums in Eqs. (4-6) extend up to  $[s/2]$ , the entire part of  $s/2$ .
8. I am here using the notation of P. Yeung, Appendix, Phys. Rev. D13, 2306 (1976).
9. G. M. Cicuta and G. Marchesini, in preparation.
10. The list of previous investigations is long and this is a subset:  
M. Grisaru, H. Schnitzer, H. Tsao, Phys. Rev. D8, 4498 (1973);  
H. T. Nieh and Y. P. Yao, Phys. Rev. D13, 1082 (1976); L. N. Lipatov, Sov. J. Nucl. Phys. 23, 338 (1976); E. A. Kuraev, L. N. Lipatov and V. S. Fadin, Sov. J. Phys. JETP 45, 199 (1975); B. M. McCoy and T. T. Wu, Phys. Rev. D13, 1076 (1976); L. Tyburski, Phys. Rev. D13, 1107 (1976); C. Y. Lo and H. Cheng, Phys. Rev. D15, 2959 (1977); H. Cheng et al., Phys. Lett. 76B, 129 (1978); preprints by C. Y. Lo, whom I thank for making me available his unpublished work.  
J. Bartels, Phys. Lett. 68B, 258 (1977); J. B. Bronzan and R. L. Sugar, Phys. Rev. D16, 466 (1977).
11. See for instance, R. J. Eden et al., The Analytic S-Matrix, Cambridge University Press, 1966, Chapter 3.
12. To compare the present results with the previous ones, one should imagine the computation with double infrared regularization, that is the SU(N) model with Higgs particles<sup>10</sup> in d-dimensional space-time. To work with dimensionless coupling constant, in d-dimensional space-time, one usually replaces  $g \rightarrow g(m^2)^{2-d/2}$ . However, in the Regge trajectory there is an effective coupling  $\gamma = g^2 N/2$  which may be replaced by a dimensionless coupling constant  $\gamma \rightarrow \gamma(m^2)^{2-d/2}$ .



One would then obtain the reggeized gluon with the trajectory

$$\alpha(t) = 1 + (m^2)^{2-d/2} (t - \mu^2) K_{d-2}(t)$$

where  $K_{d-2}(t)$  is the usual bubble graph in an Euclidian  $d-2$  dimensional space-time.

$$K_{d-2}(t) = \frac{1}{2\pi} \Gamma\left(\frac{d-2}{2}\right) \Gamma\left(3 - \frac{d}{2}\right) \int_0^1 \frac{d\alpha}{[\alpha(1-\alpha)t + \mu^2]^{3-d/2}}$$

This trajectory smoothly approaches Eq. (7) by removing the Higgs or it goes smoothly into the usual  $\alpha(t) = 1 + g^2 \frac{N}{2} (t - \mu^2) K_2(t) + \dots$  by removing the dimensional regularization, that is, by replacing  $d=4$ . Either way, one has a singular limit if one removes the second regularization. Still our way may have some advantage because by letting  $d$  approach four from above in Eq. (7), the slope of the trajectory becomes infinite while the intercept is always one, while in letting  $\mu \rightarrow 0$  in the customary Yang-Mills-Higgs system, the slope becomes infinite while the intercept is one only in the limit. I thank A. White for a discussion on this and for mentioning this possible advantage for a supercritical Pomeron.

A. White, CERN preprints TH 2592 (1978) and TH 2629 (1979).

Equation (7) may be written in a more explicit way:

$$\alpha(t) = 1 + \left[ \gamma \left( \frac{t}{m^2} \right)^{\frac{d}{2}-2} \left( \frac{\sqrt{\pi}}{2^{d-6}} \right) \frac{\Gamma\left(\frac{d}{2}-1\right)}{\left(3 - \frac{d}{2}\right) \left[ \sin\left(\frac{\pi d}{2}\right) \right] \Gamma\left(\frac{d-5}{2}\right)} \right]$$

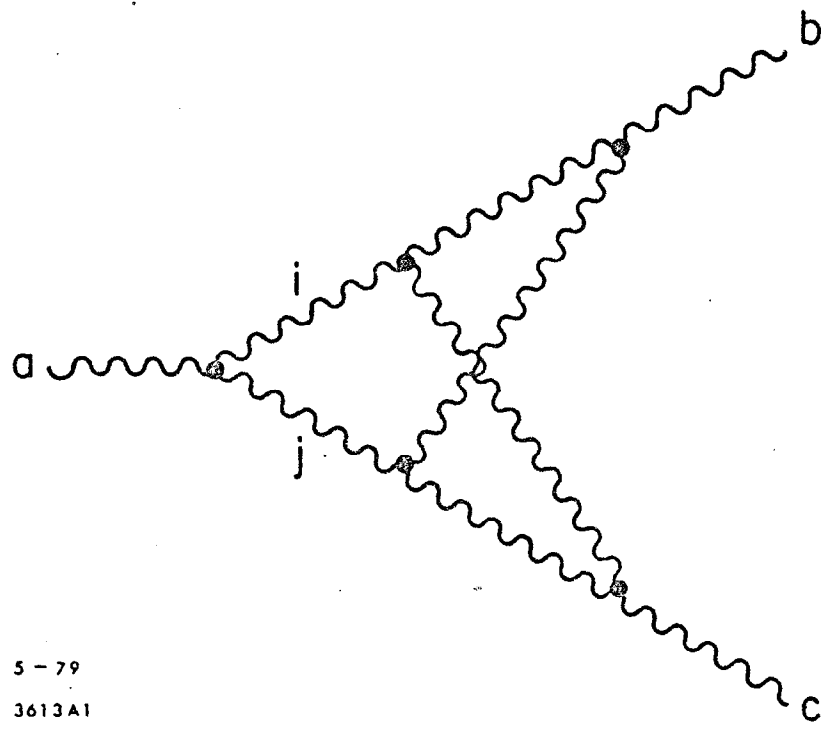
The unusual behavior of the trajectory for  $d < 4$  is perhaps related to the non-adequacy of the lowest order expansion of the trajectory in a situation in which the infrared divergence becomes more severe.

I thank R. Sugar for comments about it and encouragement.

13. It has long been known that the infinite momentum techniques usually employed in Ref. 10 do not correctly evaluate the asymptotic behavior of each Feynman graph, but may produce the correct result for well chosen sets of graphs.
14. Here strictly planar graphs indicate the planar Feynman graphs that have the double spectral function  $\rho(s,t)$ .

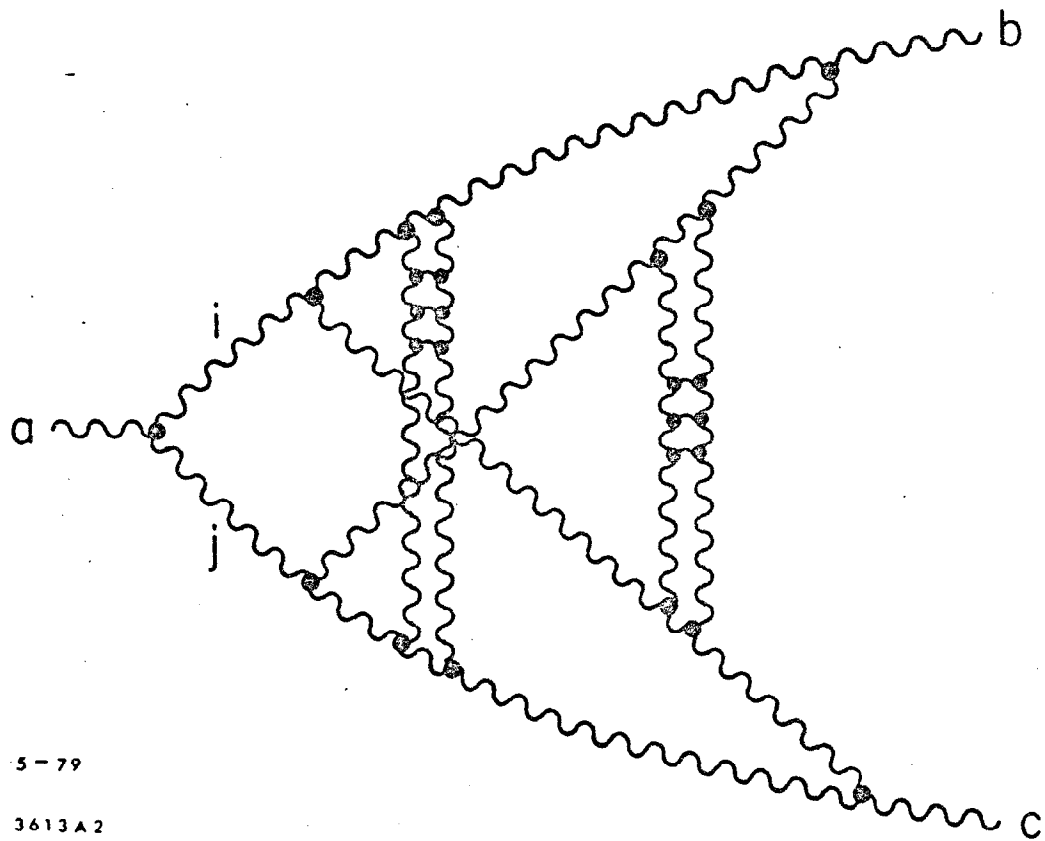
FIGURE CAPTIONS

1. The lowest order graph with vanishing group weight.
2. A general class of graphs with vanishing weight.
3. An example of the relation  $T_1 - T_2 = (N/2)^n T_{Ad}$ , pertinent to the leading log s planar graphs.
4. An example of the cancellation that occurs in leading log s, in nonplanar graphs, related to the existence of the vanishing graphs.



5 - 79  
3613A1

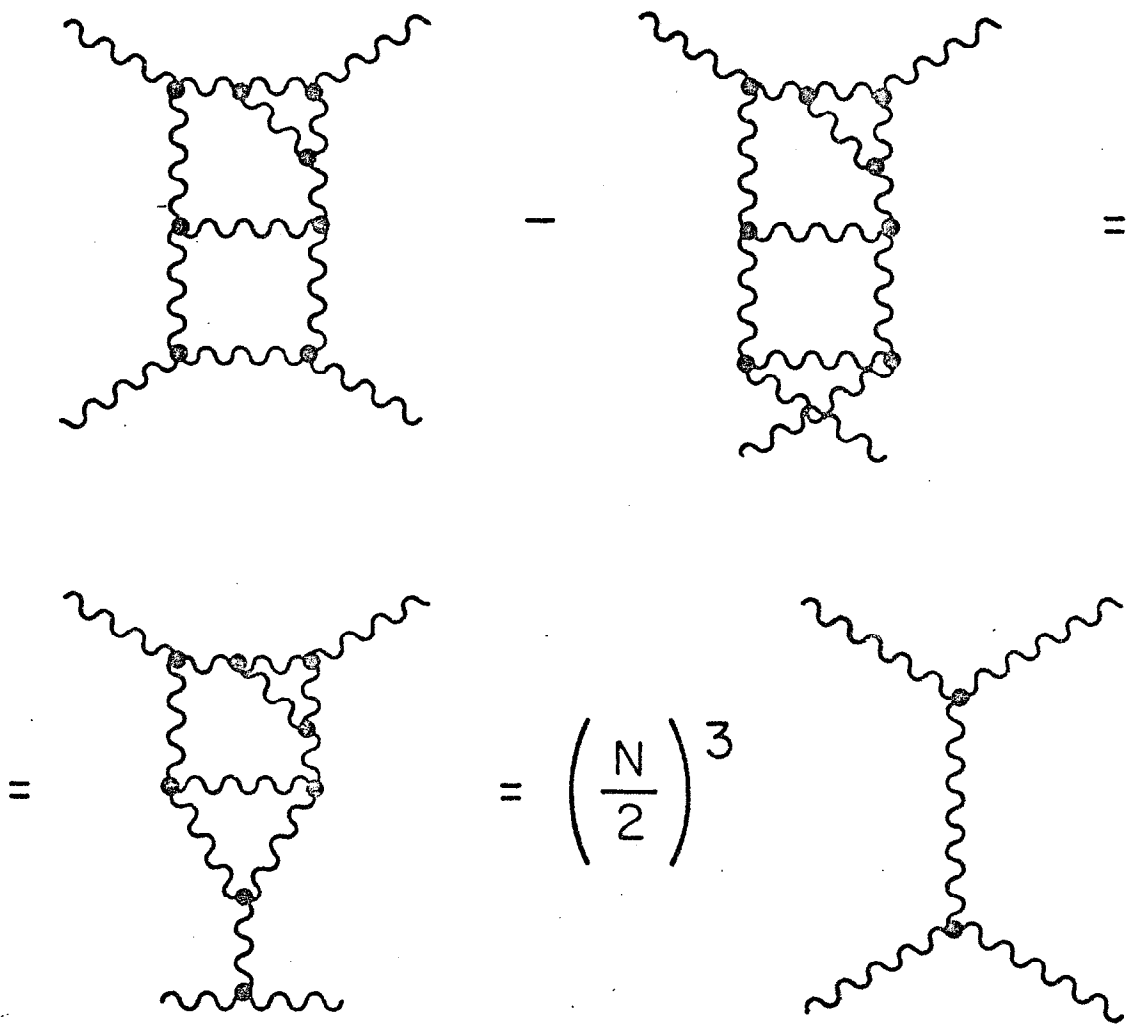
Fig. 1



5-79

3613A2

Fig. 2

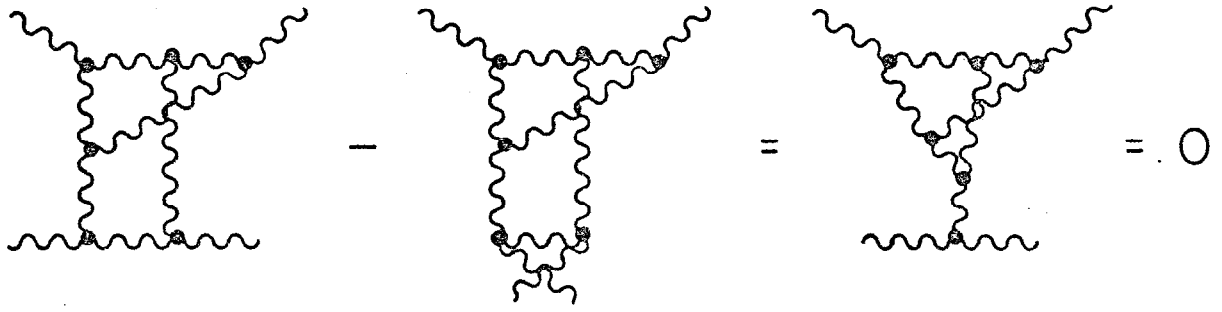


$$= \left( \frac{N}{2} \right)^3$$

5-79

3613A4

Fig. 3



5-79

3613A5

Fig. 4