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The Distribution Function of the Weak Beam Taking the Interaction with the Strong One into Account *

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ABSTRACT

Developing the idea of fast particle mixing due to a strong nonlinearity of the beam-beam interaction, the distribution function of the weak bunch in the phase space of vertical motion is found. The features of this distribution are discussed and compared to the Gaussian one.

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1. INTRODUCTION

In the previous work by the author[1] an approach to the problem of the interaction of a weak bunch with a strong one was developed, in which the interaction was assumed to be the source of an additional diffusion force. The derivation essentially uses averaging of the force (or the "kick") and its square over the perturbed distribution function of the weak bunch. The average of the first power of the force was assumed to be zero whereas the average of the square of the kick resulted in increase of the bunch size. The calculation of the equilibrium bunch size was the subject of that paper.

Under these conditions, the distribution function of the weak bunch in vertical article coordinates appears to be Gaussian in contradiction with known experimental observations (see, for example, Reference 2).

As the next step in developing the above mentioned idea, in this paper we find the distribution function of the perturbed weak bunch. The calculations are done for the same model as in Reference 1, namely one dimensional vertical motion in an electron storage ring. Damping as well as other sources of stochastic forces in the motion are also taken into account.

Section 2 is almost word-for-word the same as Section 2 in Reference 1; it is repeated here for the convenience of the reader. In Section 3 we find the distribution function, and in Section 4 we discuss its features.

2. EQUATION FOR THE DISTRIBUTION FUNCTION

Let Y and $\dot{Y} = dY/dt$ be the excursion from the median plane and corresponding velocity of a particle of the weak bunch at the interaction point. It is convenient to consider particle coordinates in units of the vertical size Σ of the strong bunch ($\Sigma = \sqrt{2\langle Y^2\rangle}$):

$$y = Y/\Sigma \quad , \tag{1}$$

$$\dot{y} = \dot{Y}/\Sigma \quad . \tag{2}$$

If the length of the strong bunch is much less than the wave length of the vertical oscillations then by one passage through the strong bunch the coordinates of the particle are changed by:

$$\Delta y = 0 \quad , \tag{3}$$

$$\Delta \dot{y} = F(y) \quad . \tag{4}$$

The actual dependence F(y) can be found at least in principle for any given particle distribution of the strong beam. Taking this change into account we can write the following equation for the particle distribution function $f(t, y, \dot{y})$ of the weak bunch:

$$\frac{\delta f}{\delta t} + \dot{y}\frac{\delta f}{\delta y} - 2\alpha \frac{\delta}{\delta \dot{y}} (\dot{y}f) - \omega^2 Q^2 y \frac{\delta f}{\delta \dot{y}}$$

$$= q_0 \frac{\delta^2 f}{\delta \dot{y}^2} + \sum_k \delta(t - t_k) \left\{ f[t_k, y, \dot{y} + F(y)] - f(t_k, y, \dot{y}) \right\} . \tag{5}$$

The left hand side of this equation describes the change of the function f due to particle oscillations with a frequency ωQ and a damping rate α. The right hand side represents the change of particle density in the phase space $(y, \dot{y})$ due to all possible reasons but beam-beam interaction (the first term) and due to the interaction occurring at the times $t_k = t_0 + 2\pi k/\omega n$, $k = 1, 2, \ldots$, (the second term; n is the number of interactions on one revolution).

If we are not interested in details of the fast time variations of the distribution function which are of the order of magnitude of one revolution period or less than the sum on the right hand side of (5) can be simplified:

$$\sum_k \delta(t-t_k) \left\{ f[t_k, y, \dot{y}+F(y)] - f(t_k, y, \dot{y}) \right\}$$

$$\simeq \frac{n\omega}{2\pi} \left\{ f[t, y, \dot{y}+F(y)] - f(t, y, \dot{y}) \right\} \quad . \qquad (6)$$

We can further expand the difference in (6) into series in $F(y)$ and retain the first two terms of the expansion. The term proportional to the first power of $F(y)$ can be considered as part of the restoring force acting on a particle. The parabolic potential well is then distorted, which causes deviation of the distribution function from the Gaussian form.

The second term we treat in exactly the same way as in Ref. 1. Using the idea of the fast mixing of particles due to strong nonlinear force, we can substitute the coefficient $F^2(y)$ by the values obtained from averaging it over an ensemble of the particles of the weak bunch.

Let us define the diffusion coefficient due to interaction as

$$q_{int} = \frac{n\omega}{4\pi} \langle F^2(y) \rangle \quad , \qquad (7)$$

where brackets stand for averaging the function $F^2(y)$ over the distribution function $f(t, y, \dot{y})$, which satisfies the following Focker-Planck equation:

$$\frac{\delta f}{\delta t} + \dot{y}\,\frac{\delta f}{\delta y} - 2\alpha\,\frac{\delta}{\delta \dot{y}}\,(\dot{y}f) - \left[\omega^2 Q^2 y + \frac{n\omega}{2\pi}\,F(y)\right]\frac{\delta f}{\delta \dot{y}} = (q_0 + q_{int})\,\frac{\delta^2 f}{\delta \dot{y}^2}\ . \qquad (8)$$

3. THE DISTRIBUTION FUNCTION

Let us denote

$$\frac{dU}{dy} = \omega^2 Q^2 y + \frac{n\omega}{2\pi}\,F(y) \qquad , \qquad (9)$$

$$q = q_0 + q_{int} \qquad (10)$$

From (9) we find

$$U(y) = \frac{\omega^2 Q^2 y^2}{2} + \frac{n\omega}{2\pi}\int_0^y F(y)\,dy \qquad , \qquad (11)$$

when we choose the integration constant in (11) in such a way that $U(0) = 0$.

It is easy to check directly that the function

$$f(y, \dot{y}) = C\,\exp\left\{-\,\frac{\alpha\dot{y}^2}{q} - \frac{2\alpha U}{q}\right\} \qquad (12)$$

is the stationary ($\partial f/\partial t = 0$) solution of equation (8), C being a normalization constant.

It can be evaluated from the condition

$$\iint_{-\infty}^{\infty} d\dot{y}\,dy\,f(y, \dot{y}) = 1 \qquad . \qquad (13)$$

Another unknown constant in (12), namely q, should be found from (10) and (7):

$$q = q_o + \frac{n\omega}{4\pi} \frac{\int_{-\infty}^{\infty} F^2(y) e^{-\frac{2\alpha U(y)}{q}} dy}{\int_{-\infty}^{\infty} e^{-\frac{2\alpha U(y)}{q}} dy} \quad . \qquad (14)$$

Let us instead of q introduce a dimensionless variable, d:

$$d^2 = q/\alpha\omega^2 Q^2 \quad . \qquad (15)$$

Then for d we have the following transcendental equation:

$$d^2 = 1 + \eta \Phi (d) \quad , \qquad (16)$$

where

$$\eta = n\omega / 4\pi\alpha\omega^2 Q^2 \qquad (17)$$

$$\Phi(d) = \int_{-\infty}^{\infty} F^2(y) e^{-V(y)} dy \Big/ \int_{-\infty}^{\infty} e^{-V(y)} dy \quad , \qquad (18)$$

$$V(y) = (y^2 + \frac{n}{\pi\omega Q^2} \int_0^y F(y)\, dy) \Big/ d^2 \quad . \qquad (19)$$

Substituting the solution d of equation (16) into (12) we finally get the distribution function:

$$f(y, \dot{y}) = \frac{\exp\left\{-\frac{\dot{y}^2}{\omega^2 Q^2 d^2} - V(y)\right\}}{2\sqrt{\pi}\, d\, \omega\, Q \int_0^\infty e^{-V(y)}\, dy} \qquad (20)$$

4. DISCUSSION

The resulting distribution function (20) is still Gaussian in velocity $\dot{y}$. The characteristic constant d^2 of the distribution no longer has the simple meaning of the dispersion, as it has in the limit of vanishing current.

Let us look at this more closely, assuming that the distribution function of the strong bunch is Gaussian in all three dimensions. In this case the function F(y) from (4) is known to be:[3]

$$F(y) = \xi \phi_b(y) \quad , \qquad (21)$$

where

$$\xi = 2\pi\omega Q \Delta Q\left[\left(\sqrt{1+b^2}+b\right)\Big/\left(\sqrt{1+b^2}-b\right)\right]^{\frac{1}{2}} \quad , \qquad (22)$$

$$\phi_b(y) = y\int_0^1 \frac{du}{\sqrt{u+b^2}} \exp\left(-uy^2\right) \quad . \qquad (23)$$

In these expressions

$$\Delta Q = e^2 N\beta \Big/ \left[2E(\Sigma_h + \Sigma)\Sigma\right] \qquad (24)$$

is the Courant parameter giving the linear tune shift of the vertical betatron oscillations of the weak beam particle due to the electromagnetic interaction with the strong bunch containing N particles.[4] E is the particle energy and β is the value of betafunction at the interaction point. Σ_h and Σ are the dispersions of the strong bunch distribution in horizontal and vertical planes.

Parameter b is defined as follows:

$$b = (\Sigma/\Sigma_h) \,/\, \sqrt{1-(\Sigma/\Sigma)^2} \quad . \tag{25}$$

For a small aspect ratio of the beam,

$$b \simeq \Sigma/\Sigma_h \quad .$$

The function $\phi_b(y)$ describes y-dependence of the force acting on a particle from the side of the strong beam. For small values of y

$$\phi_b^{\ell}(y) \simeq 2(\sqrt{1+b^2} - b)y \tag{26}$$

gives the linear part of the force.

It is interesting to find $\langle y^2 \rangle$ using (20), assuming the force F(y) to be linear:

$$F(y) = \xi\phi_b^{\ell}(y) \quad . \tag{27}$$

From (19) we get in this case:

$$V(y) = \frac{y^2(1+2n\Delta Q/Q)}{d^2} \tag{28}$$

Averaging y^2 over the function (20), it is easy to find

$$\langle y^2 \rangle = \frac{d^2}{2[1 + 2n\Delta Q/Q]} \quad , \tag{29}$$

as it should be. The term in brackets in the denominator of (29) describes the change of the oscillation tune due to beam-beam interaction.

In general the changes of the function (20) in comparison to the distribution function at zero current are the following:

1. The characteristic constant d^2 is bigger than one [cf., (16)], making the distribution function broader in y as the current of the strong beam increases.
2. The potential well is narrower than corresponding parabola for an attractive force [cf., (19)], making the distribution narrower in y (compared to the corresponding Gaussian shape) for the given value of the current of the strong beam.
3. The change in V(y) also changes the normalization constant of the distribution function.

The last comment is connected with equation (18), which determines the characteristic constant d by equation (16). From the function F(y) in the integrand of equation (18) should be subtracted its linear part, since only the nonlinear part of the force can bring the particle motion to stochasticity. One way to do this subtraction was discussed in detail in Reference 1. In contrast to this, such a subtraction should not be done in the integrand of equation (19) since that part of the calculations does not use any averaging over the distribution function.

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REFERENCES

1. S. A. Kheifets, "Weak-Strong Instability as a diffusion process," to be published in the Proceedings of the 1979 Particle Accelerator Conference, San Francisco, California, March 12-14, 1979; See alternately: S. A. Kheifets, SLAC-PUB-2252, PEP-Note-278, February 1979.

2. M. Cornacchia, "Report on some beam-beam functional dependencies in SPEAR," in BNL 25703, pp. 102-123, Brookhaven National Laboratory, Upton, New York, 11973.

3. J. E. Augustin, "Potentiel dû à Une Répartition de Charge Bigaussienne," Rapport Technique 3-70, Orsay (1970). S. Kheifets, "Potential of a Three-Dimensional Gaussian Bunch," PETRA - Kurzmitteilung 113 (1976).

4. M. Sands, "The Physics of Electron Storage Rings: An Introduction," Stanford Linear Accelerator Report SLAC-121, University of California at Santa Cruz Report UC-28 (ACC); p. 65 (1970).