SLAC-PUB-2300
March 1979
(T/E)

ON THE CALCULATION OF GLUON BREMSSTRAHLUNG IN QCD

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ABSTRACT

It is shown that several published calculations of the process

$$e^+e^- \to Gq\bar{q}$$

in the one-photon exchange approximation are in error. The calculations are all (at least) missing a factor of $2/x_3$ in the differential cross section

$$d\sigma/dx_1dx_2$$

in the rest frame of the virtual photon, where $x_i\sqrt{s}/2$ is the energy of the particle i in the final state in this frame and particle 3 is the gluon G. The reason for the omission is traced to the failure of these authors to heed the constraint of four-momentum conservation when changing to the parameters x_i. Phenomenological consequences of this omission will appear elsewhere.

(Submitted to Phys. Rev. D.)

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*Work supported by the Department of Energy under contract number EY-76-C-03-0515.

In view of the popularity of the QCD theory as a candidate for the strong interaction dynamical theory, it is not surprising that a number of authors[1-4] have considered the process

$$e^+e^- \to Gq\bar{q}$$

in the one photon exchange approximation. The diagrams of lowest order in the QCD coupling constant g are illustrated in Fig. 1, along with the kinematics. For those not familiar with QCD, the Lagrange function is written here as

$$\mathscr{L} = -\frac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu} + \bar{\psi}(i\not{\partial} - g\vec{t}\cdot\vec{\not{A}})\psi - m_q\bar{\psi}\psi \tag{1}$$

where

$$\vec{F}_{\mu\nu} = \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu - g\varepsilon_{\to bc}A^b_\mu A^c_\nu \quad . \tag{2}$$

In (1) and (2), $\vec{A}_\mu$ is the gluon gauge field and $\varepsilon_{\to bc}$ are the color gauge group structure constants (we will use SU(3) for the color group). Thus, g is the gauge coupling constant and $\vec{t}$ are the generators of the color group in the fermion representation, which we call R. Notice that we take the mass matrix to be trivial (a multiple of I) in color and in flavor. The flavor group will be suppressed throughout. Thus, a sum over flavor is to be understood in (1). The ghost and gauge fixing terms are not relevant for Fig. 1 and, hence, are not shown in (1).

From the diagrams of Fig. 1 one sees immediately that, aside from the matrix elements

$$t^i_{ab}$$

at the gluon-q-$\bar{q}$ vertices, the computation is entirely analogous to the computation of photon bremsstrahlung in $e^+e^- \to \mu^+\mu^-\gamma$.[5] Thus, we will ultimately wish to compare any answer we get with the known results of Ref. 5, for example.

Our purpose for the present communication is to point out how all of the calculations in Refs. 1-4 are missing a factor of $2/x_3$ in the e^+e^- center of momentum frame, where (see Fig. 1 for the kinematics), for $m_q = 0$,

$$(p_i + p_j)^2 \equiv s(1 - x_k) \quad , \quad i \neq j \neq k, \; i, j, k = 1, 2, 3; \tag{3}$$

here

$$s = (q_1 + q_2)^2 \tag{4}$$

is the square of the e^+e^- center of momentum frame total energy. We will always have in mind the limit $m_q \to 0$ in (1) so as to facilitate comparison with Refs. 1-4. Further, we will always work in the center of momentum frame so that

$$Q \equiv q_1 + q_2 = (\sqrt{s}\,, \vec{0}) \quad . \tag{5}$$

Thus, from (3) we see that $x_i \sqrt{s}\,/\,2$ is the energy of final state particle i in this frame.

Since we intend to depart from the results of Refs. 1-4, we will be thorough in deriving the correct expression for

$$d\sigma/dx_1 dx_2 \quad . \tag{6}$$

To begin, we note that in the Bjorken and Drell conventions[6] the amplitude of Feynman for the process in Fig. 1 is

$$M = (2\pi)^4 \delta(Q - p_1 - p_2 - p_3)\sqrt{\frac{m_\ell}{E}}\,\bar{v}(q_1)\,(-ie\gamma^\alpha)\,u(q_2)\sqrt{\frac{m_\ell}{E}}\left(\frac{-ig_{\alpha\beta}}{s}\right)$$

$$\times \Big\{ \bar{u}_a(p_1)\,(-ieQ_f\gamma^\beta)\,\frac{i}{-\not{p}_2 - \not{p}_3 - m_q + i\epsilon}\,(-igt^j_{ab}\gamma_\lambda\epsilon^\lambda_j)\,v_b(p_2)$$

$$+ \bar{u}_a(p_1)\,(-igt^j_{ab}\gamma_\lambda\epsilon^\lambda_j)\,\frac{i}{\not{p}_1 + \not{p}_3 - m_q + i\epsilon}\,(-ieQ_f\gamma^\beta)\,v_b(p_2)\Big\}$$

$$\times\sqrt{\frac{m_q}{E_1}}\sqrt{\frac{m_q}{E_2}}\,\frac{1}{\sqrt{2E_3}} \quad . \tag{7}$$

Here

$$p_i^\mu = (E_i, \vec{p}_i) \quad , \quad i = 1, 2, 3 \quad , \quad q_2^\mu = (E, \vec{q}_2) \quad , \quad q_1^\mu = (E, -\vec{q}_2) \quad ,$$

and, to repeat,

$$Q^\mu = q_1^\mu + q_2^\mu = (\sqrt{s}, \vec{0}) \quad ; \tag{8}$$

and the spinors u and v are those of Bjorken and Drell.[6] The gluon wavefunctions ϵ^λ_j will ultimately be summed over and we have written the photon propagator in the gauge of Feynman. Clearly, $E = \sqrt{s}/2$. Also, eQ_f is the charge of quarks of flavor f — we intend to sum over such flavors. The masses of the quarks, m_q, and the masses of the leptons, m_ℓ, will ultimately be taken to zero. We keep them at $m_q \neq 0 \neq m_\ell$ only for a technical convenience in the standard manner of Bjorken and Drell.[6]

The cross section is defined as follows:[6]

$$d\sigma = [(\text{Rate/unit volume}) / (\text{incident flux/unit volume})]$$

$$\times \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{d^3p_3}{(2\pi)^3} \quad . \tag{9}$$

Following Bjorken and Drell,[6] we identify the rate per unit volume with

$$|M|^2/VT \tag{10}$$

where VT is the total volume of space V times the total time T in space-time. Thus, technically

$$VT = (2\pi)^4\delta^4(0) \quad , \tag{11}$$

in agreement with Fermi. The incident flux, in these normalizations, is simply $|\vec{v}_1 - \vec{v}_2|$, as explained in Bjorken and Drell,[6] where $\vec{v}_1$ is the velocity of lepton i, $i = 1, 2$. Thus, for PEP and PETRA energies, the flux factor is simply 2 in the relevant units. We thus arrive at the textbook[6] result

$$d\sigma = \frac{\left[(2\pi)^4\delta^4(Q - p_1 - p_2 - p_3)\right]^2}{V\,T\,|\vec{v}_1 - \vec{v}_2|} \left(\frac{m_\ell^2}{E^2}\right) \left|(-ie)^2(-ig)\right|^2 \frac{|-ii|^2}{s^2} \frac{m_q^2 \sum_f Q_f^2}{E_1E_2(2E_3)}$$

$$\times\ \bar{u}(q_2)\gamma_\alpha v(q_1)\bar{v}(q_1)\gamma_\beta u(q_2) \Bigg\{ \bar{v}_b(p_2)\varepsilon_j^{\lambda*} t_{ba}^\dagger \gamma_\lambda \frac{(-\not{p}_2 - \not{p}_3 + m_q)\gamma^\alpha u_a(p_1)}{(p_2 + p_3)^2 - m_q^2 - i\varepsilon}$$

$$+ \; \bar{v}_b(p_2)\gamma^{\alpha} \frac{(\not{p}_1 + \not{p}_3 + m_q)}{(p_1+p_3)^2 - m_q^2 - i\varepsilon} t^{j\dagger}_{ba} \gamma_{\lambda}\varepsilon^{\lambda *}_{j} u_a(p_1)\Bigg\} \Bigg\{\bar{u}_a(p_1)\gamma^{\beta}$$

$$\times \frac{(-\not{p}_2 - \not{p}_3 + m_q)}{(p_2+p_3)^2 - m_q^2 + i\varepsilon} t^{j}_{ab} \gamma_{\lambda}\varepsilon^{\lambda}_{j} v_b(p_2) + \bar{u}_a(p_1) t^{j}_{ab}\gamma_{\lambda}\varepsilon^{\lambda}_{j}$$

$$\times \frac{(\not{p}_1 + \not{p}_3 + m_q)}{(p_1+p_3)^2 - m_q^2 + i\varepsilon} \gamma^{\beta} v_b(p_2)\Bigg\}\left(\frac{d^3p_1 \; d^3p_2 \; d^3p_3}{(2\pi)^9}\right), \qquad (12)$$

where $\varepsilon \downarrow 0$ is Feynman's epsilon and should not be confused with the gluon wavefunction $\varepsilon^{\lambda}_{j}$. At this point, we have retained trivial factors like $|-ii|^2 = 1$, for we want this calculation to be pedagogic.

We intend to sum the cross section (12) over all final states and to average over the initial states. Thus, following Bjorken and Drell[6] we divide (12) by $(2s_1 + 1)(2s_2 + 1)$, where s_1 is the spin of the initial particle i , i = 1,2. Here, $s_1 = s_2 = 1/2$, and $(2s_1 + 1)\,(2s_2 + 1) = 4$. Further, we will need the relations (in the Bjorken and Drell[6] metric)

$$\sum_{\text{helicity}} \varepsilon^{\lambda *}_{j} \varepsilon^{\lambda'}_{j} \underset{\text{effectively}}{=} -\delta_{jj'}\, g^{\lambda\lambda'} \qquad (13)$$

and

$$\sum_{\text{spin}} v_{\alpha}(q_1)\bar{v}_{\beta}(q_1) = \frac{(\not{q}_1 - m_{\ell})}{2m_{\ell}}{}_{\alpha\beta} \qquad (14)$$

$$\sum_{\text{spin}} u_\alpha(q_2)\bar{u}_\beta(q_2) = \frac{(\not{q}_2 + m_\ell)}{2m_\ell}\alpha\beta \tag{15}$$

$$\sum_{\text{spin}} \left(v_{b'}(p_2)\right)_\alpha\left(\bar{v}_b(p_2)\right)_\beta = \frac{(\not{p}_2 - m_q)}{2m_q}\alpha\beta\ \delta_{b'b} \tag{16}$$

$$\sum_{\text{spin}} \left(u_{a'}(p_1)\right)_\alpha\left(\bar{u}_a(p_1)\right)_\beta = \frac{(\not{p}_1 + m_q)}{2m_q}\alpha\beta\ \delta_{a'a} \quad , \tag{17}$$

together with $\vec{t}^\dagger = \vec{t}$ so that (we are using the Einstein summation convention)

$$\sum_{a,b} t^{j\dagger}_{ba}\, t^j_{ab} = r\,T\,(R) \tag{18}$$

where r is the number of generators for SU(3) and T(R) is defined by

$$\text{tr}\ t^i t^j = \delta_{ij}\, T\,(R) \quad . \tag{19}$$

There is the elementary group theory result

$$r\,T\,(R) = d\,(R)\,C_2\,(R) = 4 \tag{20}$$

since R is the triplet-(vector) representation of SU(3). Here d(R) is the dimension of the representation R and $C_2(R)$ is the value of the Casimir operator for the representation R.

The word "effective" in (13) is written to remind the reader that the current to which the massless gluon in Fig. 1 is coupled is conserved, so that the replacement for $\sum_{\text{helicity}} \varepsilon^{\lambda *}_j \varepsilon^{\lambda'}_{j'}$ implied by (13) amounts to adding and subtracting the same terms from the squared matrix element sum, as explained in Bjorken and Drell.[6] Thus, the replacement does not alter the cross section $d\sigma$.

Using (13) - (20), one arrives at

$$d\sigma = \frac{\left[(2\pi)^4 \delta^4 (Q - p_1 - p_2 - p_3)\right]^2}{V\ T\ \left|\vec{v}_1 - \vec{v}_2\right|} \left(\frac{m_\ell^2}{E^2}\right) \frac{(e^4 g^2)}{s^2} \left(\sum_f Q_f^2\right) \frac{m_q^2}{E_1 E_2} \frac{1}{2E_3} \left(\frac{1}{4}\right)$$

$$\times \frac{\mathrm{tr}[\gamma_\alpha \not{\ell}_1 \gamma_\beta \not{\ell}_2]}{4m_\ell^2} (4) \; (-g^{\lambda\lambda'}) \left(\frac{1}{4m_q^2}\right) \Big\{ \mathrm{tr}\left[\gamma_\lambda \not{p}_2 \gamma_{\lambda'} (\not{p}_1 - \not{Q}) \; \gamma^\alpha \not{p}_1 \gamma^\beta (\not{p}_1 - \not{Q})\right] \Big/$$

$$\left[(Q - p_1)^2 - m_q^2\right]^2 + \mathrm{tr}\Big[\gamma_\lambda (\not{Q} - \not{p}_2) \gamma^\beta \not{p}_2 \gamma_{\lambda'} (\not{p}_1 - \not{Q}) \gamma^\alpha \not{p}_1$$

$$+ \gamma^\beta (\not{p}_1 - \not{Q}) \gamma_\lambda \not{p}_2 \gamma^\alpha (\not{Q} - \not{p}_2) \gamma_{\lambda'} \not{p}_1\Big] \Big/ \left(\left[(Q - p_1)^2 - m_q^2\right]\left[(Q - p_2)^2 - m_q^2\right]\right)$$

$$+ \mathrm{tr}\left[\gamma_\lambda \not{p}_1 \gamma_{\lambda'} (\not{p}_2 - \not{Q}) \gamma^\beta \not{p}_2 \gamma^\alpha (\not{p}_2 - \not{Q})\right] \Big/ \left[(Q - p_2)^2 - m_q^2\right]^2 \Big\}$$

$$\times \frac{d^3 p_1 \; d^3 p_2 \; d^3 p_3}{(2\pi)^9} \quad . \tag{21}$$

In arriving at (21), we have taken $m_q, m_\ell \to 0$ in traces over the γ-matrices. The relations

$$(Q - p_i)^2 - m_q^2 = s(1 - x_i) + m_q^2 \quad , \quad i = 1, 2, \tag{22}$$

then allow us to write the denominator factors

$$[(Q - p_i)^2 - m_q^2] \tag{23}$$

as

$$s(1 - x_i) \qquad ; \tag{24}$$

for, ε has been taken to zero. In this way, we arrive at

$$d\sigma = \frac{\left((2\pi)^4\delta^4(Q-p_1-p_2-p_3)\right)^2}{VT\left|\vec{v}_1-\vec{v}_2\right|} \quad \frac{1}{E^2}\left(\frac{e^4g^2}{s^4}\right)\sum_f Q_f^2\left(\frac{1}{2E_1E_2E_3}\right)$$

$$\times \quad (-)\,L_{\alpha\beta}\,H^{\alpha\beta}\left(\frac{d^3p_1\;d^3p_2\;d^3p_3}{(2\pi)^9}\right) \quad , \tag{25}$$

where (following the methods in Bjorken and Drell[6])

$$L_{\alpha\beta} \equiv \frac{1}{4}\,\mathrm{tr}\,\gamma_\alpha \not{q}_1 \gamma_\beta \not{q}_2 = \left[q_{1\alpha}q_{2\beta} + q_{1\beta}q_{2\alpha} - \frac{s}{2}\,g_{\alpha\beta}\right] \tag{26}$$

in the limit $m_\ell \to 0$, and

$$H^{\alpha\beta} \equiv \frac{1}{4}\,\mathrm{tr}\Big[\gamma_\lambda \not{p}_2 \gamma^\lambda(\not{p}_1-\not{Q})\gamma^\alpha \not{p}_1 \gamma^\beta(\not{p}_1-\not{Q}) \Big/ (1-x_1)^2$$

$$+ \Big\{\gamma_\lambda(\not{Q}-\not{p}_2)\gamma^\beta \not{p}_2 \gamma^\lambda(\not{p}_1-\not{Q})\gamma^\alpha \not{p}_1$$

$$+ \gamma^\beta(\not{p}_1-\not{Q})\gamma_\lambda \not{p}_2 \gamma^\alpha(\not{Q}-\not{p}_2)\gamma^\lambda \not{p}_1\Big\} \Big/ (1-x_1)(1-x_2)$$

$$+ \gamma_\lambda \not{p}_1 \gamma^\lambda(\not{p}_2-\not{Q})\gamma^\beta \not{p}_2 \gamma^\alpha(\not{p}_2-\not{Q}) \Big/ (1-x_2)^2\Big]$$

$$= 2s\Big\{\Big[(1-x_1)\left(p_2^\alpha\, p_1^\beta + p_2^\beta p_1^\alpha - \frac{1}{2}\,s(1-x_3)g^{\alpha\beta}\right)$$

$$- (1-x_1)\left((Q-p_1)^\alpha p_1{}^\beta + p_1{}^\alpha(Q-p_1)^\beta - \frac{1}{2}\,s\,x_1\,g^{\alpha\beta}\right)\Big] \Big/ (1-x_1)^2$$

$$+ \Big[(1-x_2)\left(p_2^\alpha\, p_1^\beta + p_2^\beta\, p_1^\alpha - \frac{1}{2}\,s(1-x_3)g^{\alpha\beta}\right)$$

$$- (1-x_2)\left(p_2{}^\alpha(Q-p_2)^\beta + p_2{}^\beta(Q-p_2)^\alpha - \frac{1}{2}\,s\,x_2\,g^{\alpha\beta}\right)\Big] \Big/ (1-x_2)^2$$

$$+ \left\{ (x_3 - 2)\left(p_1^\alpha\, p_2^\beta + p_2^\alpha\, p_1^\beta\right) + s(1-x_3)\, g^{\alpha\beta} \right.$$

$$- (1-x_3)\left[(Q-p_1-p_2)^\alpha\,(p_1+p_2)^\beta + (Q-p_1-p_2)^\beta\,(p_1+p_2)^\alpha\right]$$

$$\left. + 2p_1^\alpha\, p_1^\beta\,(1-x_1) + 2p_2^\alpha\, p_2^\beta\,(1-x_2) \right\} \Big/ (1-x_1)(1-x_2) \Big\} \qquad (27)$$

for $m_q \to 0$.

It may be verified by explicit calculation that

$$Q^\alpha H_{\alpha\beta} = 0 = H_{\alpha\beta} Q^\beta \quad . \qquad (28)$$

Thus it follows that in computing $L_{\alpha\beta}H^{\alpha\beta}$, we may replace

$$q_1$$

with

$$-q_2 \text{ in } L_{\alpha\beta} \text{— for,}$$

$$q_1 = Q - q_2 \quad . \qquad (29)$$

This gives

$$(-)\ L_{\alpha\beta}H^{\alpha\beta} = \left(2q_{2\alpha}q_{2\beta}H^{\alpha\beta} + \frac{s}{2}\, g_{\alpha\beta}H^{\alpha\beta}\right) \quad . \qquad (30)$$

With the definitions in Fig. 1, we find

$$-L_{\alpha\beta}H^{\alpha\beta} = (2s)\left(\frac{s^2}{4(1-x_1)(1-x_2)}\right)\left[x_1^2(1+\cos^2\theta_1) + x_2^2(1+\cos^2\theta_2)\right] , \qquad (31)$$

where

$$\left(\hat{q}_2 \cdot \hat{p}_i\right) \equiv \cos\theta_i \quad , \quad i = 1, 2 \ ; \qquad (32)$$

here, $\hat{q}_2$ is the unit vector in the $\vec{q}_2$ direction and $\hat{p}_i$ is the unit vector in the $\vec{p}_i$ direction, i = 1, 2.

The result (31) then gives, for $|\vec{v}_1 - \vec{v}_2| \to 2$,

$$d\sigma = \left(\frac{(2\pi)^4 \delta^4 (Q - p_1 - p_2 - p_3)}{V\ T}\right)\left(\frac{\delta^4 (Q - p_1 - p_2 - p_3)}{2(2\pi)^5}\right)\left(\frac{2}{\sqrt{s}}\right) \frac{4g^2}{s^4}\left(\sum_f Q_f^2\right)$$
$$\times\left(\frac{s^3}{2(1 - x_1)(1 - x_2)}\right)\left[x_1^2(1 + \cos^2\theta_1) + x_2^2(1 + \cos^2\theta_2)\right]\left(d^3p_1 d^3p_2 d^3p_3\right) . \qquad (33)$$

Following Fermi we have used one of the factors of $\delta^4(Q - p_1 - p_2 - p_3)$ to set the other such factor equal to $\delta^4(0)$. Thus, $d\sigma$ is being evaluated on the surface where $\delta^4(Q - p_1 - p_2 - p_3) = \delta^4(0)$. This means that we must maintain (and have been maintaining)

$$Q = \left(p_1 + p_2 + p_3\right) \qquad (34)$$

throughout our calculations. Otherwise, $\delta^4(Q - p_1 - p_2 - p_3) \neq \delta^4(0)$, and the rate/(unit volume) will vanish! In particular, if we change variables from $\vec{p}_i$, $i = 1, 2, 3$ to some other variables, we must remember that this change in variables must be carried out under the constraint that

$$\delta^4\left(Q - p_1 - p_2 - p_3\right) = \delta^4(0) \qquad (35)$$

i.e., under the constraint of four-momentum conservation:

$$Q = \left(p_1 + p_2 + p_3\right) .$$

Indeed, integrating over d^3p_3 we have

$$d\sigma = \frac{(2\pi)^4\delta^4(0)}{VT} \frac{\delta(\sqrt{s}-E_1-E_2-E_3)}{2^3} \frac{e^4g^2}{\pi^5}\left(\sum_f Q_f^2\right)$$
$$\times \frac{\left(x_1^2(1+\cos^2\theta_1) + x_2^2(1+\cos^2\theta_2)\right)}{s^{7/2}\,(1-x_1)\,(1-x_2)\,x_1x_2x_3}$$

$$\times \left|\vec{p}_1\right|^2 d\left|\vec{p}_1\right| \, d(\cos\theta_1)\, d\phi_1 \left|\vec{p}_2\right|^2 d\left|\vec{p}_2\right| \, d(\cos\theta_2)\, d\phi_2 \;, \quad (36)$$

where ϕ_i are the azimuths of $\vec{p}_i$ respectively. But, since $E_i = x_i\sqrt{s}/2$ when (35) is satisfied, we have that

$$\delta^4(0)\;\delta(\sqrt{2}-E_1-E_2-E_3) = \delta^4(0)\;\delta(\sqrt{s}-x_1\sqrt{s}/2-x_2\sqrt{s}/2-x_3\sqrt{s}/2)$$
$$= \delta^4(0)\left(\frac{2}{\sqrt{s}}\right)\delta(2-x_1-x_2-x_3) \quad . \quad (37)$$

Further, since $\left|\vec{p}_i\right| \, d\left|\vec{p}_i\right| = E_i \, d\,E_i$, we have

$$\delta^4(0)\;\left|\vec{p}_i\right| d\left|\vec{p}_i\right| = \delta^4(0)\left(\frac{\sqrt{s}}{2}\right)^2 x_i\; d\,x_i \quad , \; i = 1,2 \; . \quad (38)$$

But, for $m_q \to 0$, $\left|\vec{p}_i\right| = E_i$. It follows that for $m_q = 0$

$$\delta^4(0)\;\left|\vec{p}_i\right|^2 d\left|\vec{p}_i\right| = \delta^4(0)\left(\frac{\sqrt{s}}{2}\right)^3 x_i^2\; d\,x_i \quad , \; i = 1,2 \; . \quad (39)$$

Hence from (39) we have

$$d\sigma = \frac{(2\pi)^4\delta^4(0)}{VT} \frac{e^4g^2\sum_f Q_f^2}{\pi^5\; 2^3\, s^{7/2}}\left(\frac{2}{\sqrt{s}}\right)\left(\frac{\sqrt{s}}{2}\right)^6 \frac{\left(x_1^2(1+\cos^2\theta_1) + x_2^2(1+\cos^2\theta_2)\right)}{(1-x_1)\,(1-x_2)\,x_1x_2x_3}$$

$$\times\; \delta(2-x_1-x_2-x_3)\; x_1^2\; x_2^2\; dx_1\; dx_2\; d(\cos\theta_1)\; d\phi_1\; d(\cos\theta_2)\, d\phi_2 \; . \quad (40)$$

At this point, it is convenient to treat the term proportional $(1+\cos^2\theta_1)$ in (40) by rotating the $\vec{p}_2$ coordinates so that $d(\cos\theta_2)d\phi_2 = d(\cos\alpha)\, d\beta$, where

$$\hat{p}_2 \cdot \hat{p}_1 \equiv \cos\alpha \tag{41}$$

i.e., α is the angle between $\vec{p}_1$ and $\vec{p}_2$. It is well known that the Jacobian of a rotation is one in magnitude. Here, β is the azimuth of $\vec{p}_2$ about $\vec{p}_1$. The integral over this azimuth then gives 2π, since $(1+\cos^2\theta_1)$ is independent of β. One then needs for the $(1+\cos^2\theta_1)$ term, for x_1, x_2, $\cos\theta_1$, ϕ_1 fixed,

$$\int \delta(2-x_1-x_2-x_3)\, d(\cos\alpha) \quad . \tag{42}$$

However here one has two choices:

(a) one can use the fact that as a consequence of the integration over d^3p_3

$$\vec{p}_3 = -\vec{p}_1 - \vec{p}_2 \tag{43}$$

so that

$$x_3\sqrt{s}/2 = E_3 = \left|\vec{p}_3\right| = \left|\vec{p}_1+\vec{p}_2\right| = \sqrt{\left|\vec{p}_1\right|^2+\left|\vec{p}_2\right|^2+2\left|\vec{p}_1\right|\left|\vec{p}_2\right|\cos\alpha}\ ; \tag{44}$$

(b) one can remember that $\delta^4(0) = \delta^4(Q-p_1-p_2-p_3)$ so that for $m_q \to 0$

$$s(1-x_3) = (Q-p_3)^2 = (p_1+p_2)^2 = 2p_1\cdot p_2 = \left(\frac{s}{2}\right)x_1x_2(1-\cos\alpha) \ . \tag{45}$$

We will differentiate (45) with x_1,x_2 fixed and conclude that

$$\Big(\partial x_3/\partial(\cos\alpha)\Big)_{x_1,x_2\ \text{fixed}} = x_1x_2/2 \tag{46}$$

so that for fixed x_1, x_2

$$\int \delta^4(0)\ \delta(2-x_1-x_2-x_3)\, d(\cos\alpha) = \int \delta^4(0)\, \delta(2-x_1-x_2-x_3)\, \frac{2}{x_1 x_2}\, dx_3$$

$$= \delta^4(0)\ \frac{2}{x_1 x_2} \quad . \qquad (47)$$

If we had used (44) we would have concluded that for $m_q \to 0$

$$\frac{\sqrt{s}}{2}\, x_3 = \frac{\sqrt{s}}{2}\, (x_1^2 + x_2^2 + 2x_1 x_2 \cos\alpha)^{1/2}$$

so that

$$\left(\frac{\partial x_3}{\partial(\cos\alpha)}\right)_{x_1, x_2 \text{ fixed}} \overset{?}{=} \frac{x_1 x_2}{x_3} \quad . \qquad (48)$$

But, the derivative (48) is not at $Q = p_1 + p_2 + p_3$; for if we satisfy $\delta^4(0) = \delta^4(Q - p_1 - p_2 - p_3)$ and if we hold x_1 and x_2 fixed, then we cannot change $|\vec{p}_3| = |\vec{p}_1 + \vec{p}_2|$; for the time component of $\delta^4(0) = \delta^4(Q - p_1 - p_2 - p_3)$ requires

$$\sqrt{s} = Q^0 = x_1\sqrt{s}/2 + x_2\sqrt{s}/2 + |\vec{p}_3| \quad . \qquad (49)$$

Thus we cannot differentiate (44) when $\delta^4(0) = \delta^4(Q - p_1 - p_2 - p_3)$, as is true in $d\sigma$. For differentiating $|\vec{p}_3|$ at fixed x_1, x_2 means

$$\lim_{\delta\alpha \to 0} \left(\frac{\left|\vec{p}_3\big(\cos(\alpha+\delta\alpha)\big)\right| - \left|\vec{p}_3(\cos\alpha)\right|}{\cos(\alpha+\delta\alpha) - \cos\alpha}\right)_{x_1, x_2 \text{ fixed}} \qquad (50)$$

and hence involves changing $|\vec{p}_3|$ at fixed x_1, x_2, inconsistent with (49). We must use (46). It appears that Refs. 1-4 are using the unphysical result (48).[7]

Substituting (47) into (40) for the coefficient of $(1+\cos^2\theta_1)$ we obtain the contribution to $d\sigma$

$$d\sigma^{(1)} = \frac{(2\pi)^4\delta^4(0)}{V\ T}\ \frac{e^4 g^2 \sum_f Q_f^2}{\pi^5\ 2^8 s}\ \frac{(2\pi)(2)\ x_1^2(1+\cos^2\theta_1)}{(1-x_1)(1-x_2)\ x_3}\ dx_1 dx_2 d(\cos\theta_1) d\phi_1\ . \tag{51}$$

The coefficient of $(1+\cos^2\theta_2)$ in (40) then gives, by symmetry in $1\leftrightarrow 2$, an analogous contribution $d\sigma^{(2)}$ to $d\sigma$, obtained from (51) by interchanging the subscripts 1 and 2 :

$$d\sigma^{(2)} = \frac{(2\pi)^4\delta^4(0)}{V\ T}\ \frac{e^4 g^2 \sum_f Q_f^2}{\pi^5\ 2^8 s}\ \frac{(2\pi)(2)\ x_2^2(1+\cos^2\theta_2)}{(1-x_1)(1-x_2)\ x_3}\ dx_1 dx_2 d(\cos\theta_2) d\phi_2\ . \tag{52}$$

Adding $d\sigma^{(1)}$ and $d\sigma^{(2)}$ and integrating over $d(\cos\hat{\theta}_i)d\phi_i$ gives

$$d\sigma = \frac{(2\pi)^4\delta^4(0)}{V\ T}\ \frac{e^4 g^2}{\pi^5 2^8 s}\left(\sum_f Q_f^2\right)\frac{(4\pi)(x_1^2+x_2^2)(2\pi)(2+2/3)}{(1-x_1)(1-x_2)\ x_3}\ dx_1\ dx_2$$

$$= \frac{(2\pi)^4\delta^4(0)}{V\ T}\ \frac{2}{3}\ \alpha_e^2\ \frac{(4\sum_f Q_f^2)}{s}\ \alpha_c\ \frac{(x_1^2+x_2^2)}{(1-x_1)(1-x_2)}\left(\frac{2}{x_3}\right) dx_1\ dx_2$$

$$= \frac{2}{3}\ \frac{\alpha_e^2\ (4\sum_f Q_f^2)}{s}\ \alpha_c\ \frac{(x_1^2+x_2^2)}{(1-x_1)(1-x_2)}\left(\frac{2}{x_3}\right) dx_1\ dx_2 \tag{53}$$

where $x_3 = 2 - x_1 - x_2$ because $\delta^4(Q-p_1-p_2-p_3) = \delta^4(0) = VT$. Here $\alpha_e = e^2/4\pi$, $\alpha_c = g^2/4\pi$.

The spin averaged point-like cross section for $e^+e^- \to q\bar{q}$ is, in the same QCD theory, in the one photon exchange approximation

$$\sigma_{pt} = \frac{4\pi\left(\sum_f Q_f^2\right)\alpha_e^2}{s} \quad . \tag{54}$$

Thus we have

$$\frac{1}{\sigma_{pt}} \frac{d\sigma}{dx_1\, dx_2} = \frac{2}{3} \frac{\alpha_c}{\pi} \frac{(x_1^2 + x_2^2)}{(1-x_1)(1-x_2)} \left(\frac{2}{x_3}\right) . \tag{55}$$

As advertised, (55) differs from Refs. 1-4 at least by the factor $2/x_3 = 1/\omega_3$ where

$$\omega_3\sqrt{s} = \text{massless gluon energy } E_3 \quad . \tag{56}$$

There is a further disagreement with Ref. 1 on the numerical factor in (55).

The phenomenological consequences of (53) will be discussed elsewhere.[8] However, we do wish to mention that the famous two logarithm infrared behavior known[5,6] to characterize the bremsstrahlung of a massless gluon in tree approximation is inherent in (55), because of the $(2/x_3)$ factor; this behavior is necessarily absent from the results of Refs. 1-4.

ACKNOWLEDGMENTS

This paper was written at the suggestion of Ms. S. Cooper, who was unable to make physical sense out of Refs. 1-4. The author is grateful to Ms. Cooper for relating her troubles with Refs. 1-4 to him. The author further acknowledges the hospitality of Professor S. D. Drell of the SLAC theory group.

This work was supported by the Department of Energy under contract number EY-76-C-03-0515.

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FIGURE CAPTION

1. Feynman diagrams for the process $e^+e^- \rightarrow G\, q\bar{q}$ in the one photon exchange approximation to the lowest order in g, the gluon coupling constant. In order to facilitate comparison with Refs. 1-4, we use the notation of Ref. 2; G is the gluon.

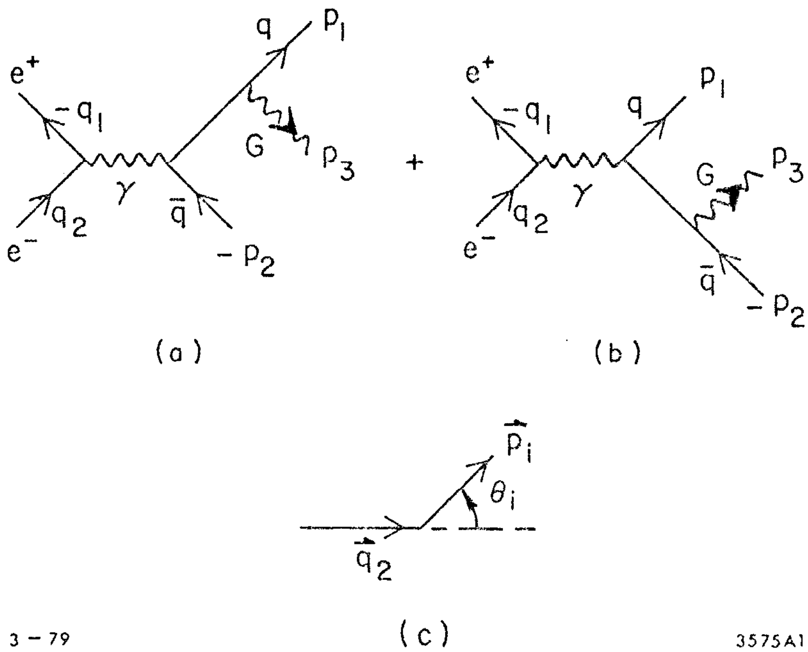

Fig. 1