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Local Chiral Symmetry for Gauge Theories on Wilson's Lattice*

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ABSTRACT

With the problem of chiral symmetry on the Wilson lattice in space-time as an objective, we introduce a definition of differentiation on a lattice which respects the product rule of Leibnitz: d(fg) = fdg + (df)g for functions f, g. The derivative is essentially a generalization of the SLAC derivative on a lattice. With this derivative, chiral (symmetry) currents have all the characteristics that they possess in the continuum quantum theory of fields. In particular, the Adler-Bell-Jackiw anomaly theorem has the same form as it does in the continuum.

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I. INTRODUCTION

In an effort to understand how it could happen that quarks are apparently confined (Ref. 1 notwithstanding), Wilson[2] introduced a lattice formalism to describe the large distance behavior of gauge theories. The motivation was the apparent success[3] of the conventional formulation of gauge theories of non-Abelian (Yang-Mills[4]) type in describing the short distance structure of the underlying quark quantum field theory. Thus, the idea was that one could cut off the short-distance phenomena in the theory if one were only interested in the large distance (confining) properties of the theory. That is to say, provided that one cuts off the short distance properties of the theory in a gauge invariant and ultravioletly attractive manner (in the conventional formulation of Wilson's short distance ideas[5,6,7]), then one should be able to learn the large distance properties of the true continuum gauge theory by studying the large distance properties of the cut-off theory. The Wilson lattice theory is presumably just such a cut-off theory.

Clearly, if the underlying quark quantum field theory is a non-Abelian gauge theory, then such a theory must confine at large distances*. Indeed, Wilson has found that, in his gauge invariant (and presumed) ultravioletly attractive lattice non-Abelian gauge theory, the strong coupling limit is a limit in which heavy quarks would be confined. We remind the reader that, in the conventional formulation of Wilson's short distance ideas, the coupling constant for non-Abelian gauge theories increases beyond computation as one looks at larger and larger distances — starting from the conventional ultravioletly attractive short distance region. Thus, it is not unreasonable to consider the strong coupling limit of the Wilson lattice Yang-Mills theory when one is studying the confinement problem.

Once one has introduced the lattice in a gauge invariant and ultra-violetly attractive manner, then one at some point must consider the attendant hadron spectroscopy, in the continuum limit. For, given the confining result for infinitely heave quarks, one must then show that when one puts dynamical interacting light** quarks (of parton-model type[8] for example) into the theory, the resulting theory, in the continuum limit at least, agrees reasonably well with the properties of the known spectrum of light hadrons. One of the nicer properties of the interactions between the light hadrons is the PCAC idea[9] (partial conservation of the axial vector current) — where we single out especially the implications for PCAC for $\pi^o \to \gamma\gamma$ in the presence of the Adler-Bell-Jackiw anomaly[10]. Clearly, one would like to feel that the Wilson lattice gauge theory was consistent with the $\pi^o \to \gamma\gamma$ prediction of the (anomalous) PCAC equation in the limit of zero lattice spacing, for example.

If one looks in detail at the Wilson lattice gauge theory, one sees that quite independent of the anomaly, the local chiral currents of massless fermions are manifestly not conserved. Specifically, in the original version of the theory[2], the Yang-Mills-fermion action is (in the Euclidean formulation)

$$A = \frac{1}{2g^2}\sum_{n\mu\nu} \mathrm{tr}\, U_A\left(b_{n\mu} b_{n+\hat{\mu},\nu} b^{-1}_{n+\hat{\nu},\mu} b^{-1}_{n\nu}\right)$$

$$+ \frac{1}{2}a^3 \sum_{n\mu}\left(\overline{\psi}_n \gamma_\mu U\,(b_{n\mu})\,\psi_{n+\hat{\mu}} - \overline{\psi}_{n+\hat{\mu}}\gamma_\mu U^\dagger(b_{n\mu})\,\psi_n\right) \qquad (1)$$

$$- a^4 \sum_n m_0\, \overline{\psi}_n\, \psi_n \quad ,$$

where $b_{n\mu}$ is an element of the Yang-Mills group, $U_a(b_{n\mu})$ is the adjoined representative of $b_{n\mu}$, $U(b_{n\mu})$ is the unitary representative of $b_{n\mu}$ in the fermion representation, g is the gauge coupling constant, a is the lattice spacing, m_o is the fermion bare mass. The remaining lattice notation follows Ref. 2. Thus, the fields ψ_n and $b_{n\mu}$ carry the group in the sense that A is invariant under the group transformation y_n:

$$\psi_n \rightarrow U(y_n)\psi_n \qquad (2)$$

$$\overline{\psi}_n \rightarrow \overline{\psi}_n\, U^\dagger(y_n) \qquad (3)$$

$$b_{n\mu} \rightarrow y_n\, b_{n\mu}\, y^{-1}_{n+\hat{\mu}} \qquad (4)$$

$$U(b_{n\mu}) \rightarrow U(y_n)\, U(b_{n\mu})\, U^\dagger(y_{n+\hat{\mu}}) \qquad (5)$$

The product $y_n b_{n\mu} y^{-1}_{n+\hat{\mu}}$ is taken from the group multiplication law. (As discussed in detail in references 11, the action (1) has the unfortunate problem that the fermion spectrum, at g=0 for example, has been doubled. To each value of the energy E, there corresponds, in one space-one time dimension, for example, two distinct values of the momentum magnitude $|k|$. Let us ignore this problem for the moment.) For m_0=0, the action (1) possesses a local γ_5-symmetry. But, if one attempts to use Noether's procedure to derive the corresponding formally conserved chiral currents, one will find that this procedure will generally lead to the conclusion that the usual currents are only formally conserved and local in the limit of zero lattice spacing, as is well-known[11]. Further, if one uses the procedures of Wilson[11] and Kogut and Susskind[11] to remedy the fermion doubling problem, even local γ_5-invariance is lost for $m_0=0$ in general.

However, here we do wish to call attention to the SLAC[11] derivative on a lattice, which, for a one-dimensional lattice, $-N \le j \le N$, is given by

$$\nabla f(j) = \sum_k ike^{ikj/\Lambda} f(k),$$
$$k = \frac{2\pi n}{L}, \; -N \le n \le N, \tag{6}$$

when

$$f(j) = \sum_k e^{ikj/\Lambda} f(k) \quad . \tag{7}$$

The parameters L, N and Λ are related by $(2N+1) = L\Lambda$ so that $1/\Lambda$ is the lattice spacing. The derivative (6) solves the fermion doubling problem at g=0 while maintaining local γ_5-invariance for $m_0=0$. But, as the SLAC group has emphasized, [11] the corresponding formally conserved chiral currents are non-local. Indeed, the fact that the SLAC chiral currents are non-local has recently been verified by Karsten and Smit.[12]

Before we proceed further with this discussion, we should like to emphasize the following. One can just as easily take the point of view that the undesirable aspects of the status of chiral (γ_5) invariance on the lattice are all resolved either by the interactions of the respective theories and/ or by taking the continuum limit. Thus, on this view, one would not necessarily expect a lattice version of the anomalous PCAC equation. Rather, only after solving the lattice theory and taking the continuum limit would one expect to see the result of Adler, Bell, and Jackiw.

One can go further in support of this view by recalling that the lattice is supposed to represent the large distance behavior of the theory of strong interactions — the short distance behavior is cut-off by the lattice spacing. Hence, since the anomaly is a short distance phenomenon[10], one could argue that one should not expect any anomalous PCAC equation on any lattice with a finite lattice spacing. From this point of view, the SLAC[11] treatment of the γ_5 invariance is quite sufficient, since, for example, the non-local parts of the conserved SLAC[11] chiral currents in a free massless fermion theory have no zero-momentum component (large distance component) for N large, where 2N+1 is the number lattice sites on a given axis in the lattice. The SLAC[11] version of the theory (1) has, for $m_0=0$, conserved chiral currents with the same property.

Returning to our main theme, however, we wish to investigate here the alternative possibility — namely that the lattice should be a perfectly good place to discuss chiral symmetry phenomena such as anomalous PCAC. Thus, in the next Section, Section II, we shall formulate lattice gauge field theory in a gauge invariant fashion in which results such as the anomalous PCAC equation will ultimately be seen to occur naturally. This will be done by using a generalization of the SLAC derivative (6). As a result, local chiral currents are conserved formally. The gauge field variables will be the usual fields $A_\mu^c(n)$, related to $U_a(b_{n\mu})$ by

$$U_A(b_{n\mu}) \equiv \exp\left\{iag\ A_\mu^c(n)\tau^c\right\} \qquad (8)$$

where τ^c carry the adjoined representation of the gauge group. Hence, the theory will appear, in all aspects, entirely isomorphic to the con- timuum theory.

The key ingredient in our construction will be seen to be the introduction a derivative on the lattice which respects the product rule of Leibnitz:

$$\nabla(fh) = (\nabla f)h + f(\nabla h) \ . \qquad (9)$$

The SLAC derivative satisfies[11]

$$\nabla(fh) = (\nabla f)h + f\nabla h + \nabla\left(\sum_{\ell_1,\ell_2} I(j;\ell_1,\ell_2)\ f(\ell_1)\ h(\ell_2)\right) \qquad (10)$$

where I(j) satisfies (supressing the ℓ_1,ℓ_2 arguments)

$$\nabla I(j) = S(j) \qquad (11)$$

and

$$S(j;\ell_1,\ell_2) = 2\pi i\Lambda \sum_{\ell_1,\ell_2} e^{i(\ell_1+\ell_2)j/\Lambda}\left\{\theta\left(\frac{2\pi N}{2N+1} - \left|\frac{\ell_1+\ell_2}{\Lambda} + 2\pi\right|\right)\right.$$
$$\left. -\theta\left(\frac{2\pi N}{2N+1} - \left|\frac{\ell_1+\ell_2}{\Lambda} - 2\pi\right|\right)\right\} \qquad (12)$$

with ∇ given by (6). Here, f(j) is defined as in (7), and

$$h(j) = \sum_k e^{ikj/\Lambda}\ h(k) \qquad (13)$$

with $k=2\pi\ n/L$, $-N \le n \le N$, and $2N+1 = \Lambda L$. For large N, the support of S is for $|\ell_1+\ell_2| > \pi\Lambda$. Thus, it is not surprising that the SLAC derivative, properly extended, will satisfy (9). We turn now to this extension.

II. LATTICE GAUGE THEORY AND CHIRAL CURRENTS

We have interest in theories of Yang-Mills type on the Wilson[2] lattice in the presence of fermions. As we wish to have local conserved chiral currents we shall work with the gauge field of the continuum limit: $A^a_\mu(n)$ is the gauge vector field at site $n = (n_0,n_1,n_2,n_3)$ on the Wilson lattice. It will be forced here to carry the adjoined representation of the gauge group G in the Yang-Mills sense: under a gauge transformation of infinitesimal type $\omega^a(n)$, $A^a_\mu(n)$ will transform as

$$A^a_\mu(n) \to A^a_\mu(n) - \varepsilon_{abc}\omega^b(n)\, A^c_\mu(n) + \frac{1}{g}\,\partial_\mu\omega^a(n) \tag{14}$$

where $\partial_\mu\omega^a(n)$ has yet to be defined and ε_{abc} are the structure constants of the gauge group — g is the gauge coupling constant.

In defining $\partial_\mu\omega^a(n)$, we wish to arrive at a derivative which respects

$$\partial_\mu(f_1 f_2) = (\partial_\mu f_1)f_2 + f_1\,\partial_\mu f_2 \tag{15}$$

for functions $f_1(n)$, $f_2(n)$ on the Wilson lattice. This we do as follows: We need to have a lattice formalism which conserves momentum, rather than conserving momentum modulo $2\pi\Lambda$, where $\Lambda = 1/a$, with a equal to the lattice spacing. For simplicity, consider the one-dimensional lattice function f(j) given in (7). Rather than Fourier analyzing f in terms of

$$k = 2\pi n/L\ , \quad -N \le n \le N \tag{16}$$

we Fourier analyze with

$$k = 2\pi n/L'\ , \quad n = 0,\ \pm 1,\ \pm 2, \ldots, \tag{17}$$

where $\Lambda L = 2N+1$, $L = (2N+1)a$, and $2\pi\ a/L'$ is not commensurate with 2π, i.e., a/L' is not rational. Hence $n(2\pi\ a/L')$ is never a rational multiple of 2π for any integer n. Thus if we write

$$f(j) = \sum_{n=-\infty}^{\infty} e^{i(ja)(2\pi n/L')-|n|\ \varepsilon}\bar{f}(n) \tag{18}$$

we have, for $N\to\infty$,

$$\sum_{j=-\infty}^{\infty} e^{-i\ ja(2\pi n'/L')\ -|j|\ \varepsilon} f(j)$$

$$= \sum_{j=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{ija(2\pi(n-n')/L')\ -\varepsilon(|j|\ +\ |n|)}\ \bar{f}(n) \tag{19}$$

But for $\varepsilon \downarrow o$,

$$\sum_{j=-\infty}^{\infty} e^{ija(n-n')(2\pi/L') - \varepsilon|j|} = \left(1 - e^{\ i(n-n')(2\pi a/L')-\varepsilon}\right)^{-1}$$

$$+ \left(1 - e^{\ -i(n-n')(2\pi a/L')-\varepsilon}\right)^{-1} -1$$

$$= \left(1 - e^{\ i(n-n')(2\pi a/L')-\varepsilon}\right)^{-1}$$

$$+ e^{\ -i(n-n')(2\pi a/L')-\varepsilon} \left(1 - e^{\ -i(n-n')(2\pi a/L')-\varepsilon}\right)^{-1} \tag{20}$$

$$= \left(1 - e^{\ i(n-n')(2\pi a/L')-\varepsilon}\right)^{-1}$$

$$+ \left(e^{\ i(n-n')(2\pi a/L')+\varepsilon}-1\right)^{-1}$$

$$= \begin{cases} 0 & , \quad n\neq n' \\ \coth\ (\varepsilon/2), & \quad n=n' \end{cases}$$

Thus,

$$\bar{f}(n')\ \coth\ (\varepsilon/2) = \sum_{j=-\infty}^{\infty} e^{-ija(2\pi n'/L')-|j|\,\varepsilon}\ f(j)\ . \tag{21}$$

More generally, if we take

$$f(j) = \eta_1 \sum_{n=-\infty}^{\infty} e^{ija(2\pi n/L')-|n|\,\varepsilon}\, \bar{f}(n) \tag{22}$$

then

$$\bar{f}(n) = \eta_2 \sum_{j=-\infty}^{\infty} e^{ija(2\pi n'/L') - |j|\varepsilon} f(j) \tag{23}$$

if

$$\eta_1\ \eta_2 \coth(\varepsilon/2) = 1 \quad \text{for } \varepsilon \downarrow 0. \tag{24}$$

The derivative of f(j) is now defined to be the generalized SLAC[11] derivative

$$\nabla f(j) = \eta_1 \sum_{n=-\infty}^{\infty} i(2\pi n/L') e^{ija(2\pi n/L')-|n|\,\varepsilon}\, \bar{f}(n) \tag{25}$$

where $\varepsilon \downarrow o$ in the sense of Feynman.[13] To check that it satisfies the Leibnitz rule (9) we compute:

$$\nabla(f(j)\ h(j)) = \eta_1 \sum_{n=-\infty}^{\infty} i(2\pi n/L') e^{ija(2\pi n/L')-|n|\,\varepsilon}\, \overline{fh}(n) \tag{26}$$

where

$$\overline{fh}(n) = \eta_2 \sum_{j=-\infty}^{\infty} e^{-ija(2\pi n/L')-|j|\,\varepsilon} f(j)\ h(j)$$

$$= \eta_2\ \eta_1\ \eta_1 \sum_{j=-\infty}^{\infty} e^{-ija(2\pi n/L')-|j|\varepsilon}$$

$$\sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} e^{ija(n_1+n_2)-(|n_1|+|n_2|)\,\varepsilon}\, \bar{f}(n_1)\ \bar{h}(n_2) \tag{27}$$

$$= \eta_2\ \eta_1\ \eta_1 \sum_{n_1,n_2=-\infty}^{\infty} \delta_{n,\ n_1+n_2} \coth(\varepsilon/2)\ \bar{f}(n_1)\ \bar{h}(n_2) e^{-(|n_1|+|n_2|)\varepsilon}$$

$$= \eta_1 \sum_{n_2=-\infty}^{\infty} \bar{f}(n-n_2)\ \bar{h}(n_2) e^{-(|n-n_2|+|n_2|)\varepsilon}$$

Thus, substituting (27) into (26) gives

$$\nabla(f(j)\ h(j)) = \eta_1 \eta_2 \sum_{n,n_2=-\infty}^{\infty} i(2\pi n/L')e^{\ ija(2\pi n/L')-|n|\ \varepsilon}\ \bar{f}(n-n_2)\ \bar{h}(n_2)\ e^{-(|n-n_2|+|n_2|)\varepsilon}$$

$$= \eta_1 \eta_1 \sum_{n,n_2=-\infty}^{\infty} \Big\{ i(2\pi(n-n_2)/L')e^{\ ija(2\pi n/L')-|n|\ \varepsilon}\ \bar{f}(n-n_2)\ \bar{h}(n_2)\ e^{-(|n-n_2|+|n_2|)\varepsilon}$$

$$+\ i(2\pi n_2/L')e^{ija(2\pi n/L')-|n|\ \varepsilon}\ \bar{f}(n-n_2)\ \bar{h}(n_2)\ e^{-(|n-n_2|+|n_2|)\varepsilon} \Big\}$$

$$= \eta_1 \sum_{n_2=-\infty}^{\infty} \bar{h}(n_2)e^{ija(2\pi n_2/L')-|n_2|\ \varepsilon}\ \eta_1 \sum_{n'=-\infty}^{\infty} i(2\pi n'/L')\ \bar{f}(n')\ \mathrm{x}e^{\ ija(2\pi n'/L')-|n'|\ \varepsilon\ -\ |n'+n_2|\ \varepsilon}$$

$$+\ \eta_1 \sum_{n_2=-\infty}^{\infty} i(2\pi n_2/L')\bar{h}(n_2)e^{ija(2\pi n_2/L')-|n_2|\ \varepsilon}\ \eta_1 \sum_{n'=-\infty}^{\infty} \bar{f}(n')e^{ija(2\pi n'/L')}\ e^{\ -|n'|\varepsilon\ -\ |n'+n_2|\varepsilon}$$

$$= h(j)\ \nabla f(j)\ +\ (\nabla h(j))\ f(j) \tag{28}$$

in agreement with (9).

In making the last step in (28), we have used Dini's theorem[14] to conclude that

$$\Big| \eta_1 \sum_{n'=-\infty}^{\infty} i(2\pi n'/L')\ \bar{f}(n')e^{\ ija(2\pi n'/L')-|n'|\ \varepsilon\ -\ |n'+n_2|\varepsilon} - \eta_1 \sum_{n'=-\infty}^{\infty} i(2\pi n'/L')\ \bar{f}(n')e^{\ ija(2\pi n'/L')-|n'|\varepsilon} \Big|$$

$$\leq \eta_1 \sum_{n'=-\infty}^{\infty} |2\pi n'/L'|\, e^{-|n'|\ \varepsilon} |e^{-|n'+n_2|\varepsilon} - 1|\ |f(n')| \ \underset{\varepsilon \downarrow 0}{\to 0} \tag{29}$$

,

assuming

$$\eta_1 \sum_{n'=-\infty}^{\infty} i(2\pi n'/L')\, \overline{f}(n') e^{\,ij(2\pi n'/L')a-|n'|\,\varepsilon} \qquad (30)$$

exists for $\varepsilon \downarrow o$.

Returning to (15) we write the general function on the Wilson lattice as

$$f_1(n_0,n_1,n_2,n_3) = \eta_1{}^4 \sum_{\ell_0=-i\infty}^{i\infty} \sum_{\ell_1,\ell_2,\ell_3=-\infty}^{\infty} e^{-in\cdot\ell(2\pi a/L')} \overline{f}_1(\ell_0,\ell_1,\ell_2,\ell_3) \qquad (31)$$

where $n\cdot\ell = n_0\ell^0 + n_1\ell^1 + n_2\ell^2 + n_3\ell^3$ (we shall henceforth take the Euclidian lattice as a Minkowski lattice with imaginary time). From (25) we have that

$$\partial_\mu f_1 = \eta_1{}^4 \sum_{\ell_0,\ell_1,\ell_2,\ell_3} -i\ell_\mu\,(2\pi/L') e^{-in\cdot\ell(2\pi a/L')}\, \overline{f}_1(\ell) \qquad (32)$$

is a Leibnitz rule respecting derivative. Thus, we take the gauge field Lagrangian as

$$\mathscr{L}_{YM} = -\frac{1}{4}\vec{F}_{\mu\nu}(n)\cdot\vec{F}^{\mu\nu}(n) \qquad (33)$$

where

$$\vec{F}_{\mu\nu}(n) \equiv \partial_\mu \vec{A}_\nu(n) - \partial_\nu \vec{A}_\mu(n) + g\varepsilon_{\vec{\ }bc} A^b_\mu(n) A^c_\nu(n) \qquad (34)$$

with ∂_μ defined by (32). Then, with $\partial_\mu\vec{\omega}(n)$ in (14) also defined by (32), we see that $\mathscr{L}_{YM}$ is gauge invariant.

Further, we introduce the fermion representation R carried by $\psi(n)$ in the fermion Lagrangian

$$\mathscr{L}_\psi = \overline{\psi}(i\partial\!\!\!/ - g\,\vec{A\!\!\!/}(n)\cdot\vec{t})\,\psi - \overline{\psi}\, m_0\psi \;. \qquad (35)$$

Here, $\vec{t}$ are the generators of $\mathcal{G}$ in R and m_0 is a gauge invariant bare mass operator. Clearly with ∂_μ given by (32), we see that

$$\partial_\mu + i\, g\, \vec{A}_\mu(n) \cdot \vec{t}$$

has the structure of the usual gauge theory covariant derivative[4] so that (35) is gauge invariant (since (32) respects the Leibnitz rule). Thus the lattice Lagrangian

$$\mathcal{L} = -\frac{1}{4}\vec{F}_{\mu\nu}(n) \cdot \vec{F}^{\mu\nu}(n) + \overline{\psi}(n)(i\partial\!\!\!/ - g\vec{A\!\!\!/}\cdot \vec{t})\psi(n) - \overline{\psi}(n)m_0\, \psi(n) \qquad (36)$$

is invariant (if ∂_μ is defined by (32)) under the simultaneous infinitesimal transformations (14) and

$$\begin{aligned} \psi(n) &\to (1 - i\vec{\omega}(n)\cdot\vec{t})\, \psi(n) \\ \overline{\psi}(n) &\to \overline{\psi}(n)(1 + i\vec{\omega}(n)\cdot\vec{t}) \end{aligned} \qquad (37)$$

We see that, for m_0=0, (36) possesses the local chiral invariance under the infinitesimal transformations

$$\begin{aligned} \psi(n) &\to (1 + i\xi\, \gamma_5)\psi \\ \overline{\psi}(n) &\to \overline{\psi}(n)(1+i\xi\,\gamma_5)\ . \end{aligned} \qquad (38)$$

Further, for m_0=0, the axial vector currents $\overline{\psi}(n)\gamma_\mu\gamma_5\vec{\lambda}\psi(n)$ satisfy the formal conservation law:

$$\partial^\mu\, (\overline{\psi}(n)\gamma_\mu\, \gamma_5\, \vec{\lambda}\psi\, (n)) = 2i\psi(n)m_0\gamma_5\vec{\lambda}\ \psi\, (n) \qquad (39)$$

where we take $\vec{\lambda}$ to commute with $\vec{t}$. To verify (39), one needs the Euler-Lagrange equations on the lattice. But since the derivative (32) respects the Leibnitz rule, clearly, the equations have the same form as in the continuum limit: The fermion field $\psi(n)$ satisfies

$$\begin{aligned} (i\partial\!\!\!/ - g\vec{A\!\!\!/}\cdot\vec{t} - m_0)\psi\,(n) &= 0 \\ \overline{\psi}(n)(-i\overleftarrow{\partial\!\!\!/} - g\vec{A\!\!\!/}\cdot\vec{t} - m_0) &= 0 \end{aligned} \qquad (40)$$

This is enough to prove the formal relation (39), as is well known.[9]

Later in this section, we shall show that (39) is invalidated in perturbation theory, as found by Adler, Bell and Jackiw. However, before turning to this, we should comment about the untraviolet behavior of (36). For, as we emphasized in the introduction, one of the main constraints on the lattice is that it not change the short distance aspect of the respective theory. That is to say, it must not change the behavior of $g_R(t)$ at short distances, where $g_R(t)$ is the Gell-Mann-Low-Callan-Symanzik[6] running coupling constant and t is the logarithm of the inverse distance scale in appropriate momentum units.[15]

Specifically, in quantizing (36), one will in general specify a gauge condition, for example,

$$\partial_\mu \vec{A}^\mu(n) = \vec{C}(n) \ . \qquad (41)$$

Then, the quantization can proceed in the standard Faddeev-Popov[16] manner with due respect to the Gribov-Mandelstam problem.[17] The net result (according to the lore) is that in each distinct Gribov copy for(41), one has to add the effective ghost related Lagrangian

$$\mathscr{L}_G = \phi^*(n)\partial^\mu D_\mu(\vec{A}(n))\,\phi(n) - \frac{1}{2\alpha}(\partial_\mu \vec{A}(n))^2 \qquad (42)$$

where the $\phi(n)$ are the usual ghost fields. Here $D_\mu(\vec{A})$ is the derivative of the variation of $\vec{A}_\mu \cdot \vec{\tau}$ due to the gauge transformation

$$\vec{A}_\mu(n)\vec{\tau} \to \Omega\,\vec{A}_\mu(n)\cdot\vec{\tau}\,\Omega^{-1} + \frac{i}{g}(\partial_\mu \Omega)\Omega^{-1} \quad , \qquad (43)$$

the derivative being taken with respect to $\vec{\omega}(n)$ and evaluated at the appropriate representative of the Gribov copy. Here,

$$\Omega \equiv \exp\left[-i\,\vec{\omega}(n)\cdot\vec{\tau}\right] \quad , \qquad (44)$$

where $\vec{\tau}$ carry the adjoined representation. According to 't Hooft, Veltman, and Lee and Zinn-Justin[18], in each Gribov copy, (36) with (41) added is a renormalizable theory.

The only possible source of a problem with these remarks is that the action, represented in our momentum space, may not generate the same Feynman rules as in Refs. 16 and 18. However, we can choose

$$\eta_1 = \frac{1}{a \coth (\varepsilon/2)}, \ \eta_2 = a \tag{45}$$

so that, for example, the continuum integrals $\int dx^1$ and $\int dk^1$ become, here,

$$\int_{-\infty}^{\infty} dx^1 \leftrightarrow a \sum_{n^1=-\infty}^{\infty} \tag{46}$$

$$\int_{-\infty}^{\infty} \frac{dk^1}{2\pi} \leftrightarrow \frac{1}{a \coth (\varepsilon/2)} \sum_{\ell^1=-\infty}^{\infty} \tag{47}$$

Similarly,

$$\int_{-i\infty}^{i\infty} i dx^0 \leftrightarrow a \sum_{n^0=-i\infty}^{\infty} \tag{48}$$

$$\int_{-i\infty}^{i\infty} \frac{dk^0}{2\pi} \leftrightarrow \frac{1}{a \coth (\varepsilon/2)} \sum_{\ell^0=-i\infty}^{i\infty} . \tag{49}$$

Thus we have the same Feynman rules as given in Refs. 16, 18, and 19 with the replacements (in the Bjorken and Drell[20] metric)

$$i\int d^4x \rightarrow a^4 \sum_{n^0=(-i\infty)}^{\infty} \ \sum_{n^1, n^2, n^3=-\infty}^{\infty} \tag{50}$$

$$\frac{\int d^4k}{(2\pi)^4} \rightarrow \frac{i}{a^4 \coth^4(\varepsilon/2)} \sum_{\ell^0=-i\infty}^{i\infty} \sum_{\ell^1,\ell^2,\ell^3=-\infty}^{\infty} \quad . \tag{51}$$

(The i's appear because we are on the Euclidean lattice, which we take as a Minkowski lattice with imaginary time.)

The question of the behavior of $g_R(t)$ for (36) may now be addressed as follows: Consider the Gribov copy near $\Omega=I$ in (42), for simplicity. Clearly, the algebraic structure of the diagrams, such as those illustrated in Fig. 1 for Z_3, which determine $\beta(g_R)$ for small g_R is the same as it would be in continuum theory. So, the only possible difference is in the value of the corresponding Feynamn integrals (which are the sums (50) and (51) in our lattice theory). In particular, for computing the logarithmic divergences which determine $\beta(g_R)$ for g_R near zero, we simply have to vertify that our lattice calculation reproduces, for zero lattice spacing,[15] the same divergences as the continuum theory. But, from (50) and (51) we see that for $a \rightarrow 0$ (zero lattice spacing) and $L' \rightarrow \infty$,

$$\lim_{a\rightarrow 0} a^4 \sum_{n^0=-i\infty}^{i\infty} \sum_{n^1,n^2,n^3=-\infty}^{\infty} = i\int d^4x \tag{52}$$

and

$$\lim_{a\rightarrow 0} \lim_{L'\rightarrow\infty} \frac{i}{(a \coth(\varepsilon/2))^4} \sum_{\ell^0=-i\infty}^{i} \sum_{\ell^1,\ell^2,\ell^3=-\infty}^{\infty} = \frac{\int d^4k}{(2\pi)^4} \tag{53}$$

if $L'/a \rightarrow \infty$ in such a way that

$$(L'/a) \rightarrow \coth(\varepsilon/2) \quad \text{for } \varepsilon\downarrow 0. \tag{54}$$

For, $d^4k = (2\pi/L')^4 d^4n$. Thus, for zero lattice spacing, our sums in our Feynman diagrams become the (Reimann-) Lebesgue continuum Feynman integrals themselves. Thus, the zero lattice spacing limit of our lattice theory is the perturbative continuum theory and hence this limit has the same perturbative $\beta(g_R)$ as the usual continuum theory.

The result that the zero lattice spacing limit of our lattice theory agrees term by term with the perturbative continuum theory means that any result of continuum Feynman diagram perturbation theory will be recovered in the continuum limit. In particular, the result of Adler-Bell-Jackiw,[10] that

$$\partial^\mu \overline{\psi}(x) \quad \gamma_\mu \gamma_5 \vec{\lambda}\psi \ (x) = 2i\, \overline{\psi}(x) m_0 \ \gamma_5 \ \vec{\lambda}\psi \ (x) + \frac{g^2}{16\pi^2} \operatorname{tr} [\vec{\lambda} t^a t^b] \\ F_{\mu\nu}{}^a(x) \ F_{\mu'\nu'}{}^b(x) \varepsilon^{\mu\nu\mu'\nu'} \quad , \tag{55}$$

is such a perturbative continuum theory result. It is therefore necessarily also a result of the zero lattice spacing limit of our lattice theory. This is the desired result.

The reader may object that we have not really verified (55) on the lattice with $a \neq 0$-we have only shown that it holds for $a \to 0$. However, if we look at the Feynman diagrams for the anomaly (Fig. 2), we can see that to be on the lattice the momenta entering the diagrams should be of the form (in the notation of Fig. 2)

$$k_i = \frac{2\pi}{L'} (\ell_i{}^0, \ell_i{}^1, \ell_i{}^2, \ell_i{}^3) \quad i = 1,2,3, \tag{56}$$

with

$$\sum_{i=1}^{3} \ell_i = 0 \tag{57}$$

Choose such k_i. Then, since L' is any number such that a/L' is not rational, replace L' with NL' and ℓ_i with $N\ell_i$, i = 1,2,3, for some integer N. Then, k_i are unchanged. Hence, we may take N to ∞. But, then, our sum over the internal momenta in Fig. 2,

$$\frac{i}{\left[a\ \coth(\varepsilon/2)\right]^4} \sum_{r^0,r^1,r^2,r^3} \tag{58}$$

becomes the Reimann-Lebesgue integral

$$\left[\frac{NL'}{a\ \coth(\varepsilon/2)}\right]^4 \frac{\int d^4r}{(2\pi)^4} = \frac{\int d^4r}{(2\pi)^4} \tag{59}$$

provided $\varepsilon \downarrow 0$, $N \to \infty$ such that

$$NL' = a\ \coth(\varepsilon/2)\ . \tag{60}$$

Thus, for each momenta set (56), we recover the momentum-space version of (55). But, this means we have the result (55) on our lattice:

$$\partial^\mu (\bar{\psi}(n)\ \ \gamma_\mu\gamma_5\ \vec{\lambda}\psi\ (n)) = 2i\ \bar{\psi}(n)m_0\ \gamma_5\ \vec{\lambda}\psi\ (n) + \frac{g^2}{16\pi^2}\ \mathrm{tr}\ [\vec{\lambda}\ t^a t^b] \times F_{\mu\nu}{}^a(n)F_{\mu'\nu'}{}^b(n)\varepsilon^{\mu\nu\mu'\nu'} \tag{61}$$

This procedure we just described then completes our construction: To define an n-point Feynman amplitude go to momentum space and evaluate the amplitude (using the usual continuum space Feynman rules and integrals) at momenta of the form (56):

$$k_i = \frac{2\pi}{L'}\ (\ell_i{}^0, \ell_i{}^1 \ell,_i{}^2 \ell,_i{}^3) \qquad i = 1,\ldots,n \tag{62}$$

for $\sum_i \ell_i = 0$, ℓ_i^α on the momentum space lattice. Then, to return to the position space lattice, simply use our lattice Fourier transform (31).

Finally, in closing we note that since our derivative respects the rule of Leibnitz,

$$\frac{g^2}{16\pi^2}\, tr(\vec{\lambda}\, t^a t^b)\, F_{\mu\nu}{}^a(n)\, F_{\mu'\nu'}{}^b(n)\, \varepsilon^{\mu\nu\mu'\nu'} = \frac{g^2}{4\pi^2}\, \partial_\mu\Big[tr\, \Big\{ \vec{\lambda}\, [A_\nu(n)\, \partial_{\mu'} A_{\nu'}(n) - \frac{2}{3}\, gi\, A_\nu(n)\, A_{\mu'}(n)\, A_{\nu'}(n)] \Big\} \varepsilon^{\mu\nu\mu'\nu'} \Big] \quad , \tag{63}$$

where

$$A_\mu = A_\mu{}^a\, t^a \quad . \tag{64}$$

Thus the current

$${}^s\vec{J}_\mu{}^5(n) = \overline{\psi}(n)\gamma_\mu\gamma_5\, \vec{\lambda}\psi\,(n) - \frac{g^2}{4\pi^2}\, tr\, \Big\{ \vec{\lambda}\, [A_\nu(n)\partial_{\mu'}\, A_{\nu'}(n) - \frac{2}{3}\, gi\, A_\nu(n)\, A_{\mu'}(n)\, A_{\nu'}(n)] \Big\}\, \varepsilon^{\mu\nu\mu'\nu'} \quad , \tag{65}$$

is conserved on our lattice, just as it is in the continuum.***

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FOOTNOTES

*The measurements of Fairbank <u>et al.</u>[1] will be viewed as systematic in what follows. Even if these measurements are not systematic effects, it is still very difficult (if not impossible) to liberate quarks!

**By "light quarks" we simply mean that the quarks have masses not comparable to the lattice spacing and that these masses remain finite as the lattice spacing approaches zero.

***Our result that the anomaly in the Adler-Bell-Jackiw theorem can be written as a divergence on our lattice appears to disagree with the work of Peskin (Cornell Reports CLNS-395, CLNS 396, 1978) for the Wilson non-Abelian lattice theory. The two results can be reconciled by viewing our lattice theory as a member of an ultravioletly attractive class which is not considered by Peskin. We have no argument against this view.

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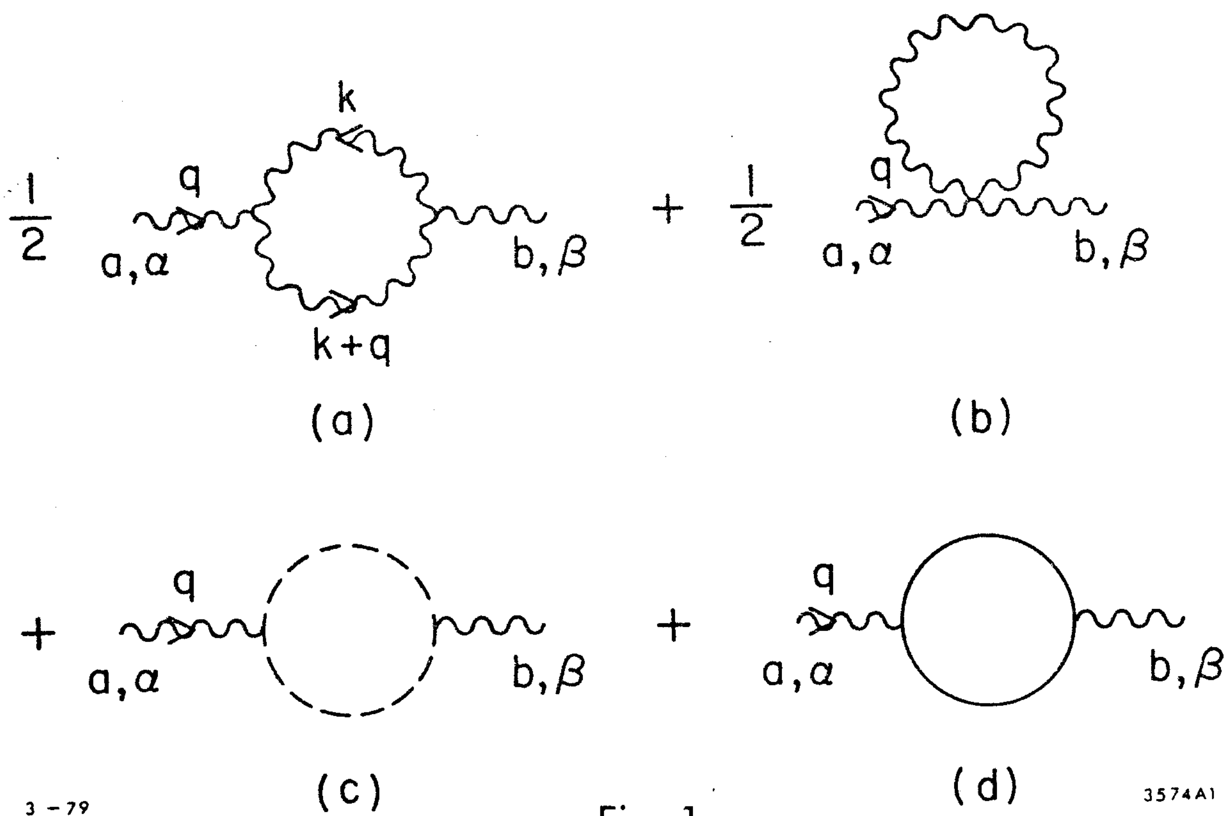

Fig. 1

Diagrams determining Z_3 for the theory (36) plus (42). The wavy lines ∿ are A_μ^a propagators. The dashed lines --- are ghost propagators. The solid lines are fermion propagators. The Feynman rules for the propagators and vertices may be found in Refs. 16, 18, 19, and 20.

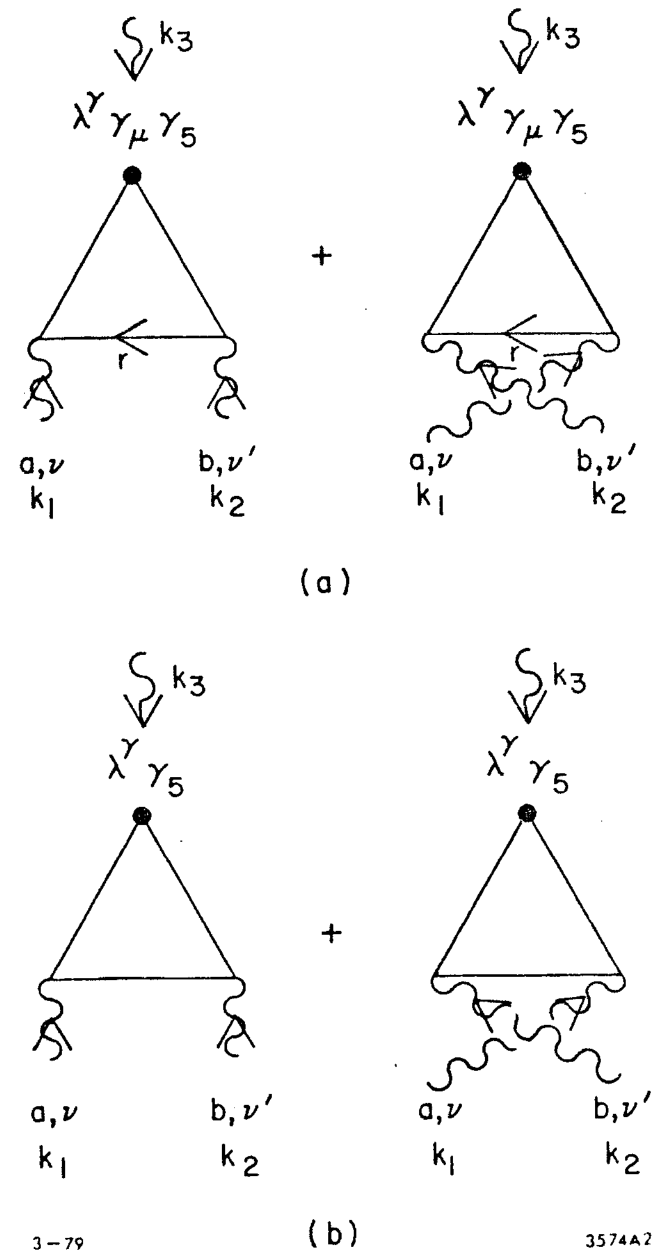

Fig. 2

Diagrams involved in the Adler-Bell-Jackiw anomaly. (a) The axial vector-vector-vector vertex. (b) The pseudoscalar-vector-vector-vertex.