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MASSLESS SU(N) YANG-MILLS THEORY IN TWO DIMENSIONS*

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ABSTRACT

An analysis of two-dimensional, massless SU(N) Yang-Mills theory initiated previously is continued and extended. The fermion propagator in the presence of a non-Abelian potential is constructed exactly, and the corresponding (induced) vacuum fermion currents and their divergences are deduced. The analysis of the color-singlet current and the associated bound states reveals the non-existence of massive color-singlet bound states. This fact together with the previously established existence of massive "colored" states characterize the spectrum, save for possible massless exitations. A consideration of the bosonized version of the theory reveals the symmetry breaking and the associated mass generation to be a Schwinger-like mechanism. An Abelian model illustrating this analogy is briefly analyzed and discussed.

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I. Introduction and Summary

In a recent work[1] we reported the outline of a calculation in two-dimensional massless SU(N) Yang-Mills theory (to be here referred to as QCD) that showed the existence of a degenerate family of N^2-1 massive "colored"[2] bosons. Since this massless theory is expected to be the strong-coupling limit $g \gg m$ (g is the coupling constant, m the bare quark mass)[3] of massive QCD, the above result is implied for the strong-coupling limit of massive QCD as well. This behavior is contrary to what is believed to happen in the 't Hooft[4] model (i.e., massive QCD in the limit of large N), and it gives support to the conjecture[3] that the model is not valid in the strong-coupling regime. In this paper a more extensive analysis of massless QCD is carried out which, among other things, establishes the non-existence of massive color-singlet bound states in the theory, thus yielding a particle spectrum basically different from that of the confined phase. How does this come about? In order to provide an answer to this question, and before entering the details of the main analysis of this paper, it is useful to consider a simple Abelian model which will essentially reproduce the behavior described above and at the same time will identify the mechanisms of symmetry breaking and mass generation which occur in the non-Abelian theory.

The model, which is a variant of the Schwinger model, contains two species of massless fermions, q_a and q_b, and is defined by the Lagrangian density

$$\mathscr{L} = \bar{q}_a i\not{\partial} q_a + \bar{q}_b i\not{\partial} q_b - \frac{e}{2}\left(\bar{q}_a\gamma^\mu q_b + \bar{q}_b\gamma^\mu q_a\right) A_\mu - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} - J^\mu A_\mu , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu , \tag{1}$$

where J is an external c-number current provided for the purpose of carrying out functional operations.[5] With reference to the currents

$$j_t^\mu = \frac{e}{2}\,(\bar{q}_a\gamma^\mu q_b + \bar{q}_b\gamma^\mu q_a), \qquad j_s^\mu = (\bar{q}_a\gamma^\mu q_a + \bar{q}_b\gamma^\mu q_b), \tag{2}$$

appropriately regularized to insure current conservation, one can calculate the corresponding vacuum currents

$$\langle j_t^\mu\rangle = \langle \mathrm{OUT}|j_t^\mu|\mathrm{IN}\rangle / \langle \mathrm{OUT}|\mathrm{IN}\rangle, \qquad \langle j_s^\mu\rangle = \langle \mathrm{OUT}|j_s^\mu|\mathrm{IN}\rangle / \langle \mathrm{OUT}|\mathrm{IN}\rangle, \tag{3}$$

induced by the "external potential"

$$\langle A_\mu\rangle = \langle \mathrm{OUT}|A_\mu|\mathrm{IN}\rangle / \langle \mathrm{OUT}|\mathrm{IN}\rangle. \tag{4}$$

In much the same way as in the original solution of the Schwinger model,[6] one obtains

$$\langle j_t^\mu\rangle = \frac{-e^2}{2\pi}\,(g^{\mu\nu} - \hat{\partial}^\mu\hat{\partial}^\nu)\,\langle A_\nu\rangle, \qquad \langle j_s^\mu\rangle = 0. \tag{5}$$

Combined with the field equation for A (in the Landau gauge), the first of these gives

$$(\partial^2 + \frac{e^2}{2\pi})\,(g^{\mu\nu} - \hat{\partial}^\mu\hat{\partial}^\nu)\,\langle A_\nu\rangle = J^\mu. \tag{6}$$

Again, as in the Schwinger model, and for the same reasons, fermions have disappeared and a massive boson has appeared. Moreover, this boson couples to the "triplet" $q\bar{q}$ channel, with the "singlet" channel remaining non-interacting, as may be seen from the following propagators:

$$-i\,\langle 0|T\,[j_t^\mu(x)\,j_t^\nu(y)]|0\rangle = \frac{-e^2}{2\pi}\,(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)\,\Delta\,(x-y), \tag{7}$$

$$-i\,\langle 0|T\,[j_s^\mu(x)\,j_s^\nu(y)]|0\rangle = \frac{-2}{\pi}\,(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)\,D\,(x-y), \tag{8}$$

where Δ (D) denotes the (Feynman) propagator for a boson of mass $e^2/2\pi$ (zero). The designations "singlet" and "triplet" have been used in reference to an obvious interpretation of the above model as a broken SU(2) Yang-Mills theory in which all but one of the gauge degrees of freedom have been frozen out, leaving an unidirectional (in internal space), i.e. Abelian, field.

Essentially the behavior observed in the above model, appropriately generalized to an analogously broken SU(N), will emerge from the analysis of massless QCD in the following sections. Thus the color-singlet quark current will remain free,[7] while the color current will develop an axial anomaly which will serve as the source of mass generation. These properties of the currents will in turn lead to the non-existence and the existence, respectively, of massive color-singlet and colored bound states in the spectrum of the theory.

The rest of this paper is organized as follows: In Section II we formulate the theory in a functional framework and proceed to construct the quark propagator in the presence of couplings to external SU(N) and U(1) gauge potentials. In Section III this propagator is used to derive the induced vacuum currents and their divergences. An alternative derivation of these current divergence relations is also given here. The above-mentioned properties of the currents, which essentially characterize the spectrum of the massive states, are deduced in this section. The fermion-antifermion equation derived previously within a new bound-state formalism is employed in Section IV to establish the non-existence of massive color-singlet states. In Section V, the theory is bosonized, and the existence of massive colored states is deduced from the infrared behavior of the theory. Concluding remarks are presented in Section VI. To make this paper self-contained, part of the material in Ref. 1 will be repeated here, often in a more detailed manner.

II. Construction of the Quark Propagator

The Lagrangian density for massless QCD including a coupling to a c-number colored current J_i^μ and a coupling of the color-singlet quark current to a c-number potential U^μ may be written

$$\mathscr{L} = \bar{q} i \not\partial q - \frac{1}{4} G_i^{\mu\nu} G_{i\mu\nu} + B_{i\mu} (j_i^\mu + J_i^\mu) - e j_\mu U^\mu , \tag{9}$$

where

$$G_{i\mu\nu} = \partial_\mu B_{i\nu} - \partial_\nu B_{i\mu} + g f_{ijk} B_{j\mu} B_{k\nu} , \qquad j_i^\mu = g\bar{q}\gamma^\mu \lambda_i q ,$$

$$j^\mu = \bar{q}\gamma^\mu q, \; [\lambda_i, \lambda_j] = i f_{ijk} \lambda_k , \qquad \mathrm{tr}\,(\lambda_i \lambda_j) = \frac{1}{2} \delta_{ij} , \tag{10}$$

$$i = 1, \ldots, N^2 - 1 .$$

Here a contraction over color as well as spin indices is implied in the definition of the currents. The rest of the notation is defined by $g^{00} = -g^{11} = 1$, $\gamma^0 = \sigma_3$, $\gamma^1 = i\sigma_2$, $\gamma^5 = \gamma^0\gamma^1 = \sigma_1$, and $\varepsilon^{01} = 1$, where $\varepsilon^{\mu\nu}$ is the antisymmetric tensor. The extraneous objects J and U will of course serve an auxiliary function and will be set equal to zero for arriving at physical results.

The first step in the analysis is the elimination of quark variables by means of functional methods.[5] Thus we consider the vacuum expectation value

$$\bar{B}_{i\mu} = \langle \mathrm{OUT} | B_{i\mu} | \mathrm{IN} \rangle / \langle \mathrm{OUT} | \mathrm{IN} \rangle , \tag{11}$$

and proceed to calculate the vacuum currents induced by $\bar{B}_{i\mu}$ and U. Defining the quark propagator in the usual way by

$$S(x,y) = -i \langle \mathrm{OUT} | T\, [q(x)\bar{q}(y)] | \mathrm{IN} \rangle / \langle \mathrm{OUT} | \mathrm{IN} \rangle , \tag{12}$$

we obtain the Schwinger equation[5]

$$\gamma_\mu \; [i\partial_x^\mu - eU^\mu(x) + g\lambda_i \; A_i^\mu(x)] \; S(x,y) = \delta(x-y) \;, \tag{13}$$

where

$$A_i^\mu(x) = \bar{B}_i^\mu(x) - i \frac{\delta}{\delta J_{i\mu}(x)} \;. \tag{14}$$

Note that the gauges for U and B have been left unspecified.

While the solution of Eq. (13) in the absence of the non-Abelian coupling (i.e., $g = 0$) is straightforward and well known, the solution of the full equation is a non-trivial matter.[8] Thus, following Schwinger[6], we make the transformation

$$S(x,y) = \exp \; [\Phi(x) - \Phi(y)] \; S'(x,y) \;, \tag{15}$$

where

$$\Phi(x) = e\int d^2x' \; S_0(x-x') \; \gamma^\mu U_\mu(x') \;, \tag{16}$$

thereby eliminating the Abelian coupling and leaving

$$\gamma_\mu \; [i\partial_x^\mu + g\lambda_i A_i^\mu(x)] \; S'(x,y) = \delta(x-y) \;. \tag{17}$$

Note that S_0 appearing in (16) is the free fermion propagator given in Eq. (21) below.

The solution of Eq. (17) is achieved by considering its light-cone decomposition,

$$S' = S'^{+} + S'^{-} \;, \quad S'^{\pm} = \Lambda^{\pm} S' \;, \quad \Lambda^{\pm} = \frac{1}{2}\,(1 \pm \gamma^5) \;, \tag{18}$$

and by writing

$$S'(x,y) = \mathscr{T}_+(x)\, \mathscr{T}_+^{-1}(y)\; S_0^+(x-y) + \mathscr{T}_-(x)\, \mathscr{T}_-^{-1}(y)\; S_0^-(x-y) \;, \tag{19}$$

with

$$S_0^{\pm}(x) = -\frac{1}{4\pi}\gamma_{\pm}\,[P\,\frac{1}{x_{\pm}} + i\pi\delta^{(1)}(x_{\pm})\,\varepsilon\,(x_{\mp})]\ , \tag{20}$$

where P denotes the principal value, ϵ gives the sign of its argument, and the definition $a_{\pm} = a^{\pm} = a_0 \pm a_1$ for the light-cone components of a_{μ} has been adopted. Note that $\partial_{\pm} = 2\partial/\partial x_{\mp}$ and that $S_0^{\pm}$ was obtained from Eq. (18) and

$$S_0(x) = -\frac{1}{2\pi}\gamma_{\mu}x^{\mu}/(x^2 - i\varepsilon)\ . \tag{21}$$

The equations obeyed by $\mathscr{T}_{\pm}$ emerge upon inserting (19) in (17); they are, in differential and integral forms respectively,

$$i\partial_{\pm}\mathscr{T}_{\pm} = -\,g\lambda_i\,A_{i\pm}\,\mathscr{T}_{\pm}\ , \tag{22}$$

and

$$\mathscr{T}_{\pm}(x) = 1 + \int d^2x'\,s_0^{\pm}(x-x')\,[-g\lambda_i A_{i\pm}(x')]\,\mathscr{T}_{\pm}(x')\ , \tag{23}$$

where $s_0^{\pm}$ is obtained from $S_0^{\pm}$ by omitting $\gamma_{\pm}/2$ from the latter. We now observe that the Ansatz

$$\begin{aligned} \mathscr{T}_{\pm}(x) &= T_{\pm}(x)\ \tau_{\pm}(x_{\pm})\ , \\ T_{\pm}(x) &= T\left\{\exp\ [ig\lambda_i \int d^2x'\delta^{(1)}(x_{\pm} - x'_{\pm})\,\Theta\,(x_{\mp} - x'_{\mp})\ A_{i\pm}(x')]\right\}\ , \end{aligned} \tag{24}$$

where T denotes ordering with respect to the non-trivial integration in (24), and a careful partial integration in (23) using the property

$$i\partial_{\pm}T_{\pm} = -g\lambda_i A_{i\pm}T_{\pm}\ , \tag{25}$$

$$\begin{aligned} \mathscr{T}_{\pm}(x) = 1 + \mathscr{T}_{\pm}(x) - \frac{1}{2\pi}\int dx'_{\pm}\ [&\frac{1}{x_{\pm} - x'_{\pm} + i\varepsilon}\,\mathscr{T}_{\pm}(x'_{\mp} = +\infty,\ x'_{\pm}) \\ &- \frac{1}{x_{\pm} - x'_{\pm} - i\varepsilon}\,\mathscr{T}_{\pm}(x'_{\mp} = -\infty,\ x'_{\pm})\,]\ . \end{aligned} \tag{26}$$

The use of Eq. (24) in (26) in turn gives

$$\tau_\pm(x_\pm) = 1 - \frac{i}{2\pi}\int dx'_\pm \frac{1}{x_\pm - x'_\pm - i\varepsilon} [T_\pm(x'_\mp = \infty, x'_\pm) - 1]\, \tau_\pm(x'_\pm) \ . \quad (27)$$

With $\tau_\pm$ thus given by this one-dimensional integral equation, we have completed the construction of the quark propagator in terms of ordered exponentials and the solution of Eq. (27). For later reference, we record here the full expression for the quark propagator

$$S(x,y) = \exp\,[\Phi(x) - \Phi(y)]\,[T_+(x)\tau_+(x_+)\,\tau_+^{-1}(y_+)T_+^{-1}(y)\Lambda^+ + T_-(x)\tau_-(x_-)\tau_-^{-1}(y_-)T_-^{-1}(y)\Lambda^-]\; S_0(x-y) \ . \quad (28)$$

We pause here to call attention to the fact that in going from (22) to (23), the usual causal boundary conditions were incorporated into the solution. That Eq. (22) admits of a wide class of solutions only one of which satisfies (23) emphasizes the fact that as usual care must be exercised in constructing the causal propagator. Moreover, it must be pointed out that the above construction is valid for any gauge and that, the appearance of light-cone coordinates notwithstanding, it is not committed to light-cone quantization.

III. The Induced Quark Currents

In this section we shall derive expressions for the induced singlet [U(1)] and colored [SU(N)] quark currents, defined respectively as

$$\langle j^\mu \rangle = \langle \mathrm{OUT}|j^\mu|\mathrm{IN}\rangle / \langle \mathrm{OUT}|\mathrm{IN}\rangle \ , \quad (29)$$

$$\langle j^\mu_i \rangle = \langle \mathrm{OUT}|j^\mu_i|\mathrm{IN}\rangle / \langle \mathrm{OUT}|\mathrm{IN}\rangle \ . \quad (30)$$

These are in turn related to the propagator by the formulae[5]

$$\langle j^\mu(x) \rangle = -i \lim_{\varepsilon\to 0} \mathrm{tr} \left\{ \gamma^\mu [1 + ieU^\nu(x)\varepsilon_\nu]\ S(x+\varepsilon/2,\ x-\varepsilon/2) \right\} , \tag{31}$$

$$\langle j_i^\mu(x) \rangle = -ig \lim_{\varepsilon\to 0} \mathrm{tr} \left\{ \gamma^\mu \lambda_i\ [1 - ig\lambda_j A_j^\nu(x)\varepsilon_\nu]\ S(x+\varepsilon/2,\ x-\varepsilon/2) \right\} . \tag{32}$$

Note that the appropriate line-integral factor insuring gauge invariance for each current has been included.

It is at this juncture that an essential simplicity of the massless two-dimensional theory surfaces; to wit, each of the U(1) and SU(N) currents depends on its corresponding field only. Accordingly, the induced U(1) current is calculated from Eq. (31) to be

$$\langle j^\mu(x) \rangle = -\frac{Ne}{\pi} (g^{\mu\nu} - \hat{\partial}^\mu \hat{\partial}^\nu)\ U_\nu(x) , \tag{33}$$

which, save for the factor of N accounting for color multiplicity, is precisely what it would be if the non-Abelian coupling were absent. Similarly, when the implied operations in (32) are carried out, one obtains

$$\langle j_i^\mu(x) \rangle = \frac{g^2}{4\pi} A_i^\mu(x) + \frac{ig}{2\pi} \mathrm{tr}\ [\lambda_i \frac{\partial \mathscr{T}_+(x)}{\partial x_+} \mathscr{T}_+^{-1}(x)\ \alpha_-^\mu + \lambda_i \frac{\partial \mathscr{T}_-(x)}{\partial x_-} \mathscr{T}_-^{-1}(x)\ \alpha_+^\mu] , \quad \alpha_\mp^\mu = (1, \pm 1) , \tag{34}$$

where the trace is to be taken over the color indices. Clearly, this current only involves A_i^μ, as asserted above.

Indeed the decoupling just demonstrated may be used to factorize the generating functional $\langle \mathrm{OUT}|\mathrm{IN}\rangle$ in a rather trivial way. From Eqs. (9) and (29), one obtains

$$\langle j^\mu(x) \rangle = \frac{i}{e}\ \delta/\delta U_\mu(x)\ \ln(\langle \mathrm{OUT}|\mathrm{IN}\rangle) = -\frac{Ne}{\pi} (g^{\mu\nu} - \hat{\partial}^\mu \hat{\partial}^\nu) U_\nu(x) . \tag{35}$$

This functional differential equation is easily integrated to give the factorized form

$$\langle \mathrm{OUT}|\mathrm{IN}\rangle = \exp\left\{\frac{iNe^2}{2\pi}\int d^2x\, d^2x'\, U^\mu(x)(g_{\mu\nu}-\hat{\partial}_\mu\hat{\partial}_\nu)U^\nu(x')\right\}\ [\langle \mathrm{OUT}|\mathrm{IN}\rangle]_{e=0}\ . \quad (36)$$

Clearly then, the color-singlet current j^μ is (in the physical limit $e=0$ which will henceforth be enforced) as it would be in the free case.[7] Therefore, from the propagator

$$\begin{aligned} -i\,\langle 0|T[j^\mu(x)j^\nu(x')]|0\rangle &= \frac{1}{e}\,\delta/\delta U_\nu(x')\,\langle j^\mu(x)\rangle \\ &= -\frac{N}{\pi}\,(g^{\mu\nu}-\hat{\partial}^\mu\hat{\partial}^\nu)\,\delta(x-x')\ , \end{aligned} \quad (37)$$

one can see that

$$j^\mu(x) = -\sqrt{\frac{N}{\pi}}\,\varepsilon^{\mu\nu}\,\partial_\nu\,\Sigma(x)\ , \quad (38)$$

with Σ a canonical pseudoscalar massless field, in accordance with a free fermionic current.

We return now to Eq. (34), and consider the vector and axial-vector divergences for the current $\langle j_i^\mu(x)\rangle$. These are most usefully expressed in the following implicit forms

$$\partial_\mu\langle j_i^\mu\rangle = -g\,f_{ijk}\,A_j{}^\mu\,\langle j_{k\mu}\rangle\ , \quad (39)$$

$$\partial_\mu\langle \tilde{j}_i{}^\mu\rangle = \frac{g^2}{2\pi}\,\partial_\mu\tilde{A}^\mu - g\,f_{ijk}\,A_j{}^\mu(\langle \tilde{j}_{k\mu}\rangle - \frac{g^2}{4\pi}\,\tilde{A}_{k\mu})\ , \quad (40)$$

where the dual vector $\tilde{a}^\mu$ is defined to be $\varepsilon^{\mu\nu}a_\nu$. As expected, these relations may be rewritten concisely in terms of the gauge-covariant derivative

$$D^{\mu}_{ij} = \delta_{ij}\, \partial^{\mu} + g\, f_{ikj}\, A_k{}^{\mu} \quad . \tag{41}$$

The result is

$$D^{\mu}_{ik}\, \langle j_{k\mu} \rangle = 0 \quad , \tag{42}$$

$$D^{\mu}_{ik}\, \langle \tilde{j}_{k\mu} \rangle = \frac{g^2}{2\pi} \langle \tilde{F}_i \rangle \ , \tag{43}$$

where

$$\langle \tilde{F}_i \rangle = \frac{1}{2}\, \varepsilon^{\mu\nu} (\partial_{\mu} A_{i\nu} - \partial_{\nu} A_{i\mu} + g\, f_{ijk}\, A_{j\mu}\, A_{k\nu}) \ . \tag{44}$$

Equations (42) and (43) are respectively statements of gauge invariance and axial-vector anomaly for the theory.[9]

Before proceeding further, it will be useful to provide an alternative derivation of Eqs. (42) and (43) and the corresponding result for the color-singlet current. This alternative method will avoid the elaborate construction of the quark propagator and, relying on differential properties, will lead directly to the expressions for the current divergences. Though limited in its results, it will provide an independent verification of the preceding calculations.

The alternative method is based on a "two-body" equation which is derived within the context of a formalism developed recently.[10,11,12] To introduce this equation, we define the operators $\mathscr{L}^{\pm}$ according to

$$\mathscr{L}^{+}_{x} = \gamma_{\mu}[i\partial^{\mu}_{x} + g\lambda_i A^{\mu}_i(x)] \ , \ \mathscr{L}^{-}_{x} = \gamma_{\mu}[i\partial^{\mu}_{x} - g\lambda^{*}_i A^{\mu}_i\ (x)] \ , \tag{45}$$

so that Eq. (17) may be written as

$$\mathscr{L}^{+}_{x}\, S(x,y) = \delta(x-y) \quad . \tag{17'}$$

Let n^μ be a unit vector, $n_\mu n^\mu = 1$. Then the sought-after equation may be written

$$[\not{n}^{(1)} \mathscr{L}_x^{+(1)} + \not{n}^{(2)} \mathscr{L}_y^{-(2)}] S(x,y) C = 0 \ , \tag{46}$$

where C is the charge conjugation matrix, and where the superscript (1) [(2)] indicates matrix multiplication onto the first (second) set of spin and color indices carried by SC. Furthermore, the characteristic simplicity of the spin algebra requires that SC commute with γ^5, with the consequence that (46) may be simplified to

$$i(\partial_x^\mu + \partial_y^\mu)\gamma_\mu S + g A_i^\mu(x) \lambda_i \gamma_\mu S - g A_i^\mu(y) \gamma_\mu S \lambda_i = 0 \ , \tag{47}$$

where now ordinary matrix multiplication is implied for all indices. Note that in arriving at (47), we have assumed λ_i to be Hermitean.

Equation (47) can now be used to derive expressions for the divergence of various currents. We will present the derivation for the divergence of the axial SU(N) current, and state the result for the remaining cases. To start, we rewrite the analog of the definition (32) for the axial current using the property [valid for the two-dimensional massless case; cf. Eq. (17)] that

$$\lim_{x \to y} S(x,y) = S_0(x,y) \ . \tag{48}$$

The result is

$$\langle \tilde{j}_i^\mu(\xi) \rangle = \frac{g^2}{4\pi} A_i^\mu(\xi) - ig \lim_{\varepsilon \to 0} \mathrm{tr}[\gamma^5 \gamma^\mu S\,(\xi + \varepsilon/2,\ \xi - \varepsilon/2)] \ . \tag{32'}$$

Next, we multiply Eq. (47) by $\lambda_k \gamma^5$ and take a double trace over spin and color indices while subjecting the coordinates to the conditions $x = \xi + \varepsilon/2$, $y = \xi - \varepsilon/2$, and $\varepsilon \to 0$. The equation thus obtained is

$$\begin{aligned}\lim_{\varepsilon\to 0} i\partial_\xi^\mu \operatorname{tr} [\lambda_k\gamma^5\gamma_\mu S(\xi + \varepsilon/2, \xi - \varepsilon/2)] \\ + g A_i^\mu(\xi) \lim_{\varepsilon\to 0} \operatorname{tr} \left\{[\lambda_k,\lambda_i]\gamma^5\gamma_\mu S(\xi + \varepsilon/2, \xi - \varepsilon/2)\right\} \\ + \frac{1}{2} g \varepsilon_\nu \frac{\partial A_i^\mu(\xi)}{\partial\xi_\nu} \lim_{\varepsilon\to 0} \operatorname{tr} \left\{(\lambda_k\lambda_i + \lambda_i\lambda_k)\gamma^5\gamma_\mu S(\xi+\varepsilon/2,\xi-\varepsilon/2)\right\} = 0 .\end{aligned} \tag{49}$$

As a consequence of Eq. (48), this equation reduces to

$$\begin{aligned}[\delta_{k\ell}\, \partial_\xi^\mu + g f_{ki\ell} A_i^\mu(\xi)](-ig) \lim_{\varepsilon\to 0} \operatorname{tr}[\lambda_\ell\gamma^5\gamma_\mu S(\xi+\varepsilon/2,\xi-\varepsilon/2)] \\ + \frac{g^2}{4\pi} \tilde{\partial}_\mu A_k^\mu(\xi) = 0 ,\end{aligned} \tag{50}$$

which, upon using (32'), reproduces Eq. (40). Equation (39) is obtained in an entirely analogous manner, and the corresponding divergences for the U(1) currents (in the limit e=0) are given by the free-field expressions

$$\partial_\mu \langle j^\mu \rangle = 0 , \quad \partial_\mu \langle \tilde{j}^\mu \rangle = 0 , \tag{51}$$

in confirmation of Eqs. (37) and (38).

IV. The Absence of Massive Color-Singlet Bound States

In this section we shall establish the non-existence of massive color-singlet bound states. To study these, we shall use the bound-state formalism referred to in the previous section. In particular, we rewrite the fermion-antifermion equation [Eq. (7) of Ref. 11 or 12] for the quark-antiquark system of the present theory. Using the definitions given in Eqs. (45), (46) and the explanatory remarks following them, we can write the equation as

$$[\not{n}^{(1)}\mathscr{L}_x^{+(1)} + \not{n}^{(2)}\mathscr{L}_y^{-(2)}]\ \chi(x,y) = g[\not{n}^{(1)}\lambda_i^{(1)}\not{F}_i^{(1)}(x) - \not{n}^{(2)}\lambda_i^{*(2)}\not{F}_i^{(2)}(y)]\ S(x,y)C\ , \tag{52}$$

where

$$F_i^\mu(x) = \frac{\delta}{\delta J_{i\mu}(x)}\ K(J)\ , \tag{53}$$

and

$$K(J) = \langle\, \mathrm{OUT}\,|\,q\bar{q},\mathrm{IN}\,\rangle / \langle\, \mathrm{OUT}\,|\,\mathrm{IN}\,\rangle\ . \tag{54}$$

Here $|q\bar{q},\mathrm{IN}\rangle$ stands for the bound state in question as an incoming state in the presence of the external source J_i^μ, and χ is the associated "wave function" defined by

$$\chi(x,y) = \langle\, \mathrm{OUT}\,|\,T[q(x)q^c(y)]\ Q\,|\,q\bar{q},\mathrm{IN}\,\rangle / \langle\, \mathrm{OUT}\,|\,\mathrm{IN}\,\rangle\ , \tag{55}$$

and

$$Q = 1 - |\mathrm{IN}\rangle\langle \mathrm{OUT}|\ /\langle\, \mathrm{OUT}\,|\,\mathrm{IN}\,\rangle\ , \tag{56}$$

where q^c is the charge-conjugate of q.

A useful identity[11] relates χ and the color-singlet current:

$$\lim_{\varepsilon\to 0}\ \mathrm{tr}\ [\gamma^\mu\chi(x+\varepsilon/2,\ x-\varepsilon/2)C] = -\langle\, \mathrm{OUT}\,|\,j^\mu(x)\,|\,q\bar{q},\mathrm{IN}\,\rangle / \langle\, \mathrm{OUT}\,|\,\mathrm{IN}\,\rangle\ . \tag{57}$$

The fact that, according to Eq. (38), j^μ is essentially a massless field implies that either the color-singlet states are massless, or that the left-hand side of (57) vanishes. It is therefore sufficient to pursue the latter possibility.

From the definition (55) and the construction of S in Section II, we can assemble χ in the form

$$\chi(x,y) = iKS(x,y)C + [\mathscr{T}_+(x)\Lambda^+ + \mathscr{T}_-(x)\Lambda^-]^{(1)}$$
$$\times\ [\mathscr{T}_+^*(y)\Lambda^+ + \mathscr{T}_-^*(y)\Lambda^-]\ \chi_0(x,y)\ , \tag{58}$$

where χ_0 satisfies the interaction-free equation

$$[\not{n}^{(1)}\not{\partial}_x^{(1)} + \not{n}^{(2)}\not{\partial}_y^{(2)}]\ \chi_0(x,y) = 0\ , \tag{59}$$

and can therefore depend only on x-y.

The simplicity of the γ-matrix algebra and the fact of masslessness again cause a decomposition of the space of wave functions (looked upon as matrices in spin indices) into two subspaces, one spanned by γ^μ and the other by $\gamma^\mu\gamma^\nu$ $(\mu,\nu = 0,1)$. The corresponding components of χ, to be designated by χ^{odd} and χ^{even} respectively, are

$$\chi^{odd}(x,y) = [\mathscr{T}_+^{(1)}(x)\mathscr{T}_-^{*(2)}(y)\Lambda^+ + \mathscr{T}_-^{(1)}(x)\mathscr{T}_+^{*(2)}(y)\Lambda^-]\ \chi_0^{\ odd}(x-y), \tag{60}$$

$$\chi^{even}(x,y) = iKS(x,y)C + [\mathscr{T}_+^{(1)}(x)\mathscr{T}_+^{*(2)}(y)\Lambda^+$$
$$+\mathscr{T}_-^{(1)}(x)\mathscr{T}_-^{*(2)}(y)\Lambda^-]\ \chi_0^{even}(x-y)\ , \tag{61}$$

where now ordinary matrix multiplication over spin indices is to be understood.

Let us consider χ^{odd} first. It is easy to see that it actually represents a mutually non-interacting pair by observing that $\mathscr{T}_+(\mathscr{T}_-)$ involves only $A_+(A_-)$, and that one of the latter may be taken to be zero as a choice of gauge. As for χ^{even}, we proceed by taking the trace of Eq. (61) over color indices in order to extract the color-singlet component of χ. In so doing, we utilize the fact that χ_0 must be proportional to the unit matrix in color indices. Hence

$$\chi^{\text{even}}_{\text{singlet}}(x,y) = iK\,\mathrm{tr}[S(x,y)]C + \Big\{\mathrm{tr}[\mathcal{T}_+(x)\mathcal{T}_+^{\dagger}(y)]\Lambda^+ + \mathrm{tr}[\mathcal{T}_-(x)\mathcal{T}_-^{\dagger}(y)]\Lambda^-\Big\}\,\chi^{\text{even}}_{0\,\text{singlet}}(x-y) \;, \tag{62}$$

where "†" denotes the adjoint matrix.

The mass of the state represented by (62) may be determined by examining the dependence of the wave function upon $x+y$. This dependence can be gauge-invariantly determined by considering the $x = y$ limit,[13] were it not for the possibility that χ_0 may be singular there. However, Eq. (57) and the remarks subsequent thereto assure us that in fact χ_0 vanishes in that limit since the factors

$$\lim_{x\to y} \mathrm{tr}[\mathcal{T}_+(x)\mathcal{T}_+^{\dagger}(y)] = \mathrm{tr}[\tau_+(x_+)\tau_+^{\dagger}(x_+)] \;, \tag{63}$$

$$\lim_{x\to y} \mathrm{tr}[\mathcal{T}_-(x)\mathcal{T}_-^{\dagger}(y)] = \mathrm{tr}[\tau_-(x_-)\tau_-^{\dagger}(x_-)] \;, \tag{64}$$

are positive-definite, and moreover,

$$\lim_{x\to y} \mathrm{tr}[S(x,y)] = 0 \;. \tag{65}$$

Note that in deducing (63) and (64), we have used the property that $T^{\dagger} = T^{-1}$. Finally since

$$\partial^2\, \mathrm{tr}[\tau_+(x_+)\tau_+^{\dagger}(x_+)] = \partial^2\, \mathrm{tr}[\tau_-(x_-)\tau_-^{\dagger}(x_-)] = 0 \;, \tag{66}$$

we are assured of the masslessness of the color-singlet bound states.

V. Bosonization and the Existence of Massive Colored States

The results of Section III suggest that the non-trivial part of the spectrum is to be found in the color sector. Moreover, with the induced quark color current given by (34), one can conveniently describe this sector in a bosonized form. To do this, we fix the gauge by adopting the

the light-cone condition $B_{i-} = 0$. The corresponding relations for the functional quantities are

$$\overline{B}_{i-} = 0, \quad A_{i-} = 0, \quad A_{i+} = \overline{B}_{i+} - 2i\,\delta/\delta J_{i-} \quad . \tag{67}$$

Furthermore, the field equations reduce to the constraint equation

$$\frac{1}{2}\partial_-^2 B_{i+} = j_{i-} + J_{i-} \ , \tag{68}$$

or, in functional terms,

$$\frac{1}{2}\partial_-^2 A_{i+} = \langle j_{i-} \rangle + J_{i-} \ . \tag{69}$$

Equation (34), on the other hand, implies the following pair:

$$\langle j_{i+} \rangle = \frac{g^2}{4\pi} A_{i+} \ , \tag{70}$$

$$\langle j_{i-} \rangle = \frac{ig}{\pi} \operatorname{tr}[\lambda_i \frac{\partial \mathscr{T}_+(x)}{\partial x_+} \mathscr{T}_+^{-1}(x)] \ . \tag{71}$$

Combining Eqs. (69) and (70), we arrive at

$$\frac{1}{2}\partial_-^2 A_{i+} = \frac{ig}{\pi} \operatorname{tr}[\lambda_i \frac{\partial \mathscr{T}_+(x)}{\partial x_+} \mathscr{T}_+^{-1}(x)] + J_{i-} \ , \tag{72}$$

which, in view of the relation

$$\overline{B}_{i+}(x) = -2i \frac{\delta}{\delta J_{i-}(x)} \ln \left(\langle \mathrm{OUT} | \mathrm{IN} \rangle\right) \ , \tag{73}$$

is equivalent to the following functional differential equation for the generating functional:

$$\left\{\partial_-^2 \frac{\delta}{\delta J_{i-}(x)} + \frac{g}{\pi} \operatorname{tr}[\lambda_i \frac{\partial \mathscr{T}_+(x)}{\partial x_+} \mathscr{T}_+^{-1}(x)] - iJ_{i-}(x)\right\} \langle \mathrm{OUT} | \mathrm{IN} \rangle = 0 \ , \tag{74}$$

where $A_{i+}(x)$ [which enters the definition of $\mathscr{T}_+(x)$ in Eq. (24)] is to be replaced by $-2i\,\delta/\delta J_{i-}(x)$ within the square brackets. Equation (74) is thus an equivalent boson formulation of the theory in that the vacuum expectation value of all the time-ordered products of the gauge potential may be obtained from the generating functional in the usual way.

It is actually possible to reduce the theory further to an equation similar to (74) but in terms of gauge-invariant quantities. This is most conveniently accomplished in terms of the field strength $\tilde{G}_i$, the pseudo-scalar dual to $B_{i\mu}$, which is (except for a global choice of gauge) gauge invariant and (in the light-cone gauge) simply related to the gauge potential by [cf. Eq. (33)]

$$\langle \tilde{G}_i \rangle = \frac{1}{2} \partial_- \bar{B}_{i+} \quad . \tag{75}$$

To realize a corresponding reformulation of (74), we introduce a new source function I_k defined by

$$J_{k-}(x) = - \partial_- I_k(x) \quad , \tag{76}$$

in terms of which we will have

$$\langle \tilde{G}_k(x) \rangle = - i\partial_- \frac{\delta}{\delta J_{k-}(x)} \ln (\langle \mathrm{OUT} | \mathrm{IN} \rangle) = - i \frac{\delta}{\delta I_k(x)} \ln (\langle \mathrm{OUT} | \mathrm{IN} \rangle) \tag{77}$$

Finally, Eq. (74) takes the form

$$\left\{ \partial_- \frac{\delta}{\delta I_k(x)} + \frac{g}{\pi} \operatorname{tr}[\lambda_k \frac{\partial \mathscr{T}_+(x)}{\partial x_+} \mathscr{T}_+^{-1}(x)] + i\partial_- I_k(x) \right\} \langle \mathrm{OUT} | \mathrm{IN} \rangle = 0 \quad . \tag{78}$$

For completeness, we record here the corresponding expression for T_+, in terms of which $\mathscr{T}_+$ is defined in Eqs. (24) and (27):

$$T_+(x) = T \left\{ \exp \left[g\lambda_k \int d^2x' \Theta(x_- - x'_-) \varepsilon (x_+ - x'_+) \frac{\delta}{\delta I_k(x')} \right] \right\} , \tag{79}$$

where the ordering is with respect to x'_-. Once the generating functional $\langle \mathrm{OUT} | \mathrm{IN} \rangle$ is obtained from (78) in terms of I, the various gauge-invariant Green's functions of the field strength $\tilde{G}$ may be calculated by means of functional differentiation according to (77).

Although an extensive simplification has been achieved in reducing the theory to the bosonized form (78), its exact solution is prohibited by the fact that the order of functional derivatives extends to infinity by virtue of the exponential terms. However, by considering the limiting values of the coupling constant g, we can infer the existence of massive colored states. To that end, it is convenient to consider the quark current divergence relations in the light-cone gauge. These may be obtained from Eqs. (39) and (40), or alternatively from (70) and (71); they lead to

$$(\delta_{ik}\partial_+ + gf_{ijk}A_{j+}) \langle j_{k-} \rangle = -\frac{g^2}{4\pi}\,\partial_- A_{i+} \;. \tag{80}$$

This equation, combined with the constraint equation (69), in turn leads to

$$(\delta_{ik}\partial_+ + gf_{ijk}A_{j+}) \left(\frac{1}{2}\partial_-^2 A_{k+} - J_{k-}\right) = -\frac{g^2}{4\pi}\,\partial_- A_{i+} \;. \tag{81}$$

Since g is the only dimensional parameter in this (finite) theory, we can deduce the infrared and ultraviolet behavior of the theory by considering its strong- and weak-coupling (i.e., $g \to \infty$, 0) limits respectively. Thus in the latter limit, the free-field result

$$\frac{1}{2}\,\partial_-^2 A_{k+} - J_{k-} = 0 \quad (g \to 0,\ \text{ultraviolet limit}) \;, \tag{82}$$

is obtained, which implies a bare propagator and asymptotic freedom. The infrared limit, on the other hand, is obtained by neglecting the linear term in g relative to the quadratic one; this gives

$$\frac{1}{2}\left(\partial^2 + \frac{g^2}{2\pi}\right)\partial_- A_{i+} = \partial_+ J_{i-} \quad (g \to \infty,\ \text{infrared limit}) \;. \tag{83}$$

This equation may be transcribed as one expressing the gauge-invariant Green's function for the field strength $\widetilde{G}$ according to (75) and (77):

$$-i\langle 0|T[\tilde{G}_i(x)\tilde{G}_j(y)]|0\rangle = \delta_{ij}\,\partial^2\Delta(x-y) \qquad (g|x-y|\to\infty)\ , \tag{84}$$

where Δ is the propagator for a boson of mass $g/\sqrt{2\pi}$. Equation (84) implies the existence of a (n^2-1)-fold degenerate family of massive, colored[2], pseudoscalar particles. Note that only the lightest of such states would be revealed in Eq. (84). Therefore if quarks were liberated with a mass less than half of the mass appearing above, the threshold of the diquark continuum would be revealed in Eq. (84). Massive quarks, on the other hand, would not in general be compatible with a free color-singlet current. Thus quark liberation at any mass is ruled out and quarks are confined.

VI. Concluding Remarks

The analysis of two-dimensional massless QCD with an arbitrary number of colors in this paper has characterized the theory as one whose essential properties are determined by a Schwinger-like mechanism. Whereas these properties are understandable in terms of the quark current operators and particularly in terms of the axial-vector anomaly as the source of symmetry breaking and mass generation, the essentially Abelian character of the underlying (broken) structure presents something of a puzzle. However, the observation that the large-scale properties of the theory, as seen in Eq. (81) for example, are determined by the anomaly term with the non-Abelian (i.e., involving f_{ijk}) contribution being relatively unimportant confirms the nature of the underlying structure. The small-scale properties of the theory, on the other hand, are determined by non-interacting quarks, thus making the non-Abelian nature of the coupling inconsequential for both small- and large-scale properties of the theory; hence the relevance of the Abelian analogy.

If one considers the above picture as the strong-coupling limit of massive SU(N) Yang-Mills theory in two dimensions, then indeed the 't Hooft model must be representative of a different phase, while the ususal weak-coupling limit (i.e., as given by a perturbation expansion in g/m with N considered finite and arbitrary) presumably corresponds to a third regime. The relevant lesson gained here is that the various limiting cases produced by the extreme values of the dimensionless parameters g/m and N do not commute.

Are there other states in the color sector higher in mass than the massive "gluons" of the preceding section? One would certainly expect there to be a (probably infinite) family of such states since, unlike the Abelian analog, these are <u>interacting</u> gluons. The latter fact is evidenced by the presence of a characteristically non-Abelian interaction term in the bosonized versions of Section V. The interesting feature encountered here (in contrast to the massive Schwinger model, for example) is that both the mass and the interaction terms are driven by the (original) coupling constant of the theory (e.g., see Eq. (81)).

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2. Strictly speaking, color symmetry is broken (or color "bleached") in this theory and one should more appropriately refer to these particles as objects belonging to the (N^2-1)-dimensional multiplet of SU(N). (The author is grateful to M. Weinstein for pointing this out to him.) However for want of a better word, we shall continue to refer to them as colored objects.
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