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SINGULAR CLASSICAL SOLUTIONS OF EUCLIDEAN FIELD THEORIES*

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ABSTRACT

The classical solutions of the equations of motion are studied in some Euclidean field theory models either conformal invariant or non-invariant. In the conformal invariant models a virial theorem for merons is derived which is of the same form as the known one for instantons. Some examples of singular solutions are discussed. An interesting relation seems to hold between the local symmetry properties of the singular solutions and the degree of divergence of the Euclidean action.

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I. Introduction

Classical solutions of equation of motion in quantum field models may help to understand the non-perturbative quantum properties. Although the emphasis was on regular and sourceless solutions with finite energy (solitons) or Euclidean action (instantons), also solutions with logarithmic divergent action (merons) were considered interesting in the approximate evaluation of the Feynman path integral and solutions of the gauge fields in presence of sources in studying the dynamics of heavy quarks. Then it seems desirable to understand the properties of the general classical solution in the Euclidean space.[1]

In Section II we derive virial type theorems[2] for meron-like solutions. As in the case of the instantons the theorem is useful in excluding solutions for some models in certain space-time dimensions or in finding exact or approximate solutions when there are. Virial theorems which may have a similar use are also easily derived for higher moments of the energy momentum tensor. We discuss the properties of meron solutions in two models: a scalar selfinteracting multiplet and the CP^{N-1} model. In Section III we indicate that some of the meron properties are lost by the singular solutions of non-conformal models. This is illustrated by two examples in $\mathbb{R}^3$: a scalar selfinteracting massless multiplet and a similar multiplet interacting with a Yang-Mills field.

Finally let us remark that there seems to be a connection between the symmetry properties of classical solution and the degree of divergence of its Euclidean action. It is well known[9,10] that in the conformal ϕ^4 model in the Euclidean $\mathbb{R}^4$, the only solution invariant under $O(5)$, the instanton, has a finite action, while the meron solution, invariant under $O(4) \times O(2)$ only, has a logarithmically divergent action.

Moreover the general solution with O(4) symmetry only[11] which is computed in terms of Jacobi elliptic functions has an action at least logarithmically divergent. Similar features are observed in the pure Yang-Mills theory by using the ansatz[12] that maps solutions of the conformal ϕ^4 model into solutions of the Yang-Mills theory. Very few solutions are known[13] which have less than O(4) invariance and they have an action divergent as a power of the cut off.

It may be useful to stress that the symmetry property of the classical solutions, we are discussing, is a local symmetry property of the solution in a region where the physical densities are substantially different from zero and we do not refer to the global symmetry property which may be less significant. For instance, by use of the conformal symmetry a meron-meron pair solution that has one center at origin and the other at infinity (and therefore O(4) × O(2) global symmetry) may be converted into a solution with both centers at finite points thus reducing the global symmetry property of the solution, but not its local properties around the centers.

II. Meron Solutions in Conformal Models

Let us recall the Laue theorem[5,6] in its simplest form useful for regular time-independent solutions (static solitons and instantons). Let $\mathscr{L}[\phi,x]$ denote the Lagrangian of a set of fields ϕ in the Euclidean $\mathbb{R}^n$ space and $T_{\mu\nu}[\phi,x]$ the energy momentum tensor (it may be the canonical, Belinfante or the "improved" one). The energy momentum conservation:

$$\partial_\mu T_{\mu\nu}[\phi,x] = 0 \tag{2.1}$$

implies

$$\partial_\alpha (x_\mu T_{\alpha\nu}) = T_{\mu\nu} \tag{2.2}$$

If (2.2) is integrated over all the Euclidean $\mathbb{R}^n$ space and if the regularity and asympototic properties of the fields allow neglecting the surface contribution of the current $J^{\alpha\mu\nu} \equiv x^\mu T^{\alpha\nu}$, one obtains:

$$\int d^n x \; T_{\mu\nu} \; [\phi,x] = 0 \qquad (\mu,\nu = 1, \ldots, n) \; . \tag{2.3}$$

The usual form of the virial theorem[2] follows by taking the trace of Eq. (2.3)

$$\int d^n x \; T_{\mu\mu} \; [\phi,x] = 0 \tag{2.4}$$

and the more detailed theorems obtained by non-isotropic scale transformations[4] correspond to linear combinations of the Eq. (2.3).

Of course one may easily obtain infinitely many other relations of the virial type by partial integration:

$$\int d^n x \; T_{\mu\nu} \; \partial_\mu f_\alpha \; (x_1, \ldots, x_n) = 0 \tag{2.5}$$

provided that $f_\alpha(x_1, \ldots, x_n)$ is a set of smooth functions and the integrals converge. Equation (2.5) contains both the local property (2.1) and the asymptotic properties of the energy-momentum density tensor.

If the set f_α is chosen to be a complete set of orthonormal functions in $\mathbb{R}^n$, the Eqs. (2.5) are well defined and fully equivalent to Eq. (2.1). Sometimes the first few equations are helpful in the search for approximate solutions which do not have simple symmetry properties.[14]

We shall now extend this theorem to the singular solutions often called merons. Such solutions have only been studied in some conformal models: nonlinear σ-models in $\mathbb{R}^2$,[8] massless ϕ^4 in $\mathbb{R}^4$,[10] pure Yang-Mills in $\mathbb{R}^4$,[10,15] Yang-Mills coupled with scalar multiplet in $\mathbb{R}^4$,[16] massless scalars interacting with massless fermions in $\mathbb{R}^4$.[17] Multimeron solutions in these models have also been studied.[18]

Most of the single meron pair solutions here mentioned have the following properties:

i) the Lagrangian density is regular and has a finite integral over any region of the Euclidean space-time excluding the neighbourhoods of two points which we call centers of the meron pair. These two regions give logarithmically divergent contributions;

ii) in non-Abelian gauge models the gauge field is proportional to a pure gauge; in some models the topological charge density is concentrated at the centers of the meron pair;

iii) the meron solutions have a large group of symmetry. For instance, in models in Euclidean $\mathbb{R}^4$ space invariant under the conformal group O(5,1) (generated by $M_{\mu\nu}$, P_μ, K_μ and D) while the instanton solutions are invariant under the subgroup O(5) generated by $M_{\mu\nu}$ and $R_\mu = P_\mu + K_\mu$, the meron solutions are invariant under the subgroup generated by $M_{\mu\nu}$ and D.

Yet no single such property holds for all the above mentioned solutions (in fact no general characterization of meron solutions seems to exist).

Property ii) is probably the less appropriate not only because one may be interested in classical solutions (regular or otherwise) in models that have no topological number but mainly because it was shown that for a singular solution the topological number may be changed (by a singular gauge transformation) without affecting the type of singular behaviour of the solution[15] (this does not happen for regular solutions). One may also notice that property iii) and i) seem to be closely related. We then adopt a definition based on the property i) and furthermore we require a specific <u>local</u> behaviour of any symmetric traceless energy-momentum tensor $T^{\mu\nu}$, in the regions where the solution is singular. Such local behaviour is suggested by properties i) and iii) and indeed it holds for all known solutions where i) and (or) iii) hold. However it is more convenient than the above properties in looking for exact or approximate solutions in less simple cases (for instance multi-merons). Specifically we shall here consider the sourceless solutions of conformal field theories in the Euclidean $\mathbb{R}^n$ space everywhere regular except for two points, the centers of the meron pair. In the neighbourhoods of each center, say $x_\mu \simeq a_\mu$, we assume that the energy-momentum tensor is:

$$T_{\mu\nu}[\phi,x] \simeq \left[\delta_{\mu\nu} - n \frac{(x_\mu - a_\mu)(x_\nu - a_\nu)}{(x_\alpha - a_\alpha)^2} \right] f\left((x_\alpha - a_\alpha)^2\right) \quad . \tag{2.6}$$

The conservation of the energy-momentum tensor implies that

$$f\left((x_\alpha - a_\alpha)^2\right) = \frac{c_a}{\left[(x_\alpha - a_\alpha)^2\right]^{n/2}} \tag{2.7}$$

c_a being a constant.

One might check for example that the solutions of the conformally invariant ϕ^4 in $\mathbb{R}^4$ which have O(4) symmetry[11] (but not a higher one) do not allow the local representation [(2.6) and (2.7)] even in the cases where the action is minimally (i.e. logarithmically) divergent.

By using the conformal symmetry one of the centers of the action density can be shifted to the origin and the second to infinity. Then it is plausible that the assumed local behaviour (2.6) holds globally:

$$T_{\mu\nu}\ [\phi,x] = \left(\delta_{\mu\nu} - n \frac{x_\mu x_\nu}{r^2}\right) \frac{c}{r^n} \quad . \tag{2.8}$$

Indeed, Eq. (2.8) holds for every known meron solution but it will not be needed in the derivation of the virial theorem.

We shall now easily prove the virial theorem for merons. Let us call a_μ and b_μ the centers of the meron pair and σ_a, σ_b two small spheres with centers in a_μ and b_μ. By integrating Eq. (2.2) over all $\mathbb{R}^n$ space and using the divergence theorem we have

$$\int d^n x \; T_{\mu\nu} = \int_{S_\infty} d\sigma_\alpha \; J_{\alpha\mu\nu} + \int_{\sigma_a} d\sigma_\alpha \; J_{\alpha\mu\nu} + \int_{\sigma_b} d\sigma_\alpha \; J_{\alpha\mu\nu} \quad . \tag{2.9}$$

The integral over the surface at infinity S_∞ vanishes because of the meron property that the action density is integrable over any domain that excludes the centers of the meron pair. By using (2.6) and (2.7) we obtain

$$\begin{aligned} \int d^n x \; T_{\mu\nu} &= (1-n) \; c_a \int d\Omega_n \frac{y_\nu \, (y_\mu - a_\mu)}{y^2} \\ &\quad + (1-n) \; c_b \int d\Omega_n \frac{(y_\mu - b_\mu) \; y_\nu}{y^2} \\ &= \frac{(1-n)}{n} \, \Omega_n \delta_{\mu\nu} \; (c_a + c_b) \end{aligned} \tag{2.10}$$

where y_μ is the coordinate with respect to one of the centers of the meron pair and Ω_n is the total solid angle in $\mathbb{R}^n$. Now by taking the trace of Eq. (2.10), conformal symmetry implies $c_a = -c_b$, therefore:

$$\int d^n x \; T_{\mu\nu} \; [\phi, x] = 0 \qquad (\mu,\nu = 1, \; \ldots, \; n) \tag{2.11}$$

The virial theorem for merons then has the same form as for instantons. Of course it will not be useful here to consider the trace relation, as in Eq. (2.4), because by conformal symmetry this is trivial identity.

As an example of use of this theorem, let us consider a set of massless scalar fields ϕ_a (a = 1, ..., 4), which transforms as a vector under the internal symmetry group O(4) in the Euclidean $\mathbb{R}^4$ space with the Lagrangian

$$\mathscr{L} = \frac{1}{2}\,\partial_\mu\phi_a\,\partial_\mu\phi^a + \frac{\lambda}{4}\,(\phi_a\phi^a)^2 \tag{2.12}$$

The proper energy-momentum tensor here is the improved one[19]

$$T_{\mu\nu} = \partial_\mu\phi_a\,\partial_\nu\phi^a - \delta_{\mu\nu}\mathscr{L} + \frac{1}{6}\,(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)(\phi_a\phi^a) \tag{2.13}$$

because we need $T_{\mu\mu} = 0$, but the improvement gives no contribution after integration. Then Eq. (2.11) yields

$$\begin{aligned}\int d^4x\,(\partial_x\phi_a)^2 &= \int d^4x\,(\partial_y\phi_a)^2 = \int d^4x\,(\partial_z\phi_a)^2 \\ &= \int d^4x\,(\partial_t\phi_a)^2 = -\frac{\lambda}{4}\int d^4x\,(\phi_a\phi^a)^2\end{aligned} \tag{2.14}$$

and

$$\int d^4x\,\partial_\mu\phi_a\,\partial_\nu\phi^a = 0 \qquad (\mu \neq \nu)\quad . \tag{2.15}$$

Hence meron solutions may only exist for negative λ. If we look for a meron solution with a center in the origin and the other at infinity, the global assumption (2.8) yields

$$\left(\delta_{\mu\nu} - \frac{4x_\mu x_\nu}{r^2}\right)\frac{c}{r^4} = \partial_\mu\phi_a\,\partial_\nu\phi^a - \partial_{\mu\nu}\mathscr{L} + \frac{1}{6}\,(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu)(\phi_a\phi^a)^2 \tag{2.16}$$

Analogous equations may be obtained for Yang-Mills systems and they are of first order.

Two simple (sourceless) solutions are obtained from (2.12) and (2.16)

$$-\phi_a = \frac{\delta_{a1}}{\sqrt{-\lambda}}\,\frac{1}{r} \qquad (a = 1, \ldots, 4) \tag{2.17}$$

which is well known[10], and

$$\phi_a = \pm\frac{2}{\sqrt{-\lambda}}\,\frac{x_a}{r^2} \qquad (a = 1, \ldots, 4) \tag{2.18}$$

The topological charge of a Higgs multiplet ϕ_a is usually calculated from the Kronecker index of the normalized vectorial field $\hat{\phi}_a = \dfrac{\phi_a}{(\phi_b\phi^b)^{1/2}}$.

This however requires regularity properties for the modulus $(\phi_a \phi^a)^{1/2}$ which do not hold in the solution (2.18). Every regularization of (2.18) would produce a unit topological charge.

The meron solution (2.17) is invariant under a O(4) × O(2) group, where O(4) is generated by the four-dimensional rotation operators and O(2) is generated by $D = x_\mu \partial_\mu + 1$. In the case of meron solution (2.18), since space and internal indices are mixed, the space-time rotations have to be supplemented by similar transformations in the internal space, so that (2.18) is still invariant under a O(4) × O(2) group where now O(4) corresponds to the "complete" (i.e. space plus internal) four-dimensional rotations and O(2) is generated by $D = x_\mu \partial_\mu + 1$. As another example of meron solution in conformal models, we consider the two-dimensional Euclidean CP^{N-1} non-linear σ-models[20] with Lagrangian density

$$\mathscr{L} = \partial_\mu \bar{z}_\alpha \, \partial_\mu z^\alpha + (\bar{z}_\alpha \partial_\mu z^\alpha)^2 \tag{2.19}$$

where $z^\alpha(x)$ $(\alpha = 1, \ldots, N)$ is a complex N-component field satisfying the constraint $\bar{z}_\alpha z^\alpha = 1$.

A meron solution which is a simple generalization of that of the O(3) σ-model[8] is

$$z_\alpha(x) = \frac{1}{\sqrt{2}} \left(e^{-i\frac{\theta}{2}} u_\alpha + e^{i\frac{\theta}{2}} v_\alpha \right) \tag{2.20}$$

where $\theta = \arg(x_1 + ix_2)$ and $\bar{u}_\alpha u^\alpha = \bar{v}_\alpha v^\alpha = 1$, $\bar{u}_\alpha v^\alpha = 0$. It has a topological charge density

$$q(x) \equiv \frac{1}{2\pi i} \partial_\mu (\bar{z}_\alpha \varepsilon_{\mu\nu} \partial_\nu z^\alpha) = \frac{1}{2} \delta^2(x) \quad ; \tag{2.21}$$

the canonical energy-momentum tensor is

$$T_{\mu\nu} = \left(\delta_{\mu\nu} - 2 \frac{x_\mu x_\nu}{x^2} \right) \frac{1}{4x^2} \tag{2.22}$$

and the Lagrangian density is

$$\mathscr{L}(x) = \frac{1}{2x^2} \quad . \tag{2.23}$$

III. Singular Solutions in Non-Conformal Models

It is clear that the singular solutions of equations of motion of non-conformal models lack most of the properties of meron solutions in conformal models. However, non-conformal models also are interesting and the singular solutions that we describe in this section are of obvious relevance in the study of the general solution of the classical equations of motion.

First we consider a scalar massless multiplet ϕ_a ($a = 1, \ldots, n$) which transform as a vector under the internal symmetry group O(n) in the Euclidean $\mathbb{R}^n$ space described by the Lagrangian density

$$\mathscr{L} = \frac{1}{2} \partial_\mu \phi_a \, \partial_\mu \phi^a + \frac{\lambda}{4} (\phi_a \phi^a)^2 \tag{3.1}$$

With a radial ansatz $\phi_a = \frac{x_a}{r} f(r)$, $r = (x_\mu x_\mu)^{1/2}$, one has the equation of motion

$$\frac{d^2 f}{dr^2} + \frac{(n-1)}{r} \frac{df}{dr} - \frac{(n-1)}{r^2} f - \lambda f^3 = 0 \tag{3.2}$$

By a change of variables $g = rf$, $r = e^z$ one obtains the autonomous equation

$$\frac{d^2 g}{dz^2} + (n-4) \frac{dg}{dz} - 2(n-2) g - \lambda g^3 = 0 \tag{3.3}$$

The associated autonomous system of first order differential equations

$$\frac{dg}{dz} = y$$

$$\frac{dy}{dz} = (4-n)\,y + 2\,(n-2)\,g + \lambda g^3 \tag{3.4}$$

has three singular points $(\frac{dg}{dz} = \frac{dy}{dz} = 0)$:

$$g = 0 \ , \qquad g = \pm\sqrt{\frac{2(2-n)}{\lambda}} \tag{3.5}$$

which yield the singular solution

$$\phi_a = \pm\sqrt{\frac{2(2-n)}{\lambda}}\ \frac{x_a}{r^2} \tag{3.6}$$

For n = 4 the model is conformal invariant and the singular solution is the meron already mentioned in the last section. For n = 3 Eq. (3.3) is essentially the equation discussed by Wu and Yang[21] that describes static solutions of pure Yang-Mills system in $\mathbb{R}^3$. We note some obvious properties of the solution (3.6) in $\mathbb{R}^3$:

a) it is invariant under the group O(3) × O(2) generated by the usual "complete" operator $M_{\mu\nu}$ and $\tilde{D}$ (where $\tilde{D} = x_\mu \partial_\mu + 1$ while the canonical dimension of ϕ_a in $\mathbb{R}^3$ is 1/2);

b) the density of the canonical energy-momentum tensor vanishes;

c) The Euclidean Lagrangian density is regular everywhere in $\mathbb{R}^3$ except at the origin; it diverges linearly when integrated in a domain that includes the origin;

d) the same comment made on the topological charge of the solution (2.18) holds here.

As a second example of singular solution for a non-conformal theory, we consider the SU(2) Yang-Mills field coupled with an SU(2) Higgs field in the Euclidean $\mathbb{R}^3$ space. For convenience, as in the previous example, the scalar multiplet is taken massless. With the usual radial ansatz:

$$A_i^a = \varepsilon_{aij}\, x_j\, \frac{(1-K(r))}{er^2}$$
$$\phi_a = x_a \frac{H(r)}{er^2} \tag{3.7}$$

the Euclidean action is

$$A = -\frac{4\pi}{e^2}\int_0^\infty dr \left[(K')^2 + \frac{(K^2-1)^2}{2r^2} + \frac{H^2K^2}{r^2} + \frac{(rH'-H)^2}{2r^2} + \frac{\lambda H^4}{4e^2r^2}\right] \tag{3.8}$$

and the field equations are

$$r^2K'' = K\,(K^2-1) + K\,H^2$$
$$r^2H'' = 2\,H\,K^2 + \frac{\lambda}{e^2}H^3 \tag{3.9}$$

After the change of variable $r = e^z$ we have the autonomous system

$$\frac{dK}{dz} = Y$$
$$\frac{dY}{dz} = K\,(K^2-1) + K\,H^2$$
$$\frac{dH}{dz} = W \tag{3.10}$$
$$\frac{dW}{dz} = 2\,H\,K^2 + \frac{\lambda}{e^2}H^3 \quad .$$

There are two sets of singular points $\left(\frac{dK}{dz} = \frac{dY}{dz} = \frac{dH}{dz} = \frac{dW}{dz}\right) = 0$

$$H = 0, \qquad K = 0, \ \pm 1 \tag{3.11}$$

$$H^2 = \frac{2e^2}{2e^2-\lambda}, \qquad K^2 = \frac{-\lambda}{2e^2-\lambda} \tag{3.12}$$

The first set implies the vanishing of the Higgs fields and then yield the singular points of the self coupled Yang-Mills system discussed by Wu and Yang[21] (we just recall that $K^2 = 1$ implies $F^a_{ij} = 0$, while $K = 0$ provides the singular solution

$$H^a_\ell \equiv \frac{1}{2}\,\varepsilon_{ij\ell}\,F^a_{ij} = -\,\frac{x_a x_\ell}{er^4}\quad). \tag{3.13}$$

The second set provides, for negative λ, singular solutions that have interesting properties:

a) the non-Abelian gauge field is proportional to a pure gauge:

$$A_i \equiv e\,\frac{A^a_i\,\sigma_a}{2i} = \frac{1-K}{2}\,U^{-1}\,\partial_i\,U, \quad \text{where } U = i\,\frac{\sigma_K x_k}{r}\ ;$$

b) the solution is invariant under the group $O(3) \times O(2)$ generated by the "complete" rotation operators and by $\tilde{D} = x_K\,\partial_K + 1$;

c) the density of the symmetric (Belinfante) energy-momentum tensor vanishes;

d) the Euclidean Lagrangian density is everywhere regular except at the origin; it diverges linearly if it is integrated in a domain that includes the origin;

e) the same comment made on the topological charge of the solution (2.18) holds here;

f) this solution, like the solution of the previous example, is closely related to the Wu-Yang solutions[21] of static Yang-Mills fields and has the same source problem: it is easily seen that the radial equations have a source proportional to $\delta^3(r)$ and therefore the solution (3.12) actually solves the equation of motion of the field A^a_i in presence of a source proportional to $J_i^a = \epsilon_{aij} \frac{x_j}{r} \delta^3(r)$ which is obviously ill-defined[22].

One may remark that by taking K = 1, H ≠ 0 the gauge fields A_i^a vanish so the previous example is recovered.

As a last example we consider the same system allowing for massive Higgs fields, restricting to the simpler configuration with K = 0. Then the only equation is

$$r^2 H'' = \frac{\lambda}{e^2} H^3 - \mu^2 H r^2 \qquad (3.14)$$

where μ is the Higgs mass.

The power-like solution $H(r) = \pm \frac{e\mu}{\sqrt{\lambda}} r$ yields $\phi_a(x)$ with spherical and scale symmetry (although with the non-canonical scale dimension). Now the Euclidean action is divergent both at the origin and at infinity.

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