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WEAK-STRONG INSTABILITY AS A DIFFUSION PROCESS*

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ABSTRACT

The phenomenological analysis of the weak-strong instability for an electron storage ring is developed. The vertical size of the weak beam is found to depend on two machine parameters: $\sqrt{\eta}$, which is proportional to ΔQ, and b, which depends on the aspect ratio of the strong beam. The model also contains one fitting parameter.

Experimental consequences of such dependence are discussed.

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1. INTRODUCTION

A complete analytical solution of the problem of the beam-beam interaction is hardly possible. The difficulty is mainly due to the nonlinear character of the forces involved.

At the same time, this strong nonlinearity leads to fast mixing of the particles within the bunch. Particle coordinates in phase space become erratic and in the long run each given particle can be expected to appear at any point of the space occupied by a bunch. The behavior of the particle does not depend on its initial coordinates. The particle motion resembles a random walk rather more than movement along a trajectory.

Under these conditions, we can try to find the volume of the phase space occupied by the ensemble of particles by considering the forces due to beam-beam interaction as a diffusion force (in addition to the diffusion forces which exist in their motion due to all other reasons such as quantum excitation, residual gas scattering, intrabunch scattering and so forth).

It is extremely important in this approach to deal only with the nonlinear part of the force, since the linear part does not lead to stochasticity of the motion. The linear part can be included in the machine structure, thus giving only the changes in the closed orbit and tune shift of the particle oscillations. By no means does the linear force increase the phase volume (the Liouville theorem).

Moreover, it is not quite clear how strong the nonlinearity should be to change the character of the motion to being purely stochastic.

The only criterion now existing is the Chirikov criterion[1] which is very difficult to formulate quantitatively in the presence of many resonances acting simultaneously. This uncertainty and the more or less arbitrary procedure of subtracting the linear part of the force makes this analysis of a phenomenological type, requiring the introduction of a fitting constant. It is not quite clear yet if and how this constant can be expressed through physical parameters of the storage ring.

We restrict ourselves to the one dimensional case of a vertical motion of a particle in an electron storage ring. Further, we consider the weak-strong instability, thus assuming the particle distribution function of the strong bunch to be unaffected by beam-beam interaction. The action of the strong bunch on a probe particle in the weak bunch is approximated by a nonlinear "kick".

In Section 2 we derive an approximate equation for the distribution function. In Section 3 the beam blow up is estimated for the case of a Gaussian strong bunch. In Section 4 we discuss experimental consequences suggested by this analysis and make the comparison with an experiment.

2. EQUATION FOR THE DISTRIBUTION FUNCTION

Let Y and $\dot{Y} = dY/dt$ be the excursion from the median plane and for corresponding velocity of a particle of the weak bunch at the interaction point. It is convenient to consider particle coordinates in units of the vertical size Σ of the strong bunch

$$y = Y/\Sigma \quad , \tag{1}$$

$$\dot{y} = \dot{Y} / \Sigma \quad . \tag{2}$$

If the length of the strong bunch is much less than the wave length of the vertical oscillations then by one passage through the strong bunch the coordinates of the particle are changed by:

$$\Delta y = 0 \quad , \tag{3}$$

$$\Delta \dot{y} = F(y) \quad . \tag{4}$$

The actual dependence $F(y)$ can be found at least in principle for any given particle distribution of the strong beam. Taking this change into account we can write the following equation for the particle distribution function $f(t, y, \dot{y})$ of the weak bunch:

$$\frac{\delta f}{\delta t} + \dot{y}\frac{\delta f}{\delta y} - 2\alpha\frac{\delta}{\delta \dot{y}}(\dot{y} f) - \omega_0^2 Q^2 y \frac{\delta f}{\delta \dot{y}}$$

$$= q_0 \frac{\delta^2 f}{\delta \dot{y}^2} + \sum_k \delta(t - t_k)\left\{ f[t_k, y, \dot{y} + F(y)] - f(t_k, y, \dot{y}) \right\} \quad . \tag{5}$$

The left hand side of this equation describes the change of the function f due to particle oscillations with a frequency $\omega_0 Q$ and a damping rate α. The right hand side represents the change of particle density in the phase space $(y, \dot{y})$ due to all possible reasons but beam-beam interaction (the first term) and due to the interaction occurring at the times $t_k = t_0 + 2\pi k / \omega_0 n$, $k = 1, 2, \ldots$, (the second term, n, is the number of interactions on one revolution).

If we are not interested in details of the fast time variations of the distribution function which are of the order of magnitude of one revolution period or less) then the sum on the right hand side of (5) can be simplified:

$$\sum_k \delta(t-t_k)\left\{f[t_k, y, \dot{y}+F(y)] - f(t_k, y, \dot{y})\right\}$$

$$\simeq \frac{n\omega_0}{2\pi}\left\{f[t, y, \dot{y}+F(y)] - f(t, y, \dot{y})\right\} \quad . \tag{6}$$

We can further expand the difference in (6) into series in $F(y)$. We are ready now to use the feature of fast particle mixing discussed above. The application of this idea means that we can substitute the coefficients $F(y)$ and $F^2(y)$ by the values obtained from averaging them over an ensemble of the particles of the weak bunch.

The velocity jumps, of magnitude $\Delta\dot{y}$, occur in all phases of betatron oscillations. Hence the averaging over any distribution symmetric in y makes the first coefficient vanish.

Let us define the diffusion coefficient due to interaction as

$$q_{int} = \frac{n\omega_0}{4\pi} < F^2(y) > \quad , \tag{7}$$

where brackets stand for averaging the function $F^2(y)$ over the distribution function $f(t, y, \dot{y})$, which satisfies the following Focker-Planck equation:

$$\frac{\delta f}{\delta t} + \dot{y}\frac{\delta f}{\delta y} - 2\alpha\frac{\delta}{\delta\dot{y}}(\dot{y}f) - \omega^2 Q^2 y\frac{\delta f}{\delta\dot{y}} = \left(q_0 + q_{int}\right)\frac{\delta^2 f}{\delta\dot{y}^2} \quad . \tag{8}$$

The solution of this equation is well known.[2] In the limit as $t \to \infty$ (stationary solution) it is a Gaussian function in both variables y and $\dot{y}$.

3. THE EVALUATION OF THE BEAM BLOW UP

The dispersion σ of the distribution function in y for the stationary solution can be found from the equation:

$$\sigma^2 = \left(q_0 + q_{int}\right) / 2\alpha \qquad (9)$$

or

$$\sigma^2 = \Sigma^2 + q_{int} / 2\alpha \qquad (10)$$

by definition of Σ. Since q_{int} is an integral over the distribution function with the same dispersion σ, (10) is a transcendental equation for σ.

We solve now this equation assuming that the distribution function of the strong bunch is Gaussian in all three dimensions. In this case the function $F(y)$ from (4) is known to be[3]:

$$F(y) = \xi \phi_b(y) \quad , \qquad (11)$$

where

$$\xi = 2\pi c \Delta Q[(\sqrt{1+b^2} + b)/(\sqrt{1+b^2} - b)]^{\frac{1}{2}} / \beta \quad , \qquad (12)$$

$$\phi_b(y) = y \int_0^1 \frac{du}{\sqrt{u + b^2}} \exp(-uy^2) \qquad (13)$$

In these expressions

$$\Delta Q = e^2 N\beta \Big/ \left(2 E (\Sigma_h + \Sigma) \Sigma\right) \qquad (14)$$

is the Courant parameter giving the linear tune shift of the vertical betatron oscillations of the weak beam particle due to the electromagnetic interaction with the strong bunch containing N particles.[4] E is the particle energy and β is the value of betafunction at the interaction point. Σ_h and Σ are the dispersions of the strong bunch distribution in horizontal and vertical planes.

Parameter b is defined as follows:

$$b = \left(\Sigma/\Sigma_h\right)\Big/\sqrt{1-(\Sigma/\Sigma\ )^2} \quad . \tag{15}$$

For a small aspect ratio of the beam,

$$b \simeq \Sigma/\Sigma_h \quad .$$

The function $\phi_b(y)$ describes y-dependence of the force acting on a particle from the side of the strong beam. For small values of y

$$\phi_b^{\ell}(y) \simeq 2\left(\sqrt{1+b^2}-b\right) y \tag{16}$$

gives the linear part of the force.

Let us describe the blow up of the weak beam by a ratio $d = \sigma/\Sigma$. Equation (10) can now be rewritten as

$$d^2 = 1 + \eta\Phi_b(d) \quad , \tag{17}$$

where η and Φ are defined by the following expressions:

$$\eta = \frac{n\omega_0}{4\pi\alpha}(2\pi\Delta Q)^2 \quad , \tag{18}$$

$$\Phi_b(d) = \frac{(\sqrt{b^2+1}+b)}{\sqrt{\pi}\, d(\sqrt{b^2+1}-b)} \int_{-\infty}^{\infty} \tilde{\phi}^2(y)\, \ell^{-y^2/d^2}\, dy \quad . \tag{19}$$

If $\Delta Q \to 0$, so that $\eta \to 0$, then $d = 1$, i.e., we have an unperturbed beam.

The function $\tilde{\phi}_b$ in the integrand of (19) is related to the function ϕ_b in (11) by some reduction procedure.

The simplest possible reduction would be a subtraction from $\phi_b(y)$ of its linear part (16). Such a procedure seems to be unsatisfactory since large values of y yield unacceptably large values in (16). The introduction of a cut-off factor also does not give a reasonable description of the beam blow up.

The only reasonably good results were obtained by the following reduction, depending on an arbitrary constant, h:

$$\tilde{\phi}_b(y) = \phi_b(y) - (1-h)\ \phi_b\left(\frac{y}{1-h}\right) , \quad 0 < h < 1 \quad . \tag{20}$$

This expression does not contain the linear term for any value of h.

Figs. 1 - 3 represent the results of the solution of equation (17) for different values of the parameters h and b. The value of d is plotted as a function of the variable $\sqrt{\eta}$ which is proportional to ΔQ or, in different units, to the current of the strong beam.

The calculation of the function $\Phi_b(d)$ can be found in the appendix.

4. CONCLUSION

The analysis suggested here has the major drawback of using an arbitrary and poorly understood reduction procedure (20). This introduced into the analysis a fitting parameter h, the magnitude of which should be chosen in such a way that instability occurs at the correct value of the strong beam current or the Courant parameter ΔQ.

Fig. 4 illustrates this by comparing the vertical size of the weak beam measured[5] on SPEAR with a calculated curve fitted with the help of parameter h . The actual value of h for this case happens to be 0.04.

We can also draw some conclusions which can be checked by experiment. First of all, we see that the beam blow up depends on the variable $\sqrt{\eta}$ rather than on ΔQ, itself. Further, the model suggests a certain dependence of the instability threshold both on the number of the interaction points and on the energy of the particle. Of course one should remember that this holds only if the fitting parameter h does not depend on either of these variables. And finally, for the quantitative description of the beam blow up, one needs also to take into account the aspect ratio b of the strong bunch.

The analysis can be extended to the much more complicated case of the strong - strong interaction.

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APPENDIX

We calculate here the function $\Phi_b(d)$ which enters equation (17). Let us define $\phi_b = y\,I(y)$. (See Equation (13).)

$$I(y) = \int_0^1 \frac{du}{\sqrt{1+b^2}} \exp(-u y^2) . \tag{A1}$$

Using the reduction procedure (20) we get for Φ_b:

$$\Phi_b(d) = \frac{1}{d}\frac{(\sqrt{1+b^2}+b)}{(\sqrt{1+b^2}-b)}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dy e^{-y^2/d^2} y^2 \left[I^2(y)+I^2\left(\frac{y}{1-h}\right)-2I(y)I\left(\frac{y}{1-h}\right)\right] \tag{A2}$$

We change now the order of integration in (A2). Performing the integration over y we get:

$$\Phi_b(d) = \frac{1}{d}\frac{(\sqrt{1+b^2}+b)}{(\sqrt{1+b^2}-b)}\int_0^1 \frac{du}{\sqrt{u+b^2}}\left[\rho_{00}(u) + (1-h)^3\rho_{10}(u) - 2(1-h)^3\rho_{11}(u)\right], \tag{A3}$$

where

$$\rho_{im}(u) = \frac{1}{2}\int_0^1 \frac{dv}{\sqrt{v+b^2}\left(v+uh_m+h_i/d^2\right)^{3/2}} \tag{A4}$$

with

$$h_m = (1-mh)^2, \qquad m = 0,1 . \tag{A5}$$

It is easy to evaluate the integral in (A4):

$$\rho_{im}(u) = \frac{r_{im} - s_{im}}{(uh_m + h_i/d^2)(1-s_{im}^2)} , \quad s_{im} \neq 1 \tag{A6}$$

where

$$r_{im} = \frac{\sqrt{1+b^2}}{\sqrt{uh_m + h_i/d^2 + 1}} \quad , \tag{A7}$$

$$s_{im} = \frac{b}{\sqrt{uh_m + h_i/d^2}} \quad . \tag{A8}$$

For $s_{im} = 1$

$$\rho_{im} = \frac{1}{2(uh_m + h_i/d^2)(uh_m + h_i/d^2 + 1)} \quad . \tag{A9}$$

All three integrals in (A3) are of the same type:

$$\psi_{im}(u) = \int \frac{du}{\sqrt{u+b^2}} \; \rho_{im}(u)$$

$$= \int \frac{du}{\sqrt{u+b^2}} \; \frac{\sqrt{1+b^2}}{\sqrt{uh_m + h_i/d^2 + 1}\;(uh_m + h_i/d^2 - b^2)}$$

$$- \int \frac{du}{\sqrt{u+b^2}} \; \frac{b}{\sqrt{uh_m + h_i / d^2}\;(uh_m + h_i / d^2 - b^2)} \tag{A10}$$

We use the substitutions

$$x_1^2 = uh_m + h_i / d^2 + 1 \tag{A11}$$

for the first integral in (A10) and

$$x_2^2 = uh_m + h_i / d^2 \tag{A12}$$

for the second one.

Using now the identity:

$$\frac{2a}{x^2 - a^2} = \frac{1}{x-a} - \frac{1}{x+a}$$

we get the result:

$$\psi_{im}(u) = \frac{1}{\sqrt{h_m}} \left\{ S\left(1+\frac{h_i}{d^2} - b^2 h_m, \sqrt{1+b^2}, x_1\right) - S\left(1+\frac{h_i}{d^2} - b^2 h_m, -\sqrt{1+b^2}, x_1\right) - S\left(\frac{h_i}{d^2} - b^2 h_m, b, x_2\right) + S\left(\frac{h_i}{d^2} - b^2 h_m, -b, x_2\right) \right\} . \quad \text{(A14)}$$

Here the function S is defined as follows:

$$S(\alpha, \beta, x) = \begin{cases} \dfrac{1}{\sqrt{\alpha-\beta^2}} A(x) & \text{for } \alpha > \beta^2 \\ -\sqrt{(x+\beta)/(x-\beta)} \,/\, \beta & \text{for } \alpha = \beta^2 \\ -\dfrac{1}{\sqrt{\beta^2-\alpha}} L(x) & \text{for } \alpha < \beta^2 \end{cases} \quad \text{(A15)}$$

There are the following notations in (A15):

$$A(x) = \arctan \frac{\beta x - \alpha}{\sqrt{\alpha-\beta^2}\,\sqrt{x^2-\alpha}} , \quad \text{(A16)}$$

$$L(x) = \begin{cases} \ell n \left| \dfrac{\sqrt{\beta^2-\alpha}\,\sqrt{x^2-\alpha} - \alpha + \beta x}{x-\beta} \right| & \text{for } \alpha \neq 0 \\ \ell n \left| 1 + \beta/x \right| & \text{for } \alpha = 0 \end{cases} \quad \text{(A17)}$$

for b = 0 formula (A14) simplifies to:

$$\psi_{im}(u) = \frac{1}{\sqrt{h_m}} \left\{ S(1+\frac{h_i}{d^2}, 1, x_1) - S(1+\frac{h_i}{d^2}, -1, x_1) \right\} \quad \text{(A18)}$$

The values of x_1, x_2 in formulae (A14 - A18) should be taken equal to the values obtained by substituting the integration limits for u in formulae (A11, A12). When $b^2 < h_i/d^2$ or $b^2 > h_i/d^2 + h_m$ they are simply 0 and 1. However, if $h_i/d^2 \le b^2 \le h_i/d^2 + h_m$ then the integration should be performed excluding point $u_0 = (b^2 - h_i/d^2)/h_m$ from the integration path.

Figs. 5 - 7 give examples of the dependence $\Phi_b(d)$ for different values of parameters h and b .

FIGURE CAPTIONS

Fig. 1. Beam blow up. The ratio $d = \sigma(\sqrt{\eta})/\sigma(0)$ is plotted versus parameter $\sqrt{\eta} = \sqrt{\frac{\pi n \omega_0}{\alpha}}\,\Delta Q$, where n is the number of interaction points, ω_0 and α are revolution frequency and damping rate of vertical oscillations, ΔQ is the linear tune shift due to beam-beam interaction. The curves are calculated with different values of fitting parameter h. The left one corresponds to h = 0.01. For each subsequent curve h increments by 0.01. The value h for the last one is 0.10. The aspect ratio of the strong bunch b = 0.0 (infinitely thin flat strong Gaussian bunch).

Fig. 2. The same as Fig. 1, with the aspect ratio of the strong bunch b = 0.1.

Fig. 3. The same as Fig. 1, with the aspect ratio of the strong bunch b = 0.3.

Fig. 4. The comparison of calculated beam blow up (the curve) with the measurements[5] of the vertical size of the weak beam (points). The strong bunch aspect ratio b = 0.035. The value of the fitting parameter h = 0.04. The bars represent the measurement errors only, and do not include any instrumentation errors.

Fig. 5. Function $\Phi_b(d)$ from (17) for different values of the fitting parameter h and for b = 0.0.

Fig. 6. The same as Fig. 5, but b = 0.1.

Fig. 7. The same as Fig. 5, but b = 0.3.

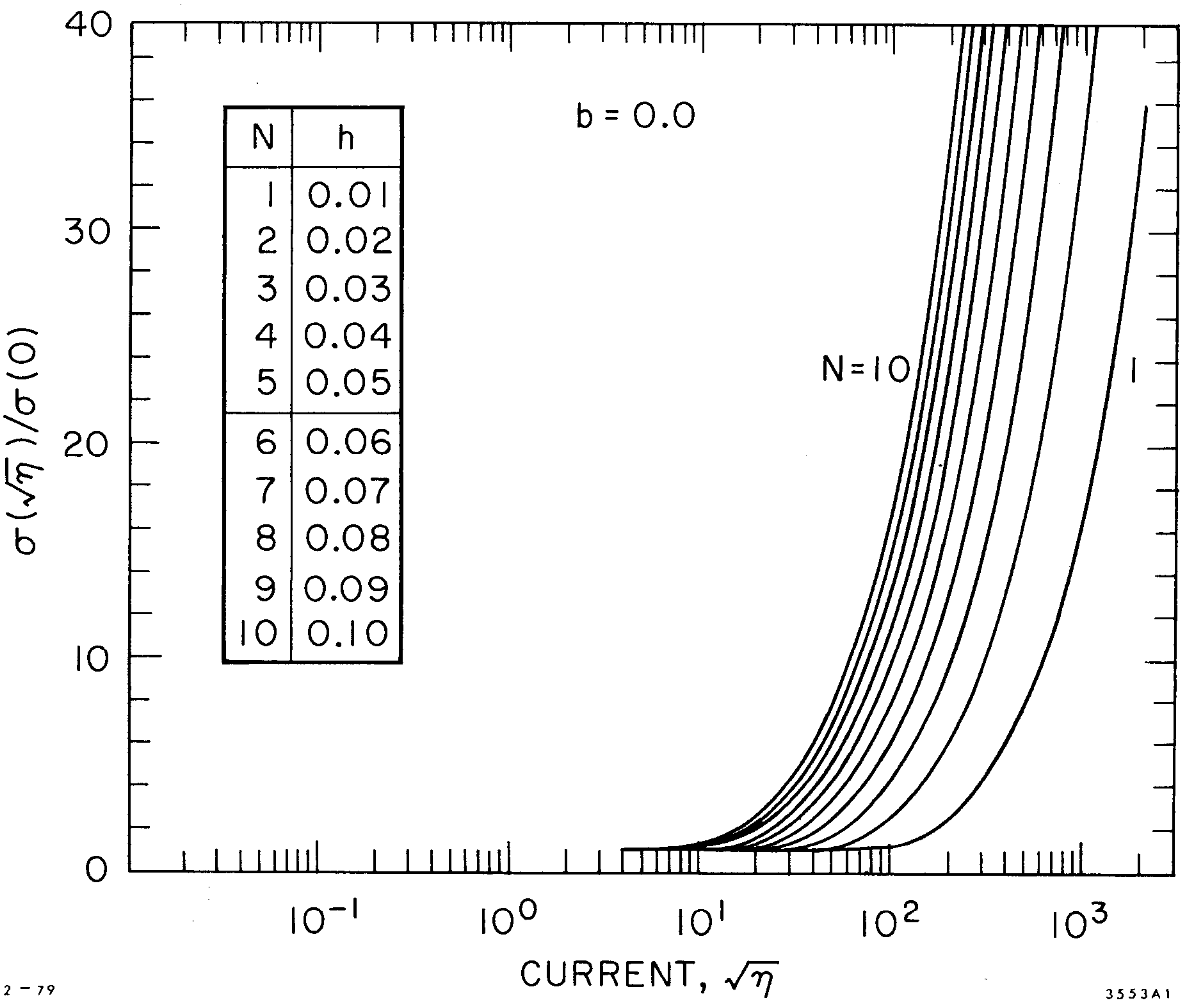
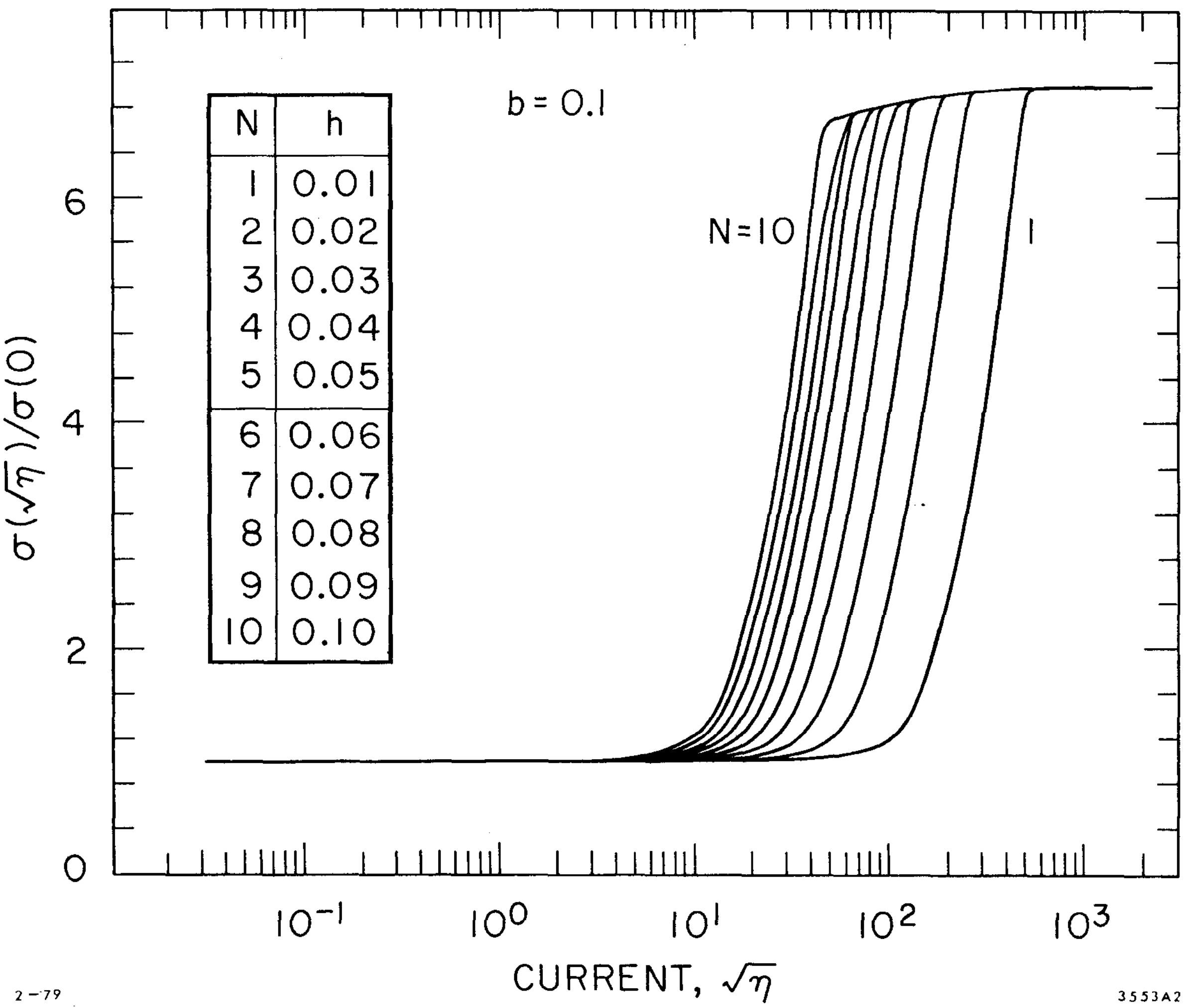
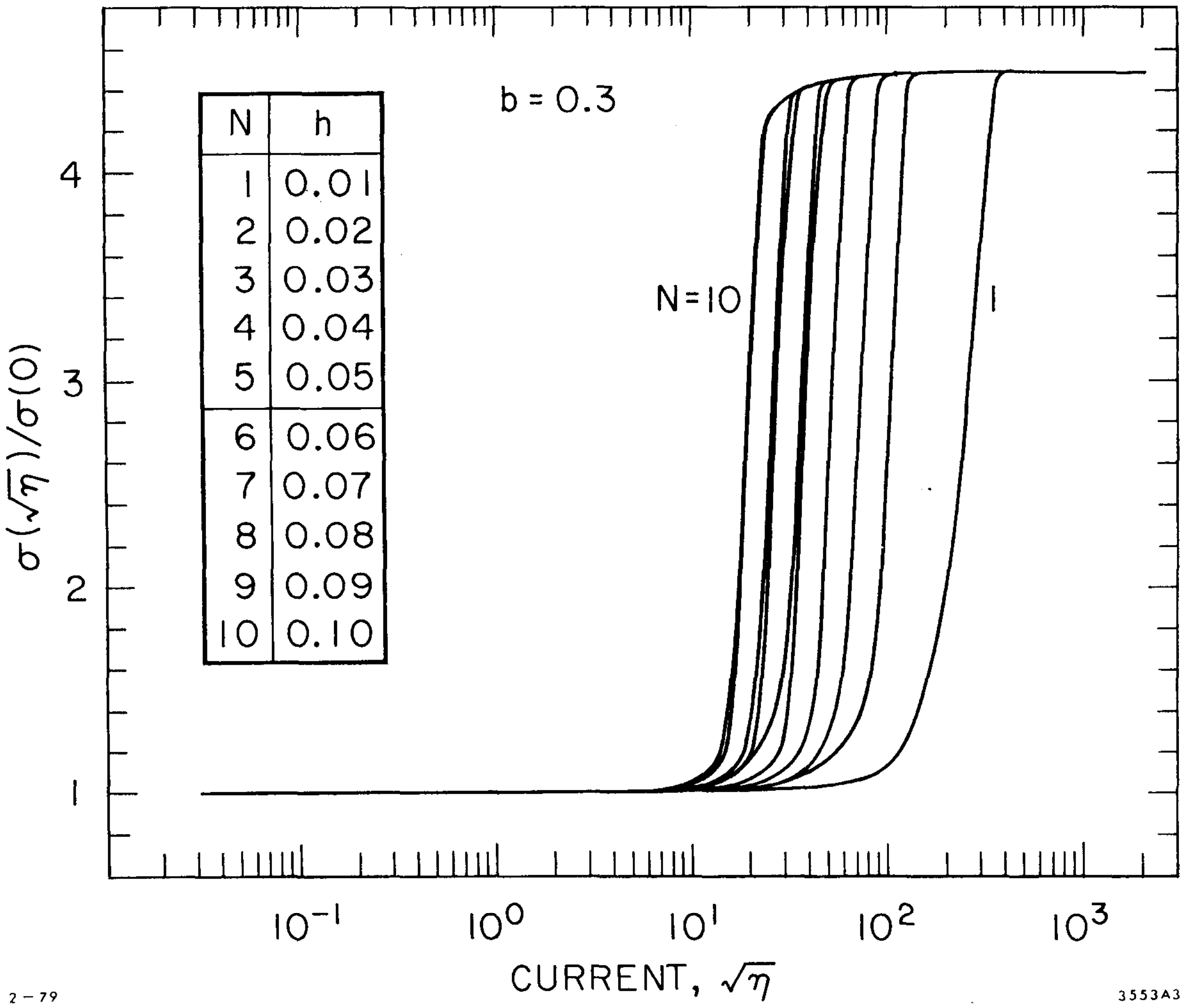

40
30
20
10
0
N
h
1 0.01
2 0.02
3 0.03
4 0.04
5 0.05
6 0.06
7 0.07
8 0.08
9 0.09
10 0.10
b = 0.0
N=10
1
$\sigma(\sqrt{\eta})/\sigma(0)$
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
CURRENT, $\sqrt{\eta}$
2 − 79
3553A1

Fig. 1



b = 0.1
N | h
1 | 0.01
2 | 0.02
3 | 0.03
4 | 0.04
5 | 0.05
6 | 0.06
7 | 0.07
8 | 0.08
9 | 0.09
10 | 0.10
N=10
1
σ(√η)/σ(0)
0
2
4
6
10^-1
10^0
10^1
10^2
10^3
CURRENT, √η
2-79
3553A2

Fig. 2



b = 0.3
N
h
1 0.01
2 0.02
3 0.03
4 0.04
5 0.05
6 0.06
7 0.07
8 0.08
9 0.09
10 0.10
N=10
1
1
2
3
4
σ(√η)/σ(0)
10⁻¹
10⁰
10¹
10²
10³
CURRENT, √η
2-79
3553A3

Fig. 3

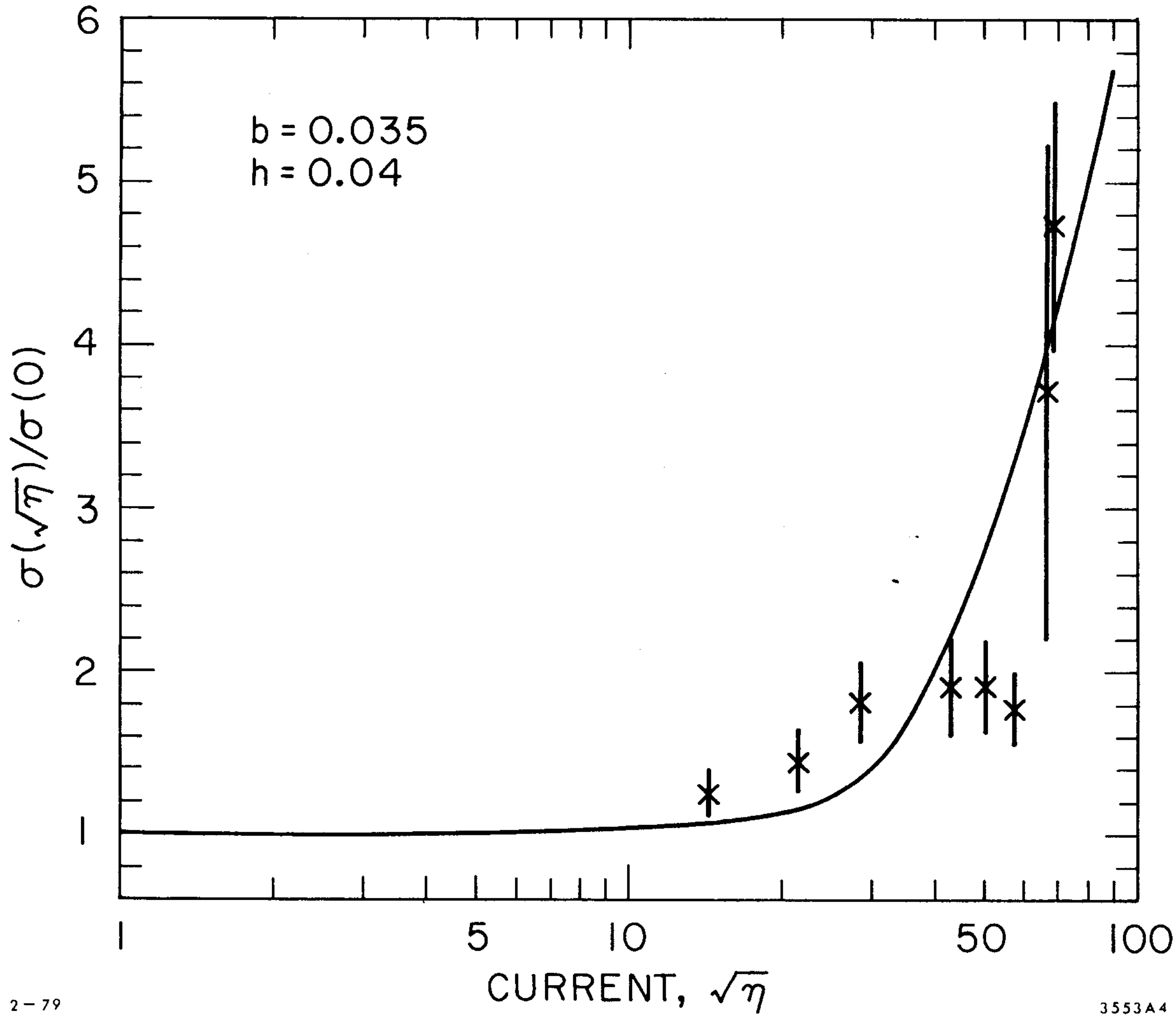

6
5
4
3
2
1
b = 0.035
h = 0.04
$\sigma(\sqrt{\eta})/\sigma(0)$
1
5
10
50
100
CURRENT, $\sqrt{\eta}$
2-79
3553A4

Fig. 4

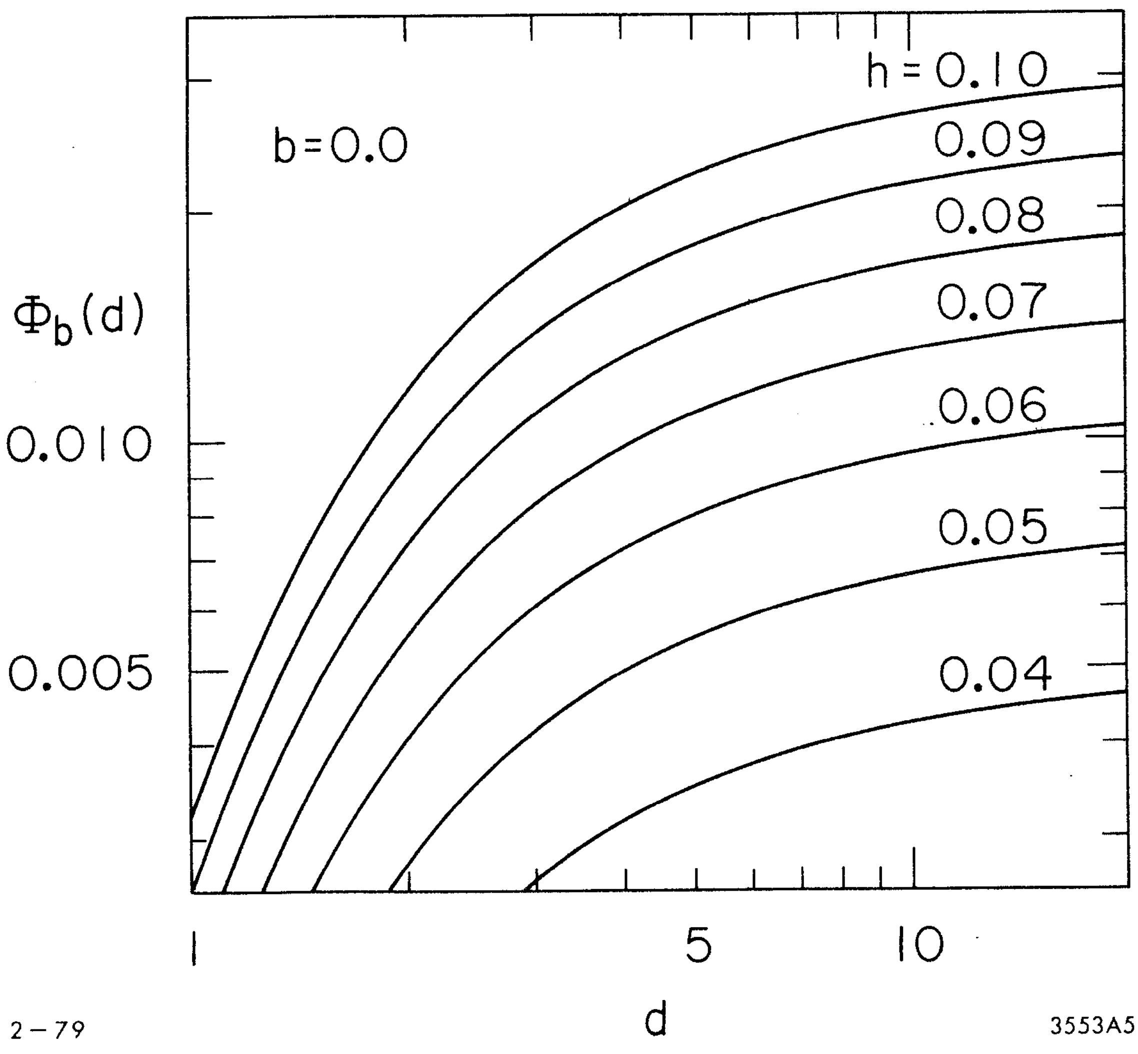

h = 0.10
0.09
0.08
0.07
0.06
0.05
0.04
b=0.0
$\Phi_b(d)$
0.010
0.005
1
5
10
d
2-79
3553A5

Fig. 5

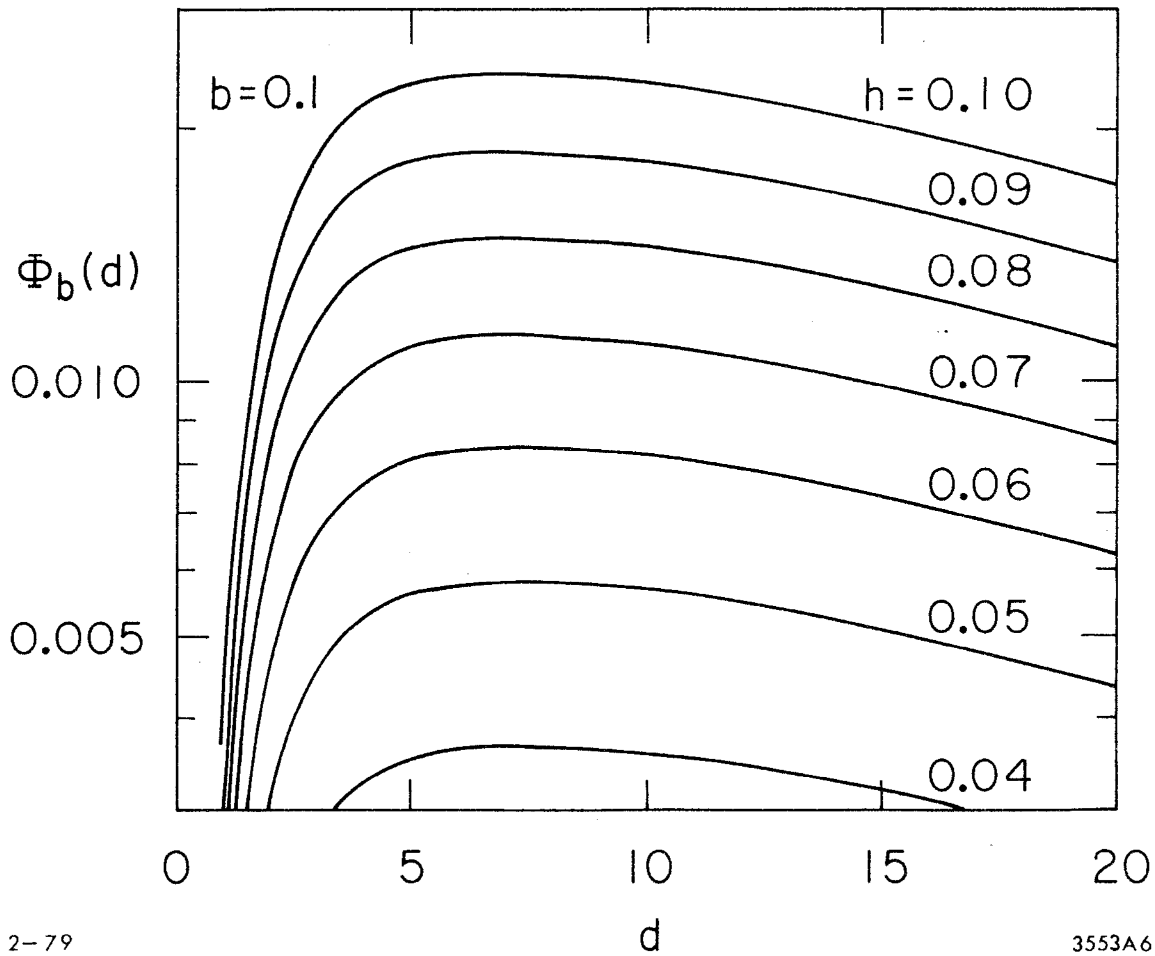
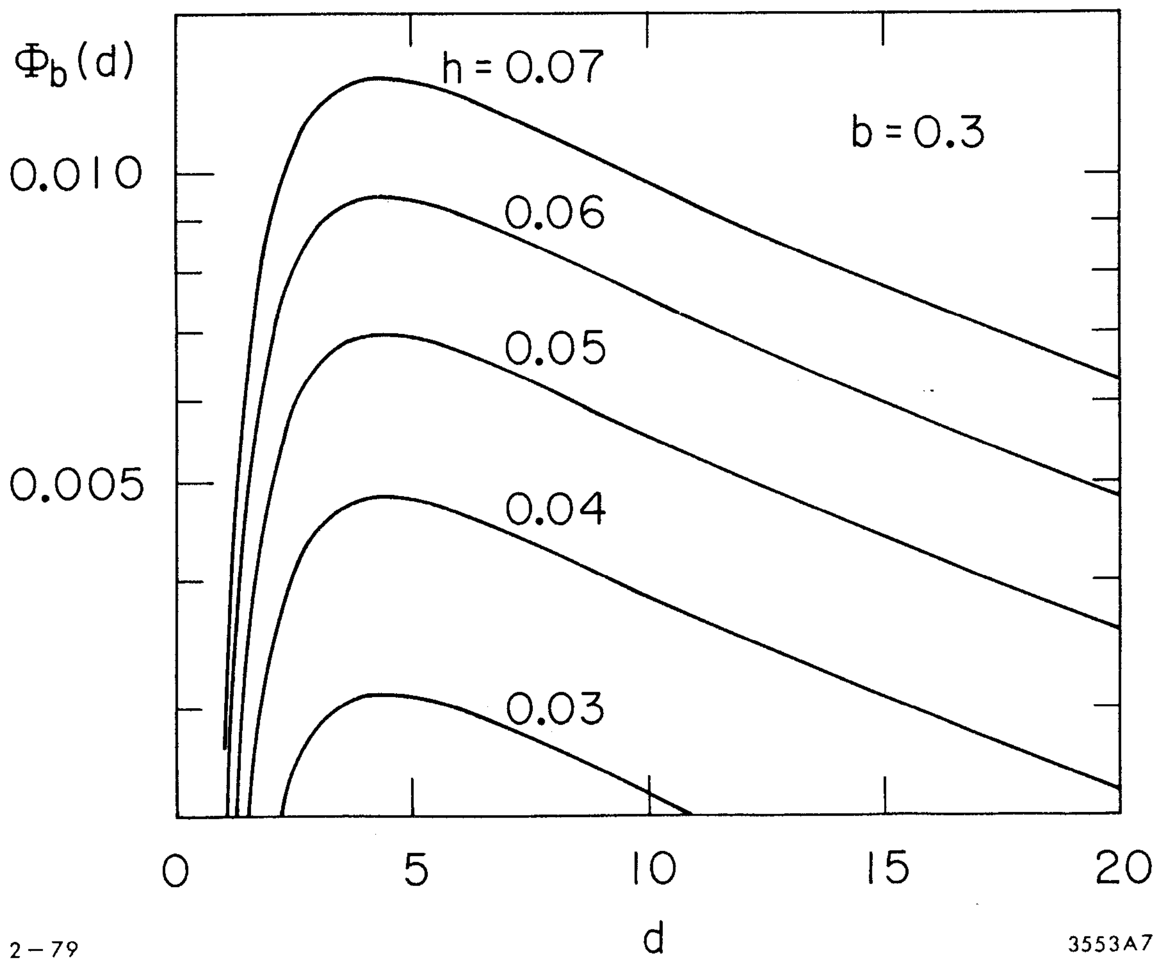

b=0.1
h=0.10
0.09
0.08
0.07
0.06
0.05
0.04
$\Phi_b(d)$
0.010
0.005
0
5
10
15
20
d
2-79
3553A6

Fig. 6



$\Phi_b(d)$
0.010
0.005
h = 0.07
0.06
0.05
0.04
0.03
b = 0.3
0
5
10
15
20
d
2-79
3553A7

Fig. 7