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A NEW APPROACH TO REGGEON QUANTUM MECHANICS

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ABSTRACT

A new method utilizing moment recursion relations is used to study Reggeon field theory in zero transverse dimensions. Accurate numerical results are given for energy levels and Green's functions for both positive and negative values of the intercept gap.

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I. Introduction

The study of Reggeon field theory in zero transverse dimensions[1-4] (Reggeon quantum mechanics) has made significant contributions to our understanding of the behavior of the theory in higher transverse dimensions. In this paper we present accurate numerical calculations of the energy levels and Green's functions in Reggeon quantum mechanics.

Our calculations are based upon moment recursion relations, which have recently proven extremely useful in the study of the anharmonic oscillator.[5] Reggeon quantum mechanics provides an interesting test of the recursion relation approach since the Hamiltonian is not hermitian and some of the standard calculational techniques are therefore not applicable. One can perform a canonical transformation of variables so that the new Hamiltonian is a simple function of the new position coordinate.[1] There are, however, subtleties in applying the proper boundary conditions in the new coordinates. In contrast, the moment approach is straightforward to apply to the original form of the Hamiltonian (written in terms of creation and destruction operators) for which the boundary conditions are simple.

The behavior of the theory when the intercept of the bare Poneron, α_0, is above one is particularly interesting. In this region the unrenormalized perturbation series in r, the triple Poneron coupling, is not Borel summable, and the energy levels and Green's functions must be defined by analytic continuation from $\alpha_0 < 1$. The moment recursion relations allow us to make this continuation and to obtain highly accurate numerical results.

We present the moment method in Section II and the method of solution in Section III. Then we briefly discuss our results in Section IV.

II. Moment Recursion Relations

We study the Hamiltonian

$$H = \Delta\psi^+\psi + \tfrac{1}{2}ir(\psi^{+2}\psi + \psi^+\psi^2), \tag{1}$$

where $\Delta = 1 - \alpha_0$ is the intercept gap of the bare Pomeron and r is the bare triple Pomeron coupling constant. ψ^+ and ψ are the creation and annihilation operators of the bare Pomeron. They satisfy the standard commutation relations

$$[\psi, \psi^+] = 1. \tag{2}$$

We have only taken into account the triplet Pomeron coupling, but the generalization of our techniques to include higher order couplings is straightforward.

Let us first consider the eigenvalue equation

$$H|\Phi_j\rangle = E_j|\Phi_j\rangle. \tag{3}$$

It is obvious that the bare vacuum state defined by $\psi|0\rangle = 0$ is an exact left and right eigenstate of H with eigenvalue zero. We therefore write $|\Phi_0\rangle = |0\rangle$, $E_0 = 0$, and focus our attention on the E_j with $j \geq 1$. We start by introducing moments for the j^{th} eigenstate:

$$F_m(E_j) \equiv \langle 0|\psi^m|\Phi_j\rangle. \tag{4}$$

Taking matrix elements of the commutator

$$[\psi^m, H] = m\Delta\psi^m + \tfrac{1}{2}ir[m\psi^{m+1} + m(m-1)\psi^{m-1} + 2m\psi^+\psi^m] \tag{5}$$

between the states $\langle 0|$ and $|\Phi_j\rangle$ gives[6]

$$E_jF_m(E_j) = m\Delta F_m(E_j) + \tfrac{1}{2}irm[F_{m+1}(E_j) + (m-1)F_{m-1}(E_j)]. \tag{6}$$

Our program for obtaining the energy levels from Eq. (6) is as follows. We start by deriving an asymptotic expansion for $F_M(E)$ and $F_{M+1}(E)$ for some large value of M and general values of E, not necessarily equal to E_j. We then use Eq. (6) to calculate recursively all the $F_m(E)$ for $m < M$. Since Eq. (6) is linear, we of course determine the $F_m(E)$ only up to an overall constant (which will be fixed by normalization). Equation (6) with $m = 2$ yields $F_1(E)$. Since $F_0(E_j)$ must vanish from orthogonality, Eq. (6) with $m = 1$ yields the eigenvalue equation

$$E = \Delta + \tfrac{1}{2}irF_2(E)/F_1(E). \tag{7}$$

This equation is, of course, only satisfied for particular values of E, namely $E = E_j$; it thus determines the eigenvalues of H.

The Green's functions

$$G_m(E) = \langle 0|\psi^m(E-H)^{-1}\psi^+|0\rangle \tag{8}$$

can be computed in a similar way. The matrix elements of the commutator $[\psi^m, E \mp H]$ between the states $\langle 0|$ and $(E - H)^{-1}\psi^+|0\rangle$ can be easily manipulated to yield the equation

$$EG_m(E) - \delta_{m,1} = m\Delta G_m(E) + \tfrac{1}{2}irm[G_{m+1}(E) + (m-1)G_{m-1}(E)]. \qquad (9)$$

Notice that for $m > 1$ Eqs. (6) and (9) are identical so that $G_m(E)$ is proportional to $F_m(E)$. The overall normalization of the $G_m(E)$ is determined by Eq. (9) with $m = 1$; this gives

$$G_1^{-1}(E) = E - \Delta - \tfrac{1}{2}irG_2(E)/G_1(E). \qquad (10)$$

Since $G_2(E)/G_1(E) = F_2(E)/F_1(E)$, we see by referring to Eq. (7) that the Green's functions will have poles if and only if $E = E_j$, just as in the case of a hermitian Hamiltonian.

III. Asymptotic Behavior

We now turn to the problem of analyzing the asymptotic behavior of $F_m(E)$ in order to determine the eigenvalues and the Green's function. We first notice from Eq. (6) that $F_m(E)$ must be an increasing function of m for large m and that

$$F_{m+1}(E)/F_{m-1}(E) \underset{m\to\infty}{\longrightarrow} -m. \qquad (11)$$

We therefore write

$$F_m(E) = (-i\sqrt{2})^{m-1}\,\Gamma(\tfrac{m}{2})\; f_m(e) \qquad (12)$$

and note that the reduced moment $f_m(e)$ satisfies the equation

$$f_{m+1}(e) - f_{m-1}(e) = -N(m)\ (d - \frac{e}{m})\ f_m(e), \tag{13}$$

where

$$N(m) \equiv \sqrt{2}\ \Gamma(\tfrac{m}{2})/\Gamma(\tfrac{m+1}{2}) \tag{14}$$

$$d \equiv \Delta/r$$

$$e \equiv E/r.$$

Since Eq. (13) is a second order difference equation, it has two independent solutions; in fact, note that if $f_m(e, d)$ satisfies (13), then so does $(-)^m f_m(-e, -d)$.

Since for large m,

$$N(m) \simeq \frac{2}{\sqrt{m}}\ [1 + \frac{1}{4m} + O(m^{-2})], \tag{15}$$

We see that these solutions have the asymptotic forms (for $m \to \infty$)

$$f_m^-(E) \simeq C_- \exp[-\Phi(m)] \tag{16}$$

$$f_m^+(E) \simeq C_+ (-)^m \exp[+\Phi(m)],$$

where $\Phi(m) = 2dm^{\frac{1}{2}} + (2e - \frac{1}{2}d + \frac{1}{3}d^3)m^{-\frac{1}{2}} + O(m^{-\frac{3}{2}})$

and the $C_\pm$ are constants independent of m.

The perturbation series (in r) for G_m, F_m and E_j are all Borel summable for $\Delta > 0$. In fact, Reggeon field theory is originally defined by its bare perturbation series for positive Δ; thus we have the following boundary conditions for $r \to 0$, $\Delta > 0$:

$$G_m(E) \to \delta_{m,1}(E - \Delta)^{-1} \tag{17}$$

$$F_m(E_j) \to \delta_{m,j}(m!)^{\frac{1}{2}}$$

$$E_j \to j\Delta.$$

As a result, it is clear that we must use the solution $f_m^-(E)$ for $r > 0$ and $f_m^+(E)$ for $r < 0$. From Eqs. (6) and (9) we see that

$$E_j(-r) = E_j(r) \tag{18}$$

$$G_m(-r) = (-)^m G_m(r),$$

so without any loss of information we may take $r > 0$, and we shall do so for the remainder of this paper.

For $\Delta < 0$ any perturbation series in r is no longer Borel summable, and the quantities of interest must be defined by analytic continuation from $\Delta > 0$. This means that (recall $r > 0$) we must use the solution with the asymptotic form f_m^- for all values of Δ.

Now for $\Delta > 0$, $r > 0$, f_m^- is the sub-dominant solution for large m and the solution of our problem is straightforward. It is convenient to introduce the quantities

$$h_m(E) = f_{m+1}(E)/f_m(E) \tag{19}$$

in terms of which Eq. (13) becomes

$$h_{m-1}(E) = [h_m(E) + N(m)(d - \tfrac{e}{m})]^{-1}. \tag{20}$$

For some large value of m, say M, we use the asymptotic form for f_m^- to write

$$h_M \simeq \exp[-dM^{-\frac{1}{2}} + (e + \tfrac{1}{6}d^3)M^{-\frac{3}{2}} + O(M^{-\frac{5}{2}})], \tag{21}$$

and then calculate all of the h_m for $m < M$ from Eq. (20).

Since we are primarily interested in the Pomeron propagator $G_1(E)$ and in the energy levels, E_j, it is convenient to write then in the dimensionless form

$$g(e, d) \equiv rG_1$$

$$e_j(d) \equiv E_j/r,$$

where $g(e, d)$ is determined by

$$g^{-1}(e, d) = e - d - (2\pi)^{-\frac{1}{2}}h_1(e). \tag{23}$$

The eigenvalues e_j are simply the zeroes of $g^{-1}(e, d)$. The calculation

of $g(e, d)$ and $e_j(d)$ converges very rapidly as is illustrated in Table I. Furthermore, $g(e, d)$ is quite insensitive to errors in h_M. The reason is that even an error in h_M of order one only corresponds to an admixture in h_1 of the unwanted solution, f_m^+, of order $\exp(-4dM^{\frac{1}{2}})$. The insensitivity of $g(e, d)$ to variations in h_M is illustrated in Table II.

The above procedure must be modified for $\Delta < 0$, because the solution of interest, f_m^-, is then the dominant one for large m. This means that $g(e, d)$ will be extremely sensitive to any error in h_M. To avoid this difficulty, we make a similarity transform of the Hamiltonian by writing

$$\hat{H} = e^{ia\psi} H e^{-ia\psi} \tag{24}$$

$$= (\Delta - ar)\psi^+\psi + \tfrac{1}{2}ir[\psi^{+2}\psi + \psi^+\psi^2] + ia(\Delta - \tfrac{1}{2}ar)\psi - \tfrac{1}{2}ar\psi^2.$$

$\hat{H}$ will of course have the same eigenvalues as H. The eigenvector corresponding to E_j is $|\hat{\Phi}_j\rangle = \exp(ia\psi)|\Phi_j\rangle$.

Introducing the new moments

$$\hat{F}_m(E_j) = \langle 0|\psi^m|\hat{\Phi}_j\rangle \tag{25}$$

and Green's functions

$$\hat{G}_m(E) = \langle 0|\psi^m(E - \hat{H})^{-1}\psi^+|0\rangle, \tag{26}$$

we obtain by use of the commutator $[\psi^m, \hat{H}]$ the recursion relations

$$E_j \hat{F}_m(E_j) = (\Delta - ar)m\hat{F}_m(E_j) \tag{27}$$

$$+ \tfrac{1}{2}ir[(m + a(2d - a))\hat{F}_{m+1}(E_j) + m(m-1)\hat{F}_{m-1}(E_j)] - \tfrac{1}{2}ar\hat{F}_{m+2}(E_j)$$

and

$$E\hat{G}_m(E) - \delta_{m,1} = (\Delta - ar)\hat{G}_m(E) \tag{28}$$

$$+ \tfrac{1}{2}ir[(m + a(2d - a))\hat{G}_{m+1}(E) + m(m-1)\hat{G}_{m-1}(E)]$$

$$- \tfrac{1}{2}ar\hat{G}_{m+2}(E).$$

Since Eqs. (27) and (28) are third-order difference equations, they have three independent solutions. These solutions have the asymptotic forms

$$\hat{F}^-_m(E) \simeq \hat{C}_-(-i\sqrt{2})^{m-1}\,\Gamma(\tfrac{m}{2})\,\exp[-\Phi(m)] \tag{29}$$

$$\hat{F}^+_m(E) \simeq \hat{C}_+(i\sqrt{2})^{m-1}\,\Gamma(\tfrac{m}{2})\,\exp[+\Phi(m)]$$

and

$$\hat{F}^0_m(E) \simeq \hat{C}_0(i/a)^m\,\Gamma(m)(1/m)[1 + O(1/m)],$$

where

$$\Phi(m) = (2d - a)m^{\frac{1}{2}} + [2e - \tfrac{1}{4}(2d - a) + \tfrac{1}{24}(2d - a)^3]m^{-\frac{1}{2}} + O(m^{-\frac{3}{2}}).$$

$\hat{F}^{\pm}_m$ are the analytic continuations in a of the two solutions defined in Eqs. (12) and (16) for a = 0. For r > 0 it is the solution $\hat{F}^-_m$ which is of interest. Clearly, for any value d, positive or negative, we can choose a so that $\hat{F}^-_m$ is the subdominant solution for large m. As a

result, the recursion relations of Eqs. (27) and (28) can be used to obtain accurate numerical results for either sign of d.

Our final formulae are obtained by defining

$$\hat{h}_m(E) = \frac{i}{2} N(m)[\hat{F}_{m+1}(E)/\hat{F}_m(E)] \tag{30}$$

in analogy with Eqs. (12) and (19). Equation (27) can then be rewritten in the form

$$\hat{h}_{m-1} = \{\hat{h}_m[1 + \frac{a}{m}(2d - a)] + N(m)[d - a - \frac{e}{m} + \tfrac{1}{2}\hat{h}_m\hat{h}_{m+1}]\}^{-1} \tag{31}$$

for $m \geq 2$. The asymptotic form for $\hat{h}_m$ for $m \to \infty$ is

$$\hat{h}_m(E) \simeq \exp\{-\tfrac{1}{2}(2d - a)m^{-\frac{1}{2}} + [e + \frac{1}{45}(2d - a)^3]m^{-\frac{3}{2}} + O(m^{-\frac{5}{2}})\}.$$

The Green's function $G_1 = r^{-1}g(e, d, a)$ is determined from the formula

$$\hat{g}^{-1}(e, d, a) = e - d + a - \tfrac{1}{2}a\hat{h}_1\hat{h}_2 - (2\pi)^{-\frac{1}{2}}\hat{h}_1[1 + a(2d - a)], \tag{33}$$

and the $\hat{G}_m$ for $m > 1$ by combining Eqs. (30) - (33).

The eigenvalues E_j are now given by the zeroes of $\hat{g}^{-1}(e, d, a)$. We have numerically verified that they are indeed independent of a. The Green's function of actual interest is $G_1 = g(e, d)/r$. It can be obtained once the $\hat{G}_m$ are known by noting that

$$G_1(E) = \langle 0|\psi e^{-ia\psi}(E - \hat{H})^{-1}e^{ia\psi}\psi^+|0\rangle = \sum_{n=0}^{\infty} \frac{(-ia)^n}{n!}\hat{G}_{n+1}(E, d, a). \tag{34}$$

IV. Conclusion

Our main results are shown in Figs. 1 - 4 and Table III. They are given for $r > 0$, but the corresponding results for $r < 0$ can be read off using Eq. (18). In Fig. 1 we plot the energy levels $e_j(d) = E_j/r$ for $j = 1, 2, \ldots 5$.[7] As we have already noted, perturbation theory must be valid for $\Delta > 0$ so $e_j(d) \to jd$ for large positive values of d. As d is decreased, $e_1(d)$ approaches zero and becomes nearly degenerate with $e_0 = 0$. In fact for $d \lesssim -1.5$, $e_1(d)$ is well approximated by its asymptotic form[2,3]

$$e_1(d) \underset{d \to -\infty}{\simeq} (2\pi)^{-\frac{1}{2}} 2d^2 e^{-2d^2} \left(1 - \frac{1}{2d^2}\right) \tag{35}$$

as can be seen from Table III. For d large and negative the other eigenvalues approach integer multiplets of $|d|$ and become almost double degenerate.[1]

In addition to the energy levels we have discussed, there is a second set, E_j' arising from the solutions of Eq. (6) with the asymptotic form f_m^+ for $r > 0$. One sees from Eq. (6) that $E_n' = -E_n$ for all $r > 0$. This set of solutions has a perturbation expansion for $\Delta < 0$, but not for $\Delta > 0$. It is therefore not relevant to the conventional discussion of Reggeon field theory.

In Figs. 2 and 3 we plot the inverse Pomeron propagator, $g^{-1}(e, d) = r^{-1} G_1^{-1}(E)$, as a function of e for two typical values of d. One of the interesting features to notice is the zeroes of g(e, d). One sees from the perturbation expansion that for $d \gg 1$, g(e, d) has two

zeroes in the range $e_2 < e < e_3$, none in the range $e_3 < e < e_4$, two in the range $e_4 < e < e_4$, etc. We see in Fig. 2 where the zeroes, of course, appear as poles in g^{-1}, that this pattern persists down to $d = 1$. However, as d decreases further, the zeroes collide and leave the real e axis. We find none at $d = -2$.

Probably the most important application of Reggeon quantum mechanics has been in the development of the analogue spin model.[8,9] This model has played an important role in our understanding of the theory in higher transverse dimensions, particularly in the supercritical region. It was introduced by putting the theory on a lattice in impact parameter space. If one starts by ignoring the couplings between lattice points, which arise from the kinetic energy terms in the Hamiltonian, then at each lattice point one must merely solve the problem of Reggeon quantum mechanics. It was then argued that for calculations in the regime $E \approx 0$ and $d << 0$, it was only necessary to include the two lowest energy levels of the single-site problem when the couplings between lattice points was taken into account. Under these circumstances the two lowest energy levels are nearly degenerate, and their mutual coupling is expected to be much stronger than their coupling to higher states. A measure of the coupling of the ground state to the j^{th} excited state is

$$S_j(d) = R_j/E_j, \tag{36}$$

where R_j is the residue of the pole of $g(e, d)$ at $e = e_j$. In Fig. 4 we plot S_1/S_2 as functions of d, and we see that $S_1/S_2 >> 1$ for $d < -1.5$.

In conclusion, we see that moment recursion relations provide a powerful method for studying certain eigenvalue problems. In particular, when applied to Reggeon quantum mechanics, they yield a very simple and rapidly convergent numerical procedure for computing eigenvalues and Green functions.

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Figure Captions

Fig. 1: The energy levels $e_j(d) = E_j/r$. Recall that $e_0(d) = 0$ for all d.

Fig. 2: The inverse Pomeron propagator for d = 1.

Fig. 3: The inverse Pomeron propagator for d = -2.

Fig. 4: The ratio S_1/S_2 as a function of d.

Table Captions

Table I: $g(0, 1)$ and $e_1(1)$ calculated with different values of M. In each case h_M is computed using Eq. (21) and the h_m for $m < M$ using Eq. (20).

Table II: $g(0, 1)$ calculated with h_M given by the asymptotic form of Eq. (21) and with h_M equal to 1, 0 and -1. Notice that for large M, $g(0, 1)$ is independent of variations in h_M.

Table III: e_1 is the energy of the first excited state calculated to eight significant figures. e_1 (Eq. 35) is the approximation to e_1 given by the asymptotic form of Eq. (35).

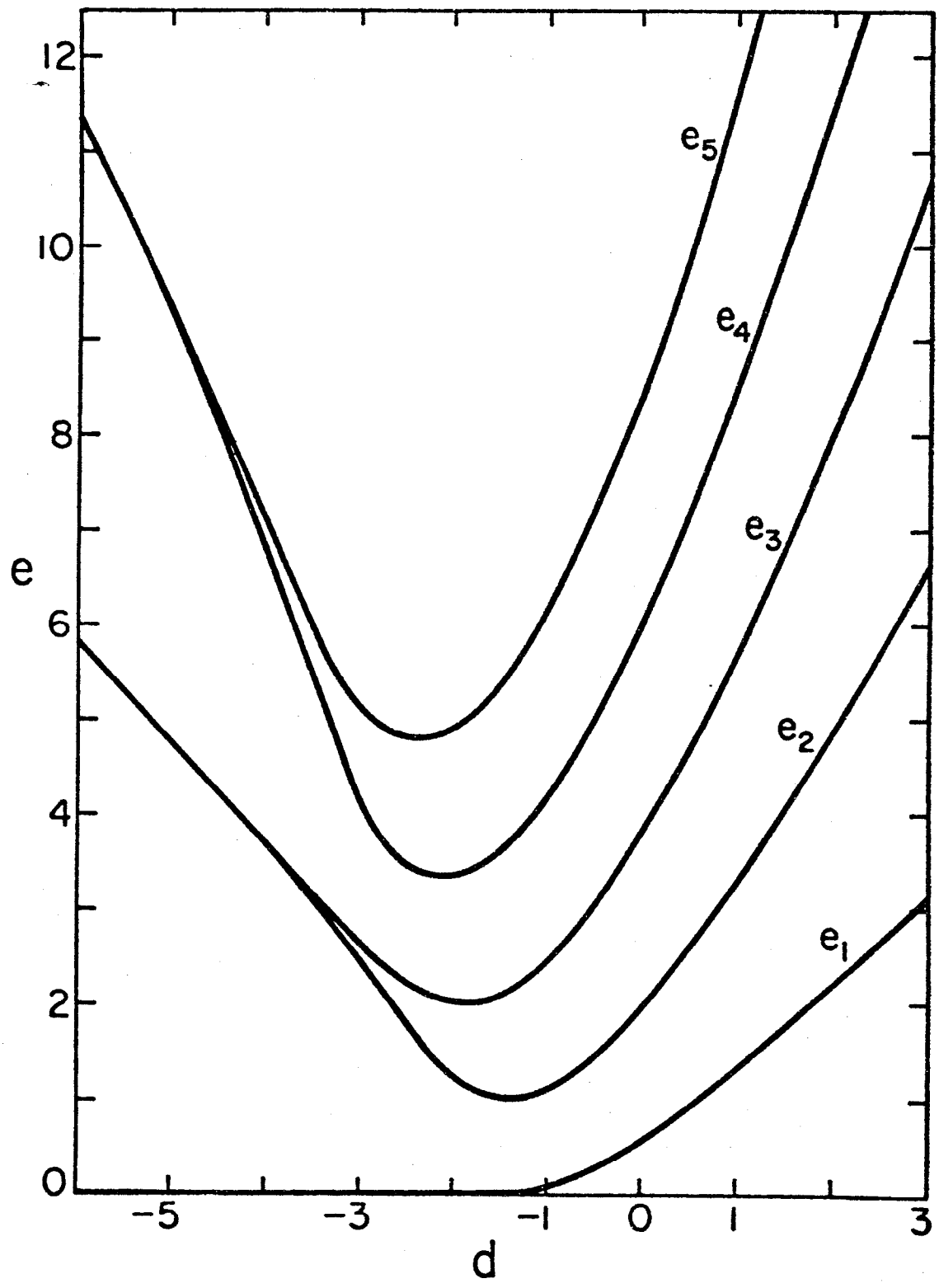

Fig. 1

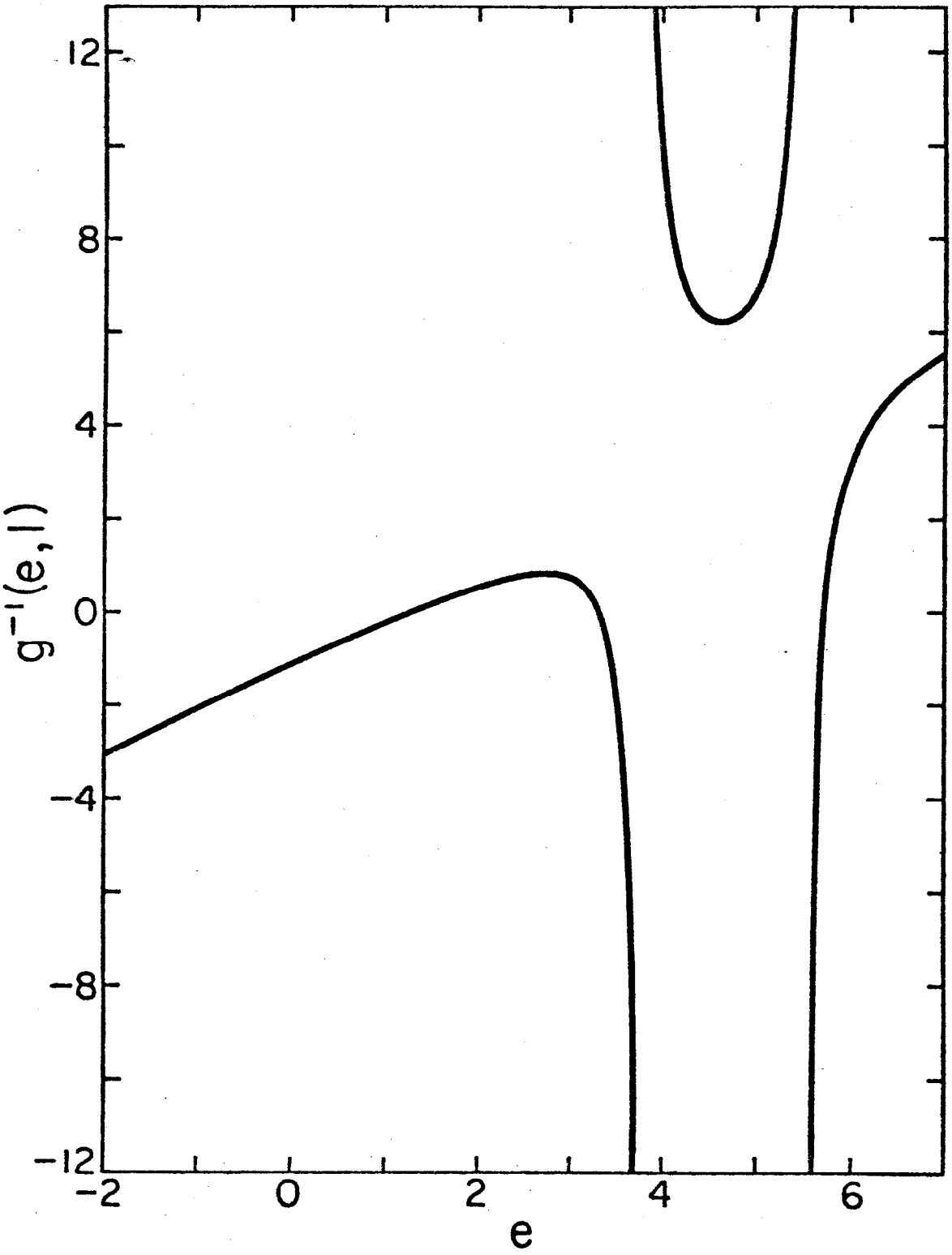

Fig. 2

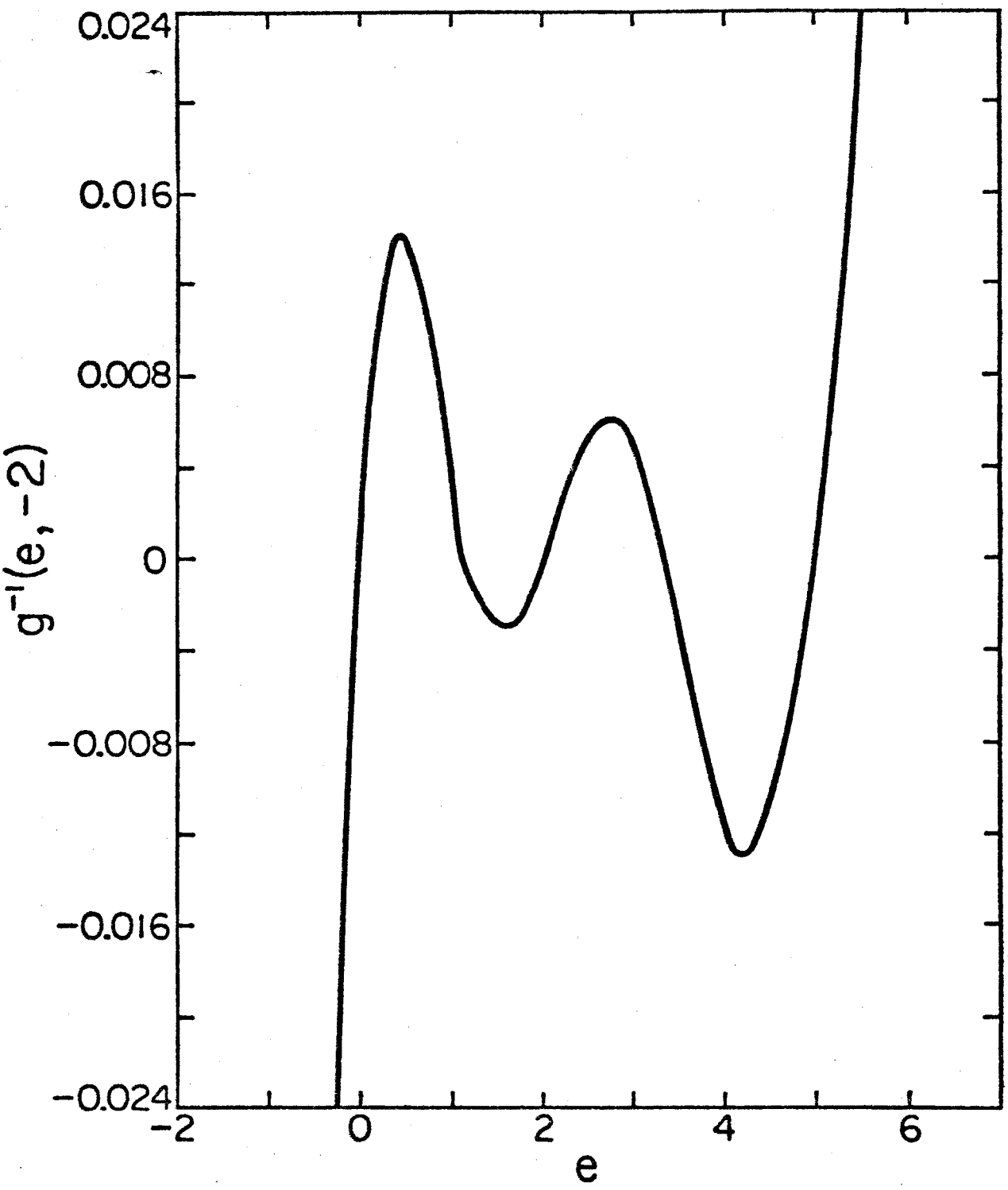

Fig. 3

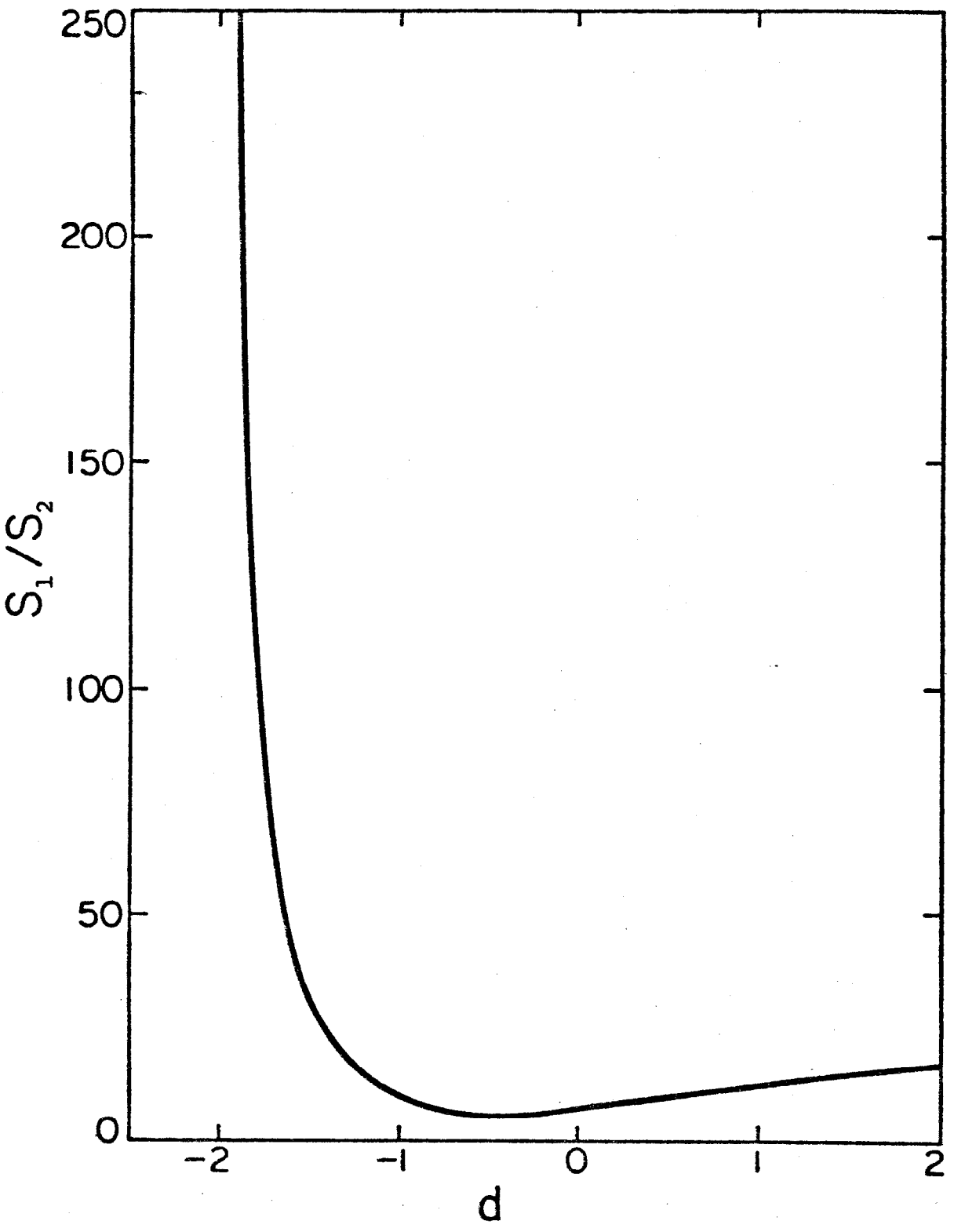

Fig. 4

TABLE I

M	g(0, 1)	$e_1(1)$
100	-.8427384586	1.3177075
50	-.8427384586	1.3177075
30	-.8427384586	1.3177075
20	-.8427384586	1.317705
10	-.84273836	1.3173
5	-.84275	1.33
3	-.8429	1.36
2	-.8417	1.25

TABLE II

g(0, 1)			
h_M	M = 100	M = 50	M = 10
h_M (Eq. 21)	-.8427384586	-.8427384586	-.84273836
1	-.8427384586	-.8427384586	-.84275
0	-.8427384586	-.8427384586	-.84267
-1	-.8427384586	-.8427384583	-.8421

TABLE III

d	e_1	e_1 (Eq. 35)
2	2.2086926	
1	1.3177075	
0	.55563898	
-1	.09167735	.054
-2	.00091942	.000937
-3	$.10298780 \times 10^{-6}$	$.1033 \times 10^{-6}$