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GENERAL RELATIVISTIC GRAVITATION AS THE THEORY OF BROKEN SYMMETRY OF INTRANSITIVE GROUPS OF TRANSFORMATIONS*

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ABSTRACT

General relativistic gravitational theories are constructed from suitable intransitive continuous groups of transformations. A minimal invariant variety forms the unperturbed universe. The formalism of the group is generalized to have the symmetry of its action on this manifold broken by gauge potentials. The theory is expressed in these potentials and it is shown how the present symmetry breaking is related to a general metric. The physical interpretation of the formalism is outlined.

I. INTRODUCTION

The generalization of Weyl's gauge formulation of electrodynamics by YANG and MILLS [1] to a non Abelian invariance group inspired Utiyamas method to obtain the general theory of relativity as a gauge theory of linear representations of the Lorentz group [2] and subsequently Kibbles generalization using the Poincaré group [3], LUBKIN [4] and later YANG [5] presented an integral formulation of gauge theory for gravitation; the Christoffel connection is the gauge potential of the linear group of transformations of parallel transported vectors. LUBKIN [4] had already defined a general dual charge analog of the magnetic monopole of which the gravitational case was analyzed by CLARKE [6]. The present author [7] obtained the electromagnetic and gravitational fields, simultaneously in the same form, from the parallel displacement of the spinor of Dirac's electron equation. The gauge group is GL(4C) in BARGMANN'S [8] covariant formulation of Dirac's equation with the electromagnetic gauge group as the invariant subgroup of the traces, or in case of the Vierbein formulation: U(1)×SL(2C). Because of these features, the combination of the two gauge groups appears artificial rather than a true unification. Subsequently the same formalism

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was also applied [9] to Dirac's DeSitter covariant spinning electron equation [10] and it was shown that general relativity with minor modifications can be based on groups other than the Poincaré group in the local limit—especially on the DeSitter group [9]. In the mentioned work of YANG [5], and of the present author [7] the gauge fields are related to the Riemann curvature tensor. YANG [5] varies the gauge potentials independent of the metric and obtains field equations of the third order in the metric tensor, whereas in [7] the metric tensor is varied together with the potential, resulting in field equations with fourth derivatives of the metric. The latter method seems justified because of the universal coupling of the gravitational field and the resulting conservation laws. Both types of equations yield all the vacuum solutions of Einstein's theory.

The approach presented here introduces inhomogenous gravitational fields by breaking the symmetry of the invariance group of space-time [11]. A general space-time metric appears only as a secondary result of the gauge formalism which is applied to generalize the mathematical apparatus of the group of transformations rather than the physical entities.

A minimal invariant variety of the manifold on which the group acts forms the universe of unperturbed space-time and the gauge group is a group of linear transformations of the base vectors of the first group. I have chosen here the general linear group rather than the adjoint group as the gauge group. A special gauge exists then for every coordinate system, which allows to express the formalism in terms of the physical potentials and thus to formulate a field theory of gravitation.

The notation used is that of Eisenhart's classical book [12].

II. GENERALIZATION OF DIRAC'S METHOD TO OBTAIN GROUP COVARIANT FIELD EQUATIONS

We consider a continuous group of transformation with r essential parameters acting on a n-dimensional space V_n. The rank of the matrix of base vectors (ξ^i_α) $(i=1\ldots n,$ $\alpha=1\ldots r)$ of the group be $q<n<r$ so that the group is intransitive. There exist then q-dimensional minimal invariant varieties.

One can in general find a metric of the n-dimensional space such that G_r is a group of motions in it and each of the q-dimensional invariant varieties is a Riemannian subspace imbedded in the n-dimensional space. We are interested in the case $q=4$ and signature +2 when one minimal invariant variety V_q forms a universe with group of motion G_r.

One can then introduce a coordinate system such that everywhere in the V_n:

$$\xi^h_\alpha = 0, \; g_{hi} = 0 \; (i \neq h), \; g_{hh} = \pm 1$$
$$(\alpha = 1 \ldots r, \; h = q+1 \ldots n, \; i = 1 \ldots n) \tag{1}$$

as follows from the Killing equations:

$$\frac{\partial g_{ik}}{\partial x^\ell} \xi^\ell_\alpha + g_e \cdot \frac{\partial \xi^\ell_\alpha}{\partial x^k} + g_{\ell k} \frac{\partial \xi^\ell_\alpha}{\partial x^i} = 0 \tag{2}$$

A nonsingular metric can be formed in group space if G_r is semisimple and the metric:

$$g^{ik} = \xi^i_\alpha \, \gamma^{\alpha\beta} \, \xi^k_\beta \tag{3}$$

with

$$\gamma_{\alpha\beta} = c^\phi_{\alpha\varepsilon} \, c^\varepsilon_{\beta\phi} \text{ and } \gamma_{\alpha\varepsilon}\gamma^{\varepsilon\beta} = \delta^\beta_\alpha \tag{3a}$$

fulfills the Killing equations (2). Group invariant Lagrangians can therefore be formed out of the Lie derivatives of tensor fields on V_q and $\gamma^{\alpha\beta}$. For example the Lagrange density of a scalar field ϕ is

$$\mathscr{L} = \sqrt{g} \; \gamma^{\alpha\beta} \xi^i_\alpha \left(\frac{\partial}{\partial x^i} \phi^*\right) \xi^k_\beta \left(\frac{\partial}{\partial x^k} \phi\right) \tag{4}$$

Such expressions are also covariant w.r.t. coordinate transformations. A Lie derivative of spinors has also been defined [11] in this context.

III. THE LIE DERIVATIVE OF A SPINOR

The metric derivative of a spinor covariant w.r.t. general coordinate and spin transformation is:

$$\psi;k = \frac{\alpha\psi}{\alpha x^k} + \Gamma_k \psi \tag{5}$$

where the metric spinor connection Γ_k is defined by the relation:

$$\frac{\partial \gamma^i}{\partial x^k} + \gamma^\ell \begin{Bmatrix} i \\ \ell \; k \end{Bmatrix} - \left[\gamma^i, \Gamma_k \right] = 0 \tag{5a}$$

in the Vierbein notation:

$$\gamma^k = h^k_{a|}\,\overset{o}{\gamma}{}^a \;,\; h^k_{a|} h^\ell_{b|}\,\overset{o}{g}{}^{ab} = g^{k\ell} \tag{6}$$

$$\{\overset{o}{\gamma}{}^a, \overset{o}{\gamma}{}^b\} = 2\overset{o}{g}{}^{ab} \;,\; \{\gamma^k, \gamma^\ell\} = 2g^{k\ell}$$

$$\Gamma_k = \frac{1}{4} h_{c|}{}^m h_{d|m;k}\,\overset{o}{\sigma}{}^{cd} \qquad (\overset{o}{\sigma}{}^{cd} = \frac{1}{2}\,[\gamma^c, \gamma^d]) \tag{6a}$$

On a space where G_r is a group of motion, I define the Lie derivative of a spinor w.r.t. ξ^i_α:

$$\psi||\alpha = \frac{\partial\psi}{\partial x^\ell}\,\xi^\ell_\alpha + \Gamma^L_\alpha\psi \tag{7}$$

with the connection Γ^L_α defined by:

$$\frac{\partial\gamma^i}{\partial x^\ell}\,\xi^\ell_\alpha - \frac{\partial\xi^i}{\partial x^\ell}\,\gamma^\ell - \left[\gamma^i, \Gamma^L_\alpha\right] = 0 \tag{7a}$$

One can supplement ξ^i_α by q-1 linear independent symbols of the group and form their algebraic complements ξ^β_k:

$$\xi^i_\alpha\,\xi^\alpha_k = \delta^i_k \;,\quad \xi^k_\alpha\,\xi^\beta_k = \delta^\beta_\alpha \;(i,k,\alpha,\beta = 1\;\ldots\;q) \tag{8}$$

for the remaining symbols we have:

$$\xi^i_\beta = \phi^\alpha_\beta(x)\,\xi^i_\alpha \begin{pmatrix}\alpha = 1\;\ldots q\\ \beta = q+1\;\ldots\; r\end{pmatrix} \tag{9}$$

One can now form the nonsymmetric connection: [12]

$$\Lambda^i_{k\ell} = -\,\xi^\alpha_\ell\,\frac{\partial\xi^i_\alpha}{\partial x^k} = \xi^i_\alpha\,\frac{\partial\xi^\alpha_\ell}{\partial x^k} \quad \begin{matrix}(\alpha,\ell = 1\;\ldots\; q)\\ \text{sum over } \alpha\end{matrix} \tag{10}$$

and finds the integrability condition for Eq. (7a):

$$\frac{\partial\Gamma^L_k}{\partial x^\ell} - \frac{\partial\Gamma^L_\ell}{\partial x^k} + \left[\Gamma^L_k, \Gamma^L_\ell\right] = \partial^{rs}\Lambda_{rs\ell k} + \frac{\partial a_k}{\partial x^\ell} - \frac{\partial a_\ell}{\partial x^k} \tag{11}$$

here $\Gamma^L_k = \Gamma^L_\alpha\,\xi^\alpha_k$ and

$$\Lambda_{rs\ell k} = g_{ri}\left(\frac{\partial\Lambda^i_{sk}}{\partial x^\ell} - \frac{\partial\Lambda^i_{s\ell}}{\partial x^k} + \Lambda^j_{sk}\Lambda^i_{j\ell} - \Lambda^j_{s\ell}\Lambda^i_{jk}\right) \tag{11a}$$

The integrability condition can only be fulfilled if

$$\Lambda_{rs\ell k} + \Lambda_{sr\ell k} = 0 \tag{11b}$$

but this relation can be shown to be a consequence of G_r being a group of motion. [12]

Using Γ^L_k as connection in the spinning electron equation covariant w.r.t. G_r, one can also show that the current is conserved. In this spinning electron equation γ^k are replaced by γ^β, the generators of G_r expressed by the γ^n (if such generators can be formed) see Ref. 10, 9.

The two connections are related as follows:

$$\Gamma^L_\alpha - \Gamma_k \xi^k_\alpha = \frac{1}{8} \sigma^{ik}\left(\frac{\partial}{\partial x^i} \xi_{\alpha k} - \frac{\partial}{\partial x^k} \xi_{\alpha i}\right) \tag{12}$$

In case of generalized Lorentz and rotation groups, the last term has itself the form of a generator in spin space and the contribution of such a term to the Dirac equation is thus only a mass term of order of magnitude determined by the inverse of the radius of the homogenous space, which one may identify with the universe.

IV. A THEOREM ON GROUP TRAJECTORIES AND GEODESICS. SPECIALIZATION ON THE GROUP COVARIANT LAW OF MOTION

The following theorem is a generalization of theorems which apply to the geometry of the group space [12]. It applies here to the invariant variety of our intransitive group.

Theorem

The trajectory of the symbol ξ^i_α of a group G_r on its minimal invariant variety V_q coincides with a geodesic of the Riemannian metric on V_q for which G_r is a group of motion, iff on every one of its points, there exist q linear independent symbols ξ^i_δ (e.g. $\delta = 1 \dots 4$) (one of which may be ξ^i_α itself) such that:

$$\left[\xi_\alpha, \xi_\delta\right]^i g_{ik}\xi^k_\alpha = c^\gamma_{\alpha\delta}\xi^i_\gamma g_{ik}\xi^k_\alpha = 0 \quad (\gamma = 1 \dots r,\ \delta = 1 \dots q) \tag{13}$$

Proof

Every symbol fulfills Killing's equation (2).

$$\frac{\partial}{\partial x^\ell} g_{ik}\xi^\ell_\delta + g_{\ell k}\frac{\partial \xi^\ell_\delta}{\partial x^i} + g_{i\ell}\frac{\partial \xi^\ell_\delta}{\partial x^k} = 0$$

contracting the indices i and k with ξ_α and making use of Eq. (13).

$$g_{i\ell}\frac{\partial \xi^\ell_\delta}{\partial x^k} \xi^k_\alpha \xi^i_\alpha = g_{i\ell}\frac{\partial \xi^\ell_\alpha}{x^k} \xi^k_\delta \xi^i_\alpha$$

results in

$$\frac{\partial}{\partial x^\ell}\left(g_{ik}\xi^i_\alpha\xi^k_\alpha\right)\xi^\ell_\delta = 0 \tag{14}$$

so that all derivatives of $g_{ik}\xi^i_\alpha \xi^k_\alpha$ vanish on the trajectory. One can choose the parameter for the trajectory such that $\dot{x}^k = \xi^k_\alpha(x)$ and thus $\ddot{x}^k = \frac{\partial}{\partial x^\ell}\left(\xi^k_\alpha\right)\xi^\ell_\alpha$ furthermore

$$\xi^\ell_\alpha \xi^m_\alpha \left\{ {k \atop \ell\ m} \right\} = \frac{1}{2} g^{ki}\left(2 \frac{\partial}{\partial x^m} g_{\ell i} - \frac{\partial}{\partial x^i} g_{\ell m}\right)\xi^\ell_\alpha \xi^m_\alpha$$

the right hand side because of Eq. (14) equals

$$g^{ki}\left(\frac{\partial}{\partial x^m} g_{\ell i}\, \xi^\ell_\alpha \xi^m_\alpha + g_{\ell m}\xi^\ell_\alpha \frac{\partial}{\partial x^i}\, \xi^m_\alpha\right)$$

which because of Killing's equation (2) equals $-\frac{\partial}{\partial x^\ell}\left(\xi^k_\alpha\right)\xi^\ell_\alpha$ such that the equation of the geodesic for our parameter:

$$\ddot{x}^k + \left\{ {k \atop \ell\ m} \right\} \dot{x}^\ell \dot{x}^m = 0$$

is fulfilled.

What is the law of motion of a macroscopic body on this manifold? Gürsey [17] in his review article on the DeSitter group points out that all the geodesics in this case are also trajectories of the group and he conjectures the motion along a time like geodesic.

One should request also for the general case that all the time like geodesics be identical or at least very well approximated by group trajectories. But even in case of the DeSitter group there are other trajectories (which do not correspond to maximal circles). Must we exclude such trajectories from the law of motion? I don't think so if we consider invariant varieties as large as the universe! A motion which does not approximate well the analog of a maximal circle as trajectory in DeSitter space for example, may be extremely rare for statistical reasons—just as we rarely can find a macroscopic system violating the second fundamental theorem of thermodynamics. The phase space of a trajectory approximating a circle of radius comparable to our solar system would be so much smaller than that approximating the radius of the Universe!

Does not the wave equation on the invariant variety constructed according to the rules outlined in Section II take account of all the degrees of freedom of the group? A detailed discussion of this conjecture will be presented in a separate publication.

V. GENERALIZATION OF THE FORMALISM AND THE GRAVITATIONAL FIELD

The formalism used is covariant w.r.t. linear transformations of the base vectors which are independent of the points of V_n. The covariance can be extended to point dependent linear transformations $\xi^i_\alpha = S^\beta_\alpha \xi^i_\beta$, by adding to the derivatives in a well known way a term linear in the derived entity for every base index. Replace for example:

$$\frac{\partial \xi^i_\alpha}{\partial x^k} \text{ by } \xi^i_{\alpha\cdot k} = \frac{\partial \xi^i_\alpha}{\partial x^k} + A^\beta_{\alpha k}(x)\xi^i_\beta \tag{15}$$

where $A^\beta_{\alpha k}$ transforms inhomogenously:

$$A'_k = SA_k S^{-1} - \frac{\partial S}{\partial x^k} S^{-1} \tag{16}$$

$\xi^i_{\alpha\cdot k}$ transforms then like ξ^i_α.

The introduction of the potentials $A^\beta_{\alpha k}$ allows to formulate Lie derivatives and Lie brackets etc. again covariantly e.g. for a vector B:

$$B^i_{\cdot\alpha} = \frac{\partial B^i}{\partial x^\ell} \xi^\ell_\alpha - \xi^i_{\alpha\cdot\ell} B^\ell \tag{17}$$

The gauge field F_{ik} transforms homogenously:

$$F_{ik} = \frac{\partial}{\partial x^i} A_k - \frac{\partial}{\partial x^k} A_i + [A_i, A_k], \quad F'_{ik} = SF_{ik}S^{-1} \tag{17a}$$

We can now restrict our considerations from V_n to V_q excluding nonvanishing derivatives and tensor components perpendicular to it. F_{ik} vanishes as long as the symmetry of G_r acting in V_q is not broken. Irrespective of this, a regular transformation $S(x)$ exists such that for every nonsingular point in a given coordinate system the base vectors become:

$$\xi^i_\alpha = \delta^i_\alpha \ (\alpha=1 \ \dots \ q=4) \quad \xi^i_\beta = 0 \ (\beta=q+1 \ \dots \ r) \tag{18}$$

Then

$$\xi^i_{\alpha\cdot\beta} = [\xi_\alpha, \xi_\beta] = A^\varepsilon_{\alpha\ell}\delta^i_\varepsilon\delta^\ell_\beta - A^\varepsilon_{\beta\ell}\delta^i_\varepsilon\delta^\ell_\alpha = c^\varepsilon_{\alpha\beta}\delta^i_\varepsilon \tag{18a}$$

Notice that $c^\varepsilon_{\alpha\beta}$ is here point dependent and cannot even be transformed into the structure constant of the group if $F_{ik} \neq 0$. One finds for example in our gauge:

$$\xi^i_{\alpha\cdot\beta\cdot\gamma} - \xi^i_{\alpha\cdot\gamma\cdot\beta} = c^\varepsilon_{\alpha\beta}c^\phi_{\varepsilon\gamma}\delta^i_\phi + c^\varepsilon_{\alpha\beta\cdot\gamma}\delta^i_\varepsilon \tag{18b}$$

$$c^\varepsilon_{\alpha\beta\cdot\gamma} = \delta^\ell_\gamma\left(\frac{\partial}{\partial x^\ell} c^\varepsilon_{\alpha\beta} + A^\phi_{\alpha\ell}\, c^\varepsilon_{\phi\beta} + A^\phi_{\beta\ell}\, c^\varepsilon_{\alpha\phi} - A^\varepsilon_{\phi\ell}\, c^\phi_{\alpha\beta}\right) \qquad \text{see } [14]$$

$$\xi^i_{\alpha\cdot[\beta\cdot\gamma\cdot\phi]} = 0 \text{ etc.} \tag{18c}$$

$\gamma_{\alpha\beta} = c^{\varepsilon}_{\alpha\phi}\, c^{\phi}_{\beta\varepsilon}$ is generalized correspondingly so that $g^{ik} = \xi^{i}_{\alpha}\, \gamma^{\alpha\beta}\, \xi^{k}_{\beta}$ is no more the metric induced originally in V_q. g_{ik} fulfills however, the generalized Killing equations—in our gauge:

$$\frac{\partial g_{ik}}{\partial x^{\ell}}\,\delta^{\ell}_{\alpha} + \left(g_{\ell k} A^{\beta}_{\alpha i} + g_{i\ell}\, A^{\beta}_{\alpha k}\right)\delta^{\ell}_{\beta} = 0 \tag{18d}$$

Care has to be taken in case of coordinate change when the base vectors have no more the values of Eq. (18).

The Lagrangians are gauge independent and we have seen that a gauge exists in which $\mathscr{L} = \sqrt{g}\, F^{\alpha}_{\beta ik} F^{\beta}_{\alpha ik}$ and the matter part can be expressed exclusively in terms of the potentials A_k and the matter fields. The metric needs not to appear. Variation w.r.t. the potentials gives field equations with second derivatives only and the matter currents as source. The gauge field has taken over the role of the symmetry breaking which used to be expressed by the metric, when a nonhomogenous distribution of sources or gravitational waves occur.

An example is the case of the DeSitter group where the principle of equivalence was shown to remain valid. [9] A consistent application to the presence of matter may require the adoption of the authors view on the physical significance of group trajectories. [11]

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