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GRAVITATION AS BROKEN GROUP SYMMETRY*

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ABSTRACT

General relativistic theories of the gravitational field are derived from a generalization of the formalism of continuous groups of transformations with nonintegrable phase factors.

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The authors who developed the integral formalism of gauge theory[1,2] had suggested also a gauge approach to gravitational theory based on the parallel displacement of vectors, with the general linear group as gauge group and the Christoffel connection as potential.[2] The present author had suggested another approach based on the parallel displacement of spinors with the spinor connection and its trace as the potentials of the gravitational and electromagnetic fields.[3] This approach has the attractive feature to formally unify the two fields with the invariance group of the generally covariant Dirac equation as gauge group (GL4C in Bargmann's formalism[7] or $SL2C \times U1$ in the more conventional tetrade formalism).

The gauge field is in every one of the mentioned approaches related to the Riemann tensor so that the term quadratic in $R_{\mu\nu\rho\sigma}$ occurs in the Lagrangian. The variation in Ref. 2 is performed independently for the gauge fields and not for the metric tensor which contracts their indices, so that equations with third derivatives of the metric tensor result. Considering the universality of the gravitational interaction (and self-interaction), in Ref. 3 the metric tensor is varied together with the potential. The resulting equations in both cases yield all the vacum solutions of general relativity. (A well known difficulty associated with the quadratic Lagrangians in the presence of matter was there avoided in Ref. 3.)

The present paper presents a more general approach in which the gauge formalism is applied to the invariance group of motion of unperturbed space. The group formalism is thus generalized to the presence of gauge fields which break the symmetry of the group.[4] The theory can be developed without a general metric of space-time but the metric tensor is expressible in terms of the potentials. These features lend further support to the method of variation adopted in Ref. 3. The gauge field in case of general groups of motion is no more the Riemann tensor; it vanishes in the unperturbed space, which is in general not flat.

Consider a continuous group G_r of transformations,[4] acting on a n-dimensional manifold V_n. The rank of the matrix of base vectors $\left(\xi^i_\alpha\right)$ ($i = 1\ldots n$, $\alpha = 1\ldots r$), is $q = 4<n<r$ so that G_r is intransitive and minimal invariant varieties V_4 exist. The existence of a metric in V_n can be assumed for which G_r is a group of motions (in most models the n-dimensions metric is flat). G_r is then also a group of motions for the imbedded V_4 with the induced metric. One V_4 of suitable extensions (and signature) may be chosen as the homogenous space-time of a universe. The base vectors of G_r expressed in coordinates on V_4 fulfill the Killing equations:

$$\frac{\partial g_{ik}}{\partial x^\ell}\,\xi^\ell_\alpha + g_{i\ell}\frac{\partial \xi^\ell_\alpha}{\partial x^k} + g_{\ell k}\frac{\partial \xi^\ell_\alpha}{\partial x^i} = 0 \tag{1}$$

g_{ik} may be expressed for semisimple G_r by:

$$g^{ik} = \xi^i_\alpha\,\gamma^{\alpha\beta}\xi^k_\beta, \qquad \gamma_{\alpha\beta} = c^\phi_{\alpha\varepsilon}c^\varepsilon_{\beta\phi}, \qquad \gamma_{\alpha\varepsilon}\,\gamma^{\varepsilon\beta} = \delta^\beta_\alpha$$
$$c^\gamma_{\alpha\beta} = \text{structure const. of } G_r \tag{2}$$

Group invariant Lagrangians can thus be formed from $\gamma^{\alpha\beta}$ and the Lie derivatives of tensor fields on V_q, e.g., for a scalar field ϕ:

$$L = \frac{1}{2}\,\gamma^{\alpha\beta}\,\xi^i_\alpha\left(\frac{\partial}{\partial x^i}\,\phi^*\right)\,\xi^k_\beta\left(\frac{\partial}{\partial x^k}\,\phi\right) \tag{3}$$

A Lie derivative of spinors has also been defined in this context.[4]

The formalism used is covariant w.r.t. linear transformations of the space of base vectors independent of the points of V_4. Covariance can formally be extended to point dependent transformations $\xi^i_\alpha = S^\beta_\alpha(x)\;\xi^i_\beta$

by a potential. Replacing:

$$\frac{\partial \xi^i_\alpha}{\partial x^k} \text{ by } \xi^i_{\alpha\cdot k} = \frac{\partial \xi^i_\alpha}{\partial x^k} + A^\beta_{\alpha k}(x)\, \xi^i_\beta \tag{4}$$

The potential A_k and field F_{ik} transform in a well known way:

$$A'_k = SA_kS^{-1} - \frac{\partial S}{\partial x^k} S^{-1}, \qquad F^1_{ik} = SF_{ik}S^{-1}$$

$$F_{ik} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k} + \left[A_i, A_k\right] \tag{4a}$$

As long as $F_{ik}=0$ the development is formal and the symmetry of the group is not broken.

For $F_{ik}\neq 0$ the symmetry of the group action on V_4 is broken and non-homogenous gravitational fields are present but the formalism remains unaltered, e.g., The Lie derivative of a vector B:

$$B^i_{\cdot\alpha} = \left[B, \xi_\alpha\right]^i = \frac{\partial B^i}{\partial x^\ell}\, \xi^\ell_\alpha - \xi^i_{\alpha\cdot\ell}\, B^\ell \tag{5}$$

For any given coordinate system there exists an $S^\alpha_\beta(x)$ such that at every regular point of V_4 the base vectors are:

$$\xi^i_\alpha = \delta^i_\alpha \quad (\alpha=1\ldots4), \qquad \xi^i_\beta = 0 \quad (\beta=q+1\ldots r)$$

$$(i=1\ldots4) \tag{6}$$

Then for example

$$\xi^i_{\alpha\cdot\beta} = \left[\xi_\alpha, \xi_\beta\right]^i = A^\varepsilon_{\alpha\ell}\delta^i_\varepsilon\delta^\ell_\beta - A^\varepsilon_{\beta\ell}\delta^i_\varepsilon\delta^\ell_\beta = c^\varepsilon_{\alpha\beta}\delta^i_\varepsilon \tag{6a}$$

$c^\varepsilon_{\alpha\beta}$ is now point dependent and so is $\gamma_{\alpha\beta} = c^\varepsilon_{\alpha\phi}c^\phi_{\beta\varepsilon}$; $g^{ik} = \xi^i_\alpha\gamma^{\alpha\beta}\xi^k_\beta$ is no more the metric induced originally in V_4 but the generalized Killing equations are fulfilled; in our special gauge they are:

$$\frac{\partial g_{ik}}{\partial x^{\ell}}\,\delta^{\ell}_{\alpha} + \left(g_{\ell k}A^{\beta}_{\alpha i} + g_{i\ell}A^{\beta}_{\alpha k}\right)\delta^{\ell}_{\beta} = 0 \qquad (6b)$$

and the field Lagrangian $\mathscr{L} = \sqrt{g}\, F_{ik\beta}^{\ \ \ \alpha}\, F^{ik\beta}_{\ \ \ \alpha}$ can in the special gauge be expressed exclusively in terms of the potentials A_k without that the metric appears. Variation w.r.t. A_k results in field equations with second derivatives and with the matter current (where g_{ik} is also expressed by A_k) as source. The gauge field has taken over the role of the symmetry breaking previously reserved to the metric.

An example is provided by the DeSitter group, a G_{10} acting on a V_5 where the principle of equivalence was shown to remain valid.[6] A detailed exposition of the subject will be submitted shortly.

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