SLAC-PUB-2162
July 1978
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HIGHER ORDER INSTANTON EFFECTS*

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ABSTRACT

We present a formalism for calculating higher order instanton effects based on a systematic expansion of the functional integrals about multiple instanton-antiinstanton configurations. We consider how various correlation functions may be constructed from the determinants and propagators in a background multiple instanton field. A multiple scattering formalism is then developed in order to express these determinants and propagators for a multi-instanton field in terms of the basic single instanton quantities. The introduction of an improved gluon propagator based on a different choice of zero mode constraints is required in order to justify the multiple scattering formalism. As examples of this approach, we consider instanton interactions, which appear as corrections to the dilute gas approximation for the instanton density, and the first order quantum contributions to the static quark potential.

(Submitted to Physical Review D)

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*Work supported by the Department of Energy, contract no. EY-76-C-03-0515.

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I. INTRODUCTION

It is widely believed that instanton effects describe the dominant non-perturbative contributions to the small scale dynamics of QCD.[1] However, almost all discussion of instantons to date have relied upon the use of the lowest order dilute gas approximation (DGA). It is natural to consider higher order effects in order to verify the validity of the lowest order computations. Consideration of these corrections has been hampered by the lack of a convenient systematic procedure for expanding about multiple instanton field configurations. Previous attempts in this direction[2] have generally been incomplete in their treatment of these effects.

In this paper, we wish to present a formalism for calculating higher order corrections to the DGA. At the same time, we solve two problems of more immediate concern; the difficulties with the definition of the single instanton vector propagator[3] and the apparent appearance of an unexpected linear term in the quantum corrections to the static quark potential.[4]

The outline of this work is as follows. In Section II, we present a general approach for approximating the full quantum theory by the contributions of fluctuations about a set of classical field configurations. We then specialize to the instanton gas in Section III, and show how propagators and determinants in a multiple instanton field may be expanded in terms of single instanton quantities. This multiple scattering expansion requires that the effect of a single instanton be sufficiently localized about its position. Unfortunately, the gluon propagator found by Brown et al.[3] does not fall off rapidly enough for this to be true. We are, therefore, forced to reconsider the definition of the propagator with the aim of modifying its long distance behavior. This leads, in Section IV, to the construction of an improved propagator, which is based on redefining the zero mode constraints which always accompany classical fields. Our construction suffers from none of the ambiguities present in the result of Ref. 3. In Section V, we consider the interactions between instantons as an example of the multiple scattering formalism. Finally, in Section VI, we investigate the first order quantum corrections to the heavy quark potential and show that it only leads to a finite renormalization of the coupling constant. A summary of the results plus suggestions for future work comprises Section VII.

We have attempted to make the discussion as self-contained as possible. Sections II and III use some recent work on semi-classical expansions in gauge theories,[5] and some knowledge of that work will be useful in understanding our formalism. Some of the results have been mentioned elsewhere[6] but the complete analysis will be given here.

II. GENERAL APPROACH

We wish to consider the calculation of the vacuum expectation value of some operator $\mathcal{O}(A, \psi, \bar{\psi})$ composed of gauge and fermion fields. We have

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \left[\mathcal{D} A \mathcal{D} \psi \mathcal{D} \bar{\psi} \right] \quad \mathcal{O}(A, \psi, \bar{\psi}) \exp - S(A, \psi, \bar{\psi}) \quad (1)$$

where $\mathcal{Z}$ is the partition function, $\int \left[\mathcal{D} A \mathcal{D} \psi \mathcal{D} \bar{\psi} \right] \exp - S(A, \psi, \bar{\psi})$. This functional integral will be approximated by expanding about some set of field configurations $\left\{ A_0^z \right\}$, such as multiple instanton-antiinstanton configurations. (The superscript "z" represents the set of collective coordinates $\left\{ z_j \right\}$ needed to parameterize the chosen field configurations.)

In order to partition the functional integral into different sectors associated with each background field A_0^z, we must impose one constraint for each continuous collective coordinate on the fluctuation $\bar{A} \equiv A - A_0^z$. Thus, we choose a set of constraint functions $\{f_j\}$ and demand

$$\int d^4x \left(A - A_0^z\right) f_j = 0 \tag{2}$$

as a requirement that the field A describe a fluctuation of the background field A_0^z. In order to fix the corresponding collective coordinate, each constraint f_j must have a non-zero overlap with the corresponding background field deformation $\frac{\partial}{\partial z_j} A_0^z$.

Following the method of Reference 5, the functional integral (1) may now be expressed as[7]

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int dz \Bigg\{ \int [\mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar{\psi}]\, \delta\left(D\left(A_0^z\right)\left(A-A_0^z\right)\right) \det\left(-D\left(A_0^z\right)D(A)\right) \cdot \delta\left(\int \left(A-A_0^z\right) f_j\right) (\det J)\, \mathcal{O}(A,\psi,\bar{\psi}) \exp - S(A,\psi,\bar{\psi}) \Bigg\} \tag{3}$$

Here $\int dz$ represents the appropriate collective coordinate integrations; and a similar expression for the partition function is understood. The collective coordinate Jacobian J is given by[5]

$$J_{jj'} = \int d^4x \left\{ \left(\frac{\partial}{\partial z_j} A_0^z\right) f_{j'} - \left(A - A_0^z\right) \frac{\partial}{\partial z_j} f_{j'} - \left(D(A) \frac{\partial}{\partial z_j} A_0^z\right) \left(-D(A) D\left(A_0^z\right)\right)^{-1} \left(D(A) f_{j'}\right) \right\} \tag{4}$$

Evaluated at $A = A_0^z$, the Jacobian becomes

$$J_{jj'}\left(A_0^z\right) = \int d^4x \left\{ \widetilde{\frac{\partial}{\partial z_j} A_0^z} \cdot \widetilde{f_{j'}} \right\} \tag{5}$$

where $\tilde{f}_j$ is the constraint f_j orthogonalized to the local gauge fluctuations $D\left(A_0^z\right)\Lambda$; that is, $\tilde{f}_j = f_j + D\left(A_0^z\right)\left(-D^2\left(A_0^z\right)\right)^{-1}\left(D\left(A_0^z\right) f_j\right)$, and $\widetilde{\partial_{z_j} A_0^z}$ is similarly the deformation $\partial_{z_j} A_0^z$ placed in background gauge. (Note that since we are only interested in perturbative expansions, the expression (3) omits the step functions limiting the space of fluctuations discussed in Reference 5.)

Exponentiating the delta function constraints, rewriting the Faddeev-Popov determinant in terms of ghost fields, and expanding the action S about $A = A_0^z$, we obtain

$$\langle \mathscr{O} \rangle = \frac{1}{\mathscr{Z}} \int dz \Big\{ \int [\mathscr{D}\bar{A}\mathscr{D}\psi\mathscr{D}\bar{\psi}\mathscr{D}c\mathscr{D}\bar{c}](\det J)\,\mathscr{O}(A_0^z + \bar{A},\psi,\bar{\psi})$$
$$\cdot \exp - \int d^4x \Big[\mathscr{L}\left(A_0^z\right) + \mathscr{J}^\nu \bar{A}^\nu + \tfrac{1}{2}\bar{A}^\mu \mathscr{M}^{\mu\nu}\bar{A}^\nu + \frac{1}{2\beta}\bar{A}^\mu f_j^\mu \left(\int d^4y\, f_j^\nu \bar{A}^\nu\right) \qquad (6)$$
$$+ \bar{\psi}\left(\not{D}\left(A_0^z\right) + m\right)\psi - \bar{c}\, D^2\left(A_0^z\right)c + \mathscr{L}_I\,(\bar{A},\psi,\bar{\psi},c,\bar{c})\Big]\Big\}$$

where

$$\mathscr{L}\left(A_0^z\right) = \frac{1}{4}\left(F^{\mu\nu}\left(A_0^z\right)\right)^2$$

$$\mathscr{J}^\nu = - D^\mu\left(A_0^z\right)F^{\mu\nu}\left(A_0^z\right)$$

$$\mathscr{M}^{\mu\nu} = - D^2\left(A_0^z\right)\delta^{\mu\nu} + \left(1 - \frac{1}{\xi}\right) D^\mu\left(A_0^z\right)D^\nu\left(A_0^z\right) - 2g\, \underset{\sim}{F}^{\mu\nu}\left(A_0^z\right)$$

$$\mathscr{L}_I = g\left(D^\mu\left(A_0^z\right)\bar{A}^\nu\right)\left[\bar{A}^\mu,\bar{A}^\nu\right] + \frac{g^2}{4}\left(\left[\bar{A}^\mu,A^{-\nu}\right]\right)^2 + g\bar{\psi}\,\bar{\not{A}}\,\psi + g\left(D^\mu\left(A_0^z\right)\bar{c}\right)\underset{\sim}{\bar{A}}^\mu\, c$$

and

$$J_{jj'} = \int d^4x \left\{ \left(\frac{\partial}{\partial z_j} A_0^z\, f_{j'}\right) - \bar{A}\frac{\partial}{\partial z_j} f_{j'} - \left(D\left(A_0^z + \bar{A}\right)\frac{\partial}{\partial z_j} A_0^z\right)\bar{c}\left[\int d^4y\, c\left(D\left(A_0^z + \bar{A}\right)f_{j'}\right)\right]\right\}$$

(A wiggly underline indicates that the object is to be considered as a matrix in the adjoint representation, e.g., $(\underset{\sim}{F}^{\mu\nu})_{ij} = f_{ikj}\, F_k^{\mu\nu}$. Lim $\beta \to 0$ is to be understood throughout.)

We are now interested in expanding the functional integral in terms of the fluctuation $\bar{A} \equiv A - A_0^z$, or equivalently, within each sector performing a perturbative expansion in the coupling g. Since the background field A_0^z is not necessarily an exact solution (or even a constrained solution), the first variation $\mathscr{J} = - D\cdot F$ need not be zero. As discussed in Ref. 5, this is acceptable so long as the point we expand about, A_0^z, is within the region covered by Gaussian fluctuations about the true constrained minimum. This means that the background field A_0^z must approach a true solution as g tends to zero; this is possible so long as in some limit of the collective coordinates $\{z\}$ the background field A_0^z becomes an exact (or constrained) minimum. (All we are saying here is that the appropriate $g \to 0$ limit is not generally $g \to 0$, z fixed, but rather involves some choice of collective coordinates z(g) such that, as $g \to 0$ A_0^z becomes a true minimum. This point will be illustrated by the specific example of multiinstanton configurations in the next section.) The end result of all this is that the first variation $\mathscr{J}^\nu$ cannot be $O(1/g)$ as might have been expected, but rather must be no more than $O(1)$ in order to obtain a valid expansion. Thus we have $A_0^z \sim O(\frac{1}{g})$, $F^{\mu\nu}\left(A_0^z\right) \sim O(\frac{1}{g})$, $S\left(A_0^z\right) \sim O\left(\frac{1}{g^2}\right)$, $\mathscr{J}^\nu \sim O(1)$, $\mathscr{M}^{\mu\nu} \sim O(1)$, $\mathscr{L}_I \sim O(g)$, and $\bar{A} \sim O(1)$.

We now consider expanding the operator $\mathscr{O}$, the collective coordinate Jacobian J, and the interactions $\exp - \mathscr{L}_I$ in powers of g (or $\bar{A}$). The lowest order contribution to $\langle \mathscr{O} \rangle$ is given by

$$\langle \mathscr{O} \rangle_0 = \frac{1}{\mathscr{Z}} \int dz \Big\{ \int [\mathscr{D}\bar{A}\, \mathscr{D}\psi\, \mathscr{D}\bar{\psi}\, \mathscr{D}c\, \mathscr{D}\bar{c}]\; \mathscr{O}\left(A_0^z\right)\left(\det J\left(A_0^z\right)\right) \tag{7}$$

$$\exp - \int d^4x \left[S\left(A_0^z\right) + \mathscr{J}^\nu \bar{A}^\nu + \frac{1}{2}\bar{A}^\mu \hat{\mathscr{M}}^{\mu\nu}\, \bar{A}^\nu + \bar{\psi}\left(\not{D}\left(A_0^z\right) + m\right)\psi - \bar{c}\left(D^2\left(A_0^z\right)\right)c \right] \Big\}$$

$$= \int dz\; n_0^z\, \mathscr{O}\left(A_0^z\right) \Big/ \int dz\; n_0^z$$

where

$$\hat{\mathcal{M}}^{\mu\nu} = \mathcal{M}^{\mu\nu} + \frac{1}{\beta} f^{\mu}_{j} \times f^{\nu}_{j}$$

and

$$n_0^z = \exp -\left(S\left(A_0^z\right) - \frac{1}{2}\mathcal{J}\hat{\mathcal{M}}^{-1}\mathcal{J}\right)(\det\hat{\mathcal{M}})^{-\frac{1}{2}} \det\left(\not{D}\left(A_0^z\right) + m\right) \det\left(D^2\left(A_0^z\right)\right) \det\left(J\left(A_0^z\right)\right) \quad (8)$$

is the lowest order "density" for the A_0^z sector. (Note that if $\mathcal{O}$ contains fermion fields, then in the lowest order contribution (7) these would be tied together with fermion propagators in the background field A_0^z. The presence of such fermion terms will be ignored for simplicity.)

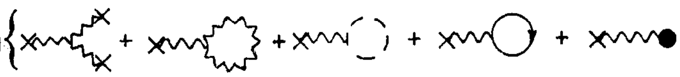
The first corrections to the above result are given by expanding either $\mathcal{O}\left(A_0^z + \bar{A}\right)$ or the interactions $\left(\det J\right) \exp \mathcal{L}_I$ to first order in g. Expanding $\mathcal{O}$, one finds the contribution

$$\langle\mathcal{O}\rangle_1 \subseteq \int dz \, \frac{\delta \mathcal{O}\left(A_0^z\right)}{\delta A_0^z} \frac{\delta}{\delta \mathcal{J}} n_0^z \Big/ \int dz \, n_0^z$$

$$= \int dz \, n_0^z \, \frac{\delta \mathcal{O}}{\delta A_0^z} \hat{\mathcal{M}}^{-1} \mathcal{J} \Big/ \int dz \, n_0^z \quad (9)$$

This term, suppressed by one power of g with respect to $\langle \mathcal{O} \rangle_0$, should really be considered as part of the "classical" result (7) since it merely shifts the background field that we expand about, A_0^z, toward the nearby constrained minimum given to this order by $A_0^z + \hat{\mathcal{M}}^{-1}\mathcal{J}$. Note that the corresponding first order shift in the action, from $S\left(A_0^z\right)$ to the minimal action $S\left(A_0^z + \hat{\mathcal{M}}^{-1}\mathcal{J}\right)$ is already contained in the lowest order density (8).

If we instead expand the interactions $\left(\det J\right) \exp \mathcal{L}_I$, then the contribution we obtain may be represented graphically as

$$\langle \mathcal{O} \rangle_1 \subset \int dz\ n_0^z \mathcal{O}\left(A_0^z\right)\Big\{$$  $$\Big\} \Big/ \Big/ \int dz\ n_0^z \qquad (10)$$

where

 is the classical shift $\mathcal{J}\hat{\mathcal{M}}^{-1}$

 is the quantum gluon propagator $\hat{\mathcal{M}}^{-1}$

 is the ghost propagator $\Delta = \left(-D^2\left(A_0^z\right)\right)^{-1}$

 is the fermion propagator $S = \left(\not{D}\left(A_0^z\right) + m\right)^{-1}$

 is the first order collective coordinate Jacobian vertex $\frac{\delta}{\delta A}$ det J

and the other vertices may be read off the interaction Lagrangian $\mathcal{L}_I$.

This term contains the second order shift in the action towards the minimal value, as well as the first order shift in the various determinants in the density (8) toward the true constrained minimum. These corrections clearly also appear in the denominator when we expand the partition function. Obviously, for an exact solution, all of these corrections vanish.

The next corrections, $O(g^2)$ w.r.t. $\langle \mathcal{O} \rangle_0$, are of several types. First, we may expand $\mathcal{O}$ to second order in $\bar{A}$. This clearly yields the following contribution to $\langle \mathcal{O} \rangle_2$,

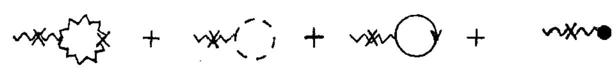
$$\langle \mathcal{O} \rangle_2 \subset \int dz \frac{\delta^2 \mathcal{O}\left(A_0^z\right)}{\delta A_0^z\ \delta A_0^z} \frac{\delta^2\ n_0^z}{\delta \mathcal{J}\ \delta \mathcal{J}} \Big/ \Big/ \int dz\ n_0^z$$
$$= \int dz\ n_0^z \frac{\delta^2 \mathcal{O}}{\delta A_0^z\ \delta A_0^z} \left(\hat{\mathcal{M}}^{-1} + \hat{\mathcal{M}}^{-1} \mathcal{J} \times \mathcal{J} \hat{\mathcal{N}}^{-1}\right) \Big/ \Big/ \int dz\ n_0^z \qquad (11)$$

corresponding to the second order classical shift along with the quantum propagator $\hat{\mathcal{M}}^{-1}$. This combination of both classical and quantum pieces will be referred to as the complete vector propagator $\mathcal{G}$;

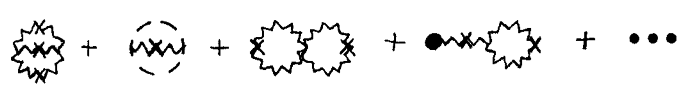
$$\mathcal{G} = \frac{1}{n_0^z} \frac{\delta^2 n_0^z}{\delta \mathcal{J} \delta \mathcal{J}} = \hat{\mathcal{M}}^{-1} + \hat{\mathcal{M}}^{-1} \mathcal{J} \times \mathcal{J} \hat{\mathcal{M}}^{-1} \qquad (12)$$

or graphically 

The second contribution to $\langle \mathcal{O} \rangle_2$ comes from expanding both $\mathcal{O}$ and the interaction (det J) exp $\mathcal{L}_I$ to first order. This yields the terms

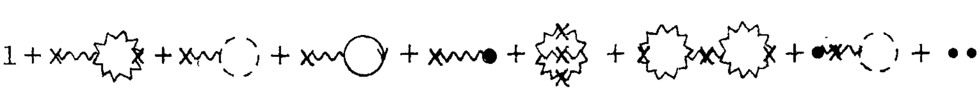
$$\langle \mathcal{O} \rangle_2 \subset \int dz\, n_0^z \frac{\delta \mathcal{O}}{\delta A_0^z} \Big\{$$  $$\Big\} \Big/ \int dz\, n_0^z \qquad (13)$$

Finally there are the terms coming from expanding the interactions to second order. This yields a variety of diagrams (in both the numerator and denominator) including, for example

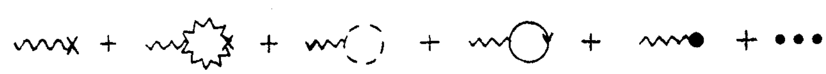
$$\langle \mathcal{O} \rangle_2 \subset \int dz\, n_0^z \mathcal{O} \Big\{$$  $$\Big\} \Big/ \int dz\, n_0^z \qquad (14)$$

These terms all consist of quantum corrections to the zero order density (8).

Examining all these corrections, we see that they may be conveniently organized if we define the full density n^z;

$$n^z = n_0^z \Big\{$$  $$\Big\} \qquad (15)$$

the one point function within each sector,

$$\langle \bar{A} \rangle^z = \Big\{$$  $$\Big\} \qquad (16)$$

and the sector two point function,

$\langle \bar{A}\,\bar{A} \rangle^{z} = \Big\{$  $+$ $+ \bullet\bullet\bullet \Big\}$ (17)

Then we see that an arbitrary vacuum expectation value $\langle \mathcal{O} \rangle$ may be computed as

$$\langle \mathcal{O} \rangle = \int dz\ n^{z} \left[\mathcal{O}\left(A_0^z\right) + \frac{\delta\, \mathcal{O}\left(A_0^z\right)}{\delta\ A_0^z} \langle \bar{A} \rangle^{z} + \frac{\delta^2 \mathcal{O}\left(A_0^z\right)}{\delta A_0^z \delta A_0^z} \langle \bar{A}\,\bar{A} \rangle^{z} + \bullet\bullet\bullet \right] \bigg/ \int dz\ n^{z} \tag{18}$$

This shows how the vacuum expectation value of any operator may be constructed from the density n^z and the sector n-point functions $\langle \bar{A}\ldots\bar{A} \rangle^{z}$. All essential information is contained in these basic objects.

III. MULTI-INSTANTON CONTRIBUTIONS

We now wish to apply the previous general formalism to the specific case of multiple instanton-antiinstanton field configurations. We choose the configurations that we will expand about to be pure superpositions of single instantons and antiinstantons,

$$A_\mu^{I} = \sum_{i=1}^{K} A_\mu^{e_i}\ (x_i,\ R_i,\ \rho_i) \tag{19}$$

Here K is the total number of instantons and antiinstantons in the superposition, e_i = + or - indicating an instanton or antiinstanton, and the single instanton field (in singular gauge) is given by

$$A_\mu^{\pm}\ (x_i,\ R_i, \rho_i) = \frac{2}{g}\, R^{i}_{ab}\ \frac{\eta^{\mp}_{b\mu\nu}\ (x - x_i)^{\nu} \rho_i^2 \left(\tau^a/2i\right)}{(x - x_i)^2 \left[(x - x_i)^2 + \rho_i^2\right]} \tag{20}$$

Each instanton is parameterized by a position x_i, scale size ρ_i, and group orientation R_i. The set of the collective coordinates will be denoted by $\left\{\Omega_\alpha^{(i)};\ 1 \le \alpha \le 4N\right\}$. (N here refers to color SU(N); there are 4N-5 group degrees of freedom.) The complete set of (continuous) collective coordinates for our multiinstanton field is thus given by

$$z = \left\{z_j;\ 1 \le j \le 4NK\right\} = \left\{\Omega_\alpha^{(i)}\right\}.$$

As discussed previously, we must choose a set of constraint functions $\left\{f_j\right\} = \left\{f_\alpha^{(i)}\right\}$, (one for each continuous collective coordinate) and impose $\int d^4x\ (A - A^I)f_j = 0$, as a requirement that the field A describe a fluctuation of the multiinstanton field A^I. The constraints $f_\alpha^{(i)}$ are naturally chosen to be determined from a basic set of single instanton constraints;

$$f_\alpha^{(i)}(x;x_i,\ R_i,\ \rho_i,\ e_i) = R^i_{ab}\ f_\alpha^{e_i}(x - x_i/\rho_i)\ \frac{\tau^a}{2i} \tag{21}$$

The basic constraints $f^\pm$ are conventionally chosen to be given by the single instanton zero modes; $f_\alpha^\pm = \frac{\partial}{\partial\Omega_\alpha} A^\pm\ (\Omega_\alpha)$; however, we will find later that this is not the most convenient choice. Until then, the basic constraints $f_\alpha^\pm$ will be left unspecified.

Now, our simple superposition (19) is of course not an exact classical solution. Nor is it even the minimal action configuration under our choice of constraints. However, if we are to expand about our simple multiinstanton configuration (19) subject to simple linear constraints (21), then we must arrange it so that our configuration is within the region covered by Gaussian fluctuations about the true constrained minimum; a region of size O(1) not O(1/g). As mentioned previously, this may be accomplished by choosing a non-trivial limit for the collective coordinates as $g \to 0$. In the case at hand, we simply require that all distances between instantons become large as $g \to 0$. Specifically, we choose to impose $g \ln \vec{x}_i \to$ const. as $g \to 0$ so that all instanton separations become exponentially large and the configuration approaches an exact solution. (From the lowest order dilute gas results one knows that the average instanton separation in the ensemble is exponentially large as $g \to 0$; what we are doing here is to arrange the $g \to 0$ limit so that the same result is true within each sector of the ensemble individually.)

Of course, having chosen this limit, all effects sensitive to the density (or separations) of instantons become exponentially suppressed as $g \to 0$, and hence all higher order density corrections are formally negligible compared to the usual quantum corrections (higher order in g). However, in the same spirit that leads one to consider non-perturbative exponentially small effects in the first place, we will not immediately discard the higher order density corrections since they may, in fact, numerically dominate the perturbative corrections for some processes of interest. In effect, we treat the quantum coupling g and the density $\sim e^{-1/g}$ as the two parameters of a double series expansion.

We have seen how any correlation function of interest may be constructed from the density n^z and sector n-point functions $\langle \bar{A} \ldots \bar{A} \rangle^z$. These objects are in turn computed from the determinants and propagators in the chosen background field. We would now like to examine how these quantities for a multiinstanton field may be expressed in terms of the basic single instanton quantities involved.

We first note the following relations:

$$A_\mu^I = \sum_i A_\mu^{(i)} \tag{22a}$$

$$F_{\mu\nu}^I = \sum_i F_{\mu\nu}^{(i)} + g \sum_{i \neq j} \left[A_\mu^{(i)},\ A_\nu^{(j)}\right] \tag{22b}$$

$$D_\mu(A^I)F_{\mu\nu}^I = g \sum_{i \neq j} \left[A_\mu^{(i)},\ F_{\mu\nu}^{(j)} + D_\mu(A^{(i)})A_\nu^{(j)} + \left[A_\mu^{(j)},\ A_\nu^{(i)}\right]\right] \tag{22c}$$

$$+ g^2 \sum_{i\neq j\neq k} \left[A_\mu^{(i)}, \left[A_\mu^{(j)},\ A_\nu^{(k)}\right]\right]$$

$$\left(F_{\mu\nu}^I\right)^2 = \sum_i \left(F_{\mu\nu}^{(i)}\right)^2 + \sum_{i \neq j} F_{\mu\nu}^{(i)} F_{\mu\nu}^{(j)} + 4g \sum_{i \neq j} F_{\mu\nu}^{(i)} \left[A_\mu^{(i)}, A_\nu^{(j)}\right] \tag{22d}$$

$$+ 2g \sum_{i\neq j\neq k} F_{\mu\nu}^{(i)} \left[A_\mu^{(j)}, A_\nu^{(k)}\right] + g^2 \left(\sum_{i \neq j} \left[A_\mu^{(i)}, A_\nu^{(j)}\right]\right)^2$$

$$\int \left(F_{\mu\nu}^I\right)^2 = \sum_i \int \left(F_{\mu\nu}^{(i)}\right)^2 - \sum_{i \neq j} \int \mathscr{F}_{\mu\nu}^{(i)} \mathscr{F}_{\mu\nu}^{(j)} - g^2 \sum_{i \neq j}$$

$$\int \left[A_\mu^{(i)},\ A_\nu^{(i)}\right]\left[A_\mu^{(j)},\ A_\nu^{(j)}\right] + \tag{22e}$$

$$+ 2g \sum_{i\neq j\neq k} \int F_{\mu\nu}^{(k)} \left[A_\mu^{(i)},\ A_\nu^{(j)}\right] + g^2 \int \left(\sum_{i \neq j} \left[A_\mu^{(i)}, A_\nu^{(j)}\right]\right)^2$$

$\mathscr{F}^{(i)}_{\mu\nu}$ is the Abelian part of $F^{(i)}_{\mu\nu}$: $\mathscr{F}^{(i)}_{\mu\nu} = \partial_\mu A^{(i)}_\nu - \partial_\mu A^{(i)}_\nu$. Relation (22e) follows from judiciously integrating by parts and is quoted from C. Bernard, ref. 2.

Next, we discuss the multiple scattering formalism for fermions. This material has been largely covered previously;[8] however, we will briefly review it here since it furnishes the simplest example of the approach.

Consider the fermion propagator in a multiinstanton field,

$$S^I = \left(\not{D}\ (A^I) + m\right)^{-1} = \left((\not{\partial} + \not{A}^I + m\right)^{-1} \tag{23}$$

Defining $\sigma_f^{(i)} = S_0^{-1}(S^{(i)} - S_0)\ S_0^{-1}$, where S_0 is the free fermion propagator $(\not{\partial} + m)^{-1}$ and $S^{(i)}$ is the fermion propagator in a single instanton field $A^{(i)}$, we may expand S^I as

$$S^I = S_0 + \sum_i S_0\sigma^{(i)}S_0 + \sum_{i \neq j} S_0\sigma^{(i)}S_0\sigma^{(j)}S_0 + \sum_{\substack{i \neq j \\ j \neq k}} S_0\sigma^{(i)}\ S_0\sigma^{(j)}\ S_0\sigma^{(k)}S_0 + \ldots \tag{24}$$

This expression may either be verified directly, or else easily derived by formally expanding S^I in terms of single instanton field insertions on free propagators, and then resumming the insertion of an individual instanton into $\sigma^{(i)}$.

Similarly, the determinant of $\left(\not{D}(A^I) + m\right)$ may be expanded as[9]

$$\det\left(\not{D}(A^I) + m\right) = \prod_i \det\left(\not{D}(A^{(i)}) + m\right)$$

$$\exp\text{-Tr}\left\{\sum_{i \neq j} S_0\sigma^{(i)}S_0\sigma^{(j)} + \sum_{\substack{i \neq j \\ j \neq k \\ k \neq i}} S_0\sigma^{(i)}S_0\sigma^{(j)}S_0\sigma^{(k)} + \ldots\right\} \tag{25}$$

These formulae obviously have a multiple scattering interpretation where the particle either freely propagates or else scatters off of a single instanton. The validity of these expressions (as asymptotic expansions in the instanton separations) presumably merely require that the "scattering amplitude" $\sigma^{(i)}$ be sufficiently localized so that the convolution integrals are finite. Note that the lowest loop in the determinant (25) is still divergent. This merely reflects the fact that a multiinstanton configuration with finite instanton separations must be renormalized at a different point than if all instantons were infinitely distant. Hence the determinant corrections (vacuum loops) require regularization and renormalization.

Next, for the scalars, we must consider

$$D^2(A^I) = \partial^2 + \sum_i \left(2g\partial \cdot \underset{\sim}{A}^{(i)} + g^2 \underset{\sim}{A}^{(i)}\underset{\sim}{A}^{(i)}\right) + g^2 \sum_{i \neq j} \underset{\sim}{A}^{(i)}\underset{\sim}{A}^{(j)} \qquad (26)$$

The complication here is the presence of the cross terms bilinear in the gluon fields. This means that the multiinstanton propagator and determinant cannot be expanded only in terms of the single instanton propagators; rather a double expansion in the usual non-local instanton vertex $\sigma^{(i)}$ as well as a local seagull vertex $V_s^{(ij)} \equiv g^2 \underset{\sim}{A}^{(i)}\underset{\sim}{A}^{(j)}$ must be used. Thus, proceeding as in the fermion case, we define $\sigma_s^{(i)} = \Delta_0^{-1}(\Delta^{(i)} - \Delta_0)\Delta_0^{-1}$ (where Δ_0 and $\Delta^{(i)}$ are the free and single instanton scalar propagators), and obtain

$$\Delta^I = \left(-D^2(A^I)\right)^{-1} = \Delta_0 + \sum_i \Delta_0\sigma^{(i)}\Delta_0 + \sum_{i \neq j}\left(\Delta_0\sigma^{(i)}\Delta_0\sigma^{(j)}\Delta_0 + \Delta_0 V^{(ij)}\Delta_0 + \ldots\right)$$

$$+ \sum_{\substack{i \neq j \\ j \neq k}} \left(\Delta_0\sigma^{(i)}\Delta_0\sigma^{(j)}\Delta_0\sigma^{(k)}\Delta_0 + \Delta_0\sigma^{(i)}\Delta_0 V^{(jk)}\Delta_0 + \Delta_0 V^{(ij)}\Delta_0\sigma^{(k)}\Delta_0 + \ldots\right)$$

$$+ \ldots \qquad (27)$$

Similarly, the determinant $\det\left(-D^2(A^I)\right)$ may be expanded as

$$\det\left(-D^2(A^I)\right) = \prod_i \det\left(-D^2(A^{(i)})\right)$$

$$\cdot \exp\text{-Tr}\left\{\sum_{i\neq j}\left(\Delta_0\sigma^{(i)}\Delta_0\sigma^{(j)} + \Delta_0 V^{(ij)} + \ldots\right)\right.$$

$$\left. + \sum_{\substack{i\neq j\\ j\neq k\\ k\neq i}}\left(\Delta_0\sigma^{(i)}\Delta_0\sigma^{(j)}\Delta_0\sigma^{(k)} + \Delta_0\sigma^{(i)}\Delta_0 V^{(jk)} + \ldots\right) + \ldots\right\} \tag{28}$$

Finally the vectors may be handled in much the same way as the scalars since the quadratic operator $\hat{\mathscr{M}}$ contains both single instanton and biinstanton terms. Here we have

$$-\hat{\mathscr{M}} = D^2(A^I)\delta^{\mu\nu} - \left(1-\frac{1}{\xi}\right)D^\mu(A^I)D^\nu(A^I) + 2g\,\underset{\sim}{F}^{\mu\nu}(A^I) - \frac{1}{2\beta}f_\mu^{(i)}\times f_\nu^{(i)}$$

$$= \left[\partial^2\delta^{\mu\nu} - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\right] + \sum_i\left\{\left(2g\partial\cdot\underset{\sim}{A}^{(i)} + g^2\,\underset{\sim}{A}^{(i)}\cdot\underset{\sim}{A}^{(i)}\right)\delta^{\mu\nu}\right. \tag{29}$$

$$\left. - \left(1-\frac{1}{\xi}\right)\left(g\left[\partial_\mu\underset{\sim}{A}_\nu^{(i)} + \underset{\sim}{A}_\mu^{(i)}\partial_\nu\right] + g^2\,\underset{\sim}{A}_\mu^{(i)}\underset{\sim}{A}_\nu^{(i)}\right) + 2g\underset{\sim}{F}^{\mu\nu}\left(A^{(i)}\right) - \frac{1}{2\beta}f_\mu^{(i)}\times f_\nu^{(i)}\right\}$$

$$+ \sum_{i\neq j} g^2\left\{\underset{\sim}{A}^{(i)}\underset{\sim}{A}^{(j)}\delta^{\mu\nu} - \left(1-\frac{1}{\xi}\right)\underset{\sim}{A}_\mu^{(i)}\underset{\sim}{A}_\nu^{(j)} + 2\left[A_\mu^{(i)}, A_\nu^{(j)}\right]\right\}$$

and exactly the same expansions as in the scalar case are valid if we replace the scalar quantities Δ_0, $\sigma_s^{(i)}$, and $V_s^{(i)}$ by the corresponding vector equivalents $G_0^{\mu\nu} = \left(\partial^2\delta^{\mu\nu} - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\right)^{-1}$, $\sigma_v^{(j)} = G_0^{-1}(G^{(i)} - G_0)G_0^{-1}$, and $V_v^{(i)} = g^2\left(\underset{\sim}{A}^{(i)}\underset{\sim}{A}^{(j)}\delta^{\mu\nu} - \left(1-\frac{1}{\xi}\right)\underset{\sim}{A}_\mu^{(i)}\underset{\sim}{A}_\nu^{(j)} + 2\left[A_\mu^{(i)}, A_\nu^{(j)}\right]\right)$.

Now, from the work of Brown, Carlitz, Creamer, and Lee,[3] the scalar and massless fermion single instanton propagators, $\Delta^{(i)}$ and $S^{(i)}$, are known and have sufficiently rapidly decreasing long distance behavior to justify the multiple scattering formulas (24) - (28). The single instanton vector propagator $G^{(i)}$ is known[3] for the case where the constraints $f^{(i)}$ are chosen to equal the zero modes, $\partial A^{(i)}/\partial\Omega$, themselves.

However, because the global gauge and dilation zero modes have a $1/r^3$ asymptotic behavior, the propagator is forced to have a $1/r$ tail. (It satisfies $\nabla^2 G^{(i)} = f^i \times f^i$ asymptotically [for $\xi = 1$].) This means that the integral of the propagator times a zero mode is logarithmically divergent. This in turn forces the propagator to contain an infinite amount of zero mode terms; $\ell n\ (\infty)\ \partial_\Omega A^{(i)} \times \partial_\Omega A^{(i)}$.

The $1/r$ tail of the BCCL propagator is a sufficiently slow fall-off that the convolution integrals involved in the vector multiple instanton expansions are divergent. Thus with this propagator it does not appear that a multiple scattering formalism is possible. Furthermore, the $1/r$ tail and the divergent pieces of the BCCL propagator are very inconvenient even for calculations in the one-instanton sector where they lead to spurious terms (such as $O(g^2)$ linear contributions to the heavy quark potential)[4] which should cancel in the final physical result.[6]

This motivates the desire to introduce an improved vector propagator, one which would be free of divergence and spurious long range tails. Since the constraints $f^{(i)}$ are actually our choice and need not equal the zero modes themselves, one might hope that by using constraints more localized than the zero modes an improved propagator would result. The problem is that we must be able to actually compute the propagator corresponding to a different set of constraints. What we must understand is how, given a propagator appropriate for one set of constraints, may we use it in order to construct the propagator appropriate for a different choice of constraints. The next section shows how the solution to this problem enables one to find an improved propagator.

IV. THE IMPROVED VECTOR PROPAGATOR

We would like to find an improved vector propagator based on a choice of constraints different from the zero modes. In order to easily find the effects of a change in the constraints, we will use the relations

$$(H + v_i \times v_i)^{-1} = H^{-1} - H^{-1} v_i \left(1 + (vH^{-1}v)\right)^{-1}_{ij} v_j \, H^{-1} \tag{30}$$

$$\det\,(H + v_i \times v_i) = (\det H) \det\left(1 + (vH^{-1}v)\right)_{ij} \tag{31}$$

which are valid for any invertible matrix H and an arbitrary set of vectors $\{v_i\}$. To apply these relations, we write the quadratic operator for the new set of constraints G_f^{-1} as

$$G_f^{-1} = \mathcal{M} + \frac{1}{\beta} f_z \times f_z = \left(\mathcal{M} + \frac{1}{\gamma}\phi_z \times \phi_z\right) + \left(\frac{1}{\beta} f_z \times f_z - \frac{1}{\gamma}\phi_z \times \phi_z\right) \tag{32}$$

$$= G_\phi^{-1} + v_i \times v_i$$

Here

$$\mathcal{M} = -D^2 \delta^{\mu\nu} + \left(1 - \frac{1}{\xi}\right) D^\mu D^\nu - 2g\, \underset{\sim}{F}^{\mu\nu},$$

ϕ_z will be taken to be the true zero modes $\partial A^{(i)}/\partial\Omega$

and

$$\{v_i\} = \left\{f_z/\sqrt{\beta},\; i\,\phi_z/\sqrt{\gamma}\right\}$$

G_ϕ, the propagator with zero mode constraints, is the BCCL propagator. Now, taking $\{\phi_z\}$ to be normalized zero modes, so that $G_\phi \phi_z = \gamma\phi_z$, we find

$$G_f = G_\phi - (G_\phi f_z)(\phi f)^{-1}_{zz'}\phi_{z'} - \phi_z (f\phi)^{-1}_{zz'}(f_{z'}G_\phi) +$$

$$+ \phi_z (f\phi)^{-1}_{zz'}(\beta + fG_\phi f)_{z'z''}(\phi f)^{-1}_{z''z'''}\phi_{z'''} \qquad (33)$$

$(f\phi)_{zz'}$ is just the overlap of the constraint f_z with the zero mode $\phi_{z'}$. In order that $\{f_z\}$ be an acceptable choice of constraints, this matrix must be nonsingular.

This result expresses the propagator for an arbitrary set of constraints in terms of the known BCCL propagator. Since the propagator G_f satisfies the differential equation $\mathcal{M} G_f = 1 - \hat{f}_z \times \hat{f}_z$, then as long as the new constraints fall faster than $1/r^3$ at large distance, the improved propagator G_f will have no $1/r$ long range tail. Similarly, one may easily verify from (33) that any amount of zero mode terms $\phi\times\phi$ contained in G_ϕ will be removed when constructing G_f. Thus for any choice of short-ranged constraints, the improved propagator G_f should suffer from none of the problems of the BCCL propagator.

The determinants used in the zero order density n_0^z are also modified by a change in constraints. Using (2), we find that the new determinant $\det\left(\mathcal{M} + \frac{1}{\beta} f\times f\right)$ is equal to the old determinant times a correction $\sim \left(\det\, (f\phi)_{zz'}\right)^2$. This correction is exactly canceled by the corresponding change in the collective coordinate Jacobian J, Eq. (5). Thus, changing the choice of constraints has no effect on the DGA instanton density n_0^z.

We would now like to explicitly examine our improved propagator for the case of a single instanton located at the origin. In order to conveniently construct a propagator with the best possible long distance behavior, we will choose our constraints to have the same tensor structure as the zero modes multiplied by some function $f(x^2)$ of compact support. (Choosing $f(x^2) = \delta(x^2 - \rho^2)$ so that the constraints are localized on spherical shells would be particularly simple.)

We begin by quoting the appropriate portions of the work of Brown et al.[3] The vector propagator satisfying the equation

$$\left(-D^2\delta^{\mu\nu} + \left(1 - \frac{1}{\xi}\right) D^\mu D^\nu - 2\underset{\sim}{F}^{\mu\nu}\right) G^{\nu\lambda} (x,y) = \\ = \delta^{\mu\lambda}\delta(x,y) - \hat{\phi}_z^\mu (x)\, \hat{\phi}_z^\lambda (y)$$

is given by

$$G^{BCCL}_{\mu\nu} = - \left(q_{\mu\nu\alpha\beta} + (\xi - 1)\ \delta_{\mu\alpha}\ \delta_{\nu\beta}\right) \vec{D}_\alpha \frac{1}{D^4} \overleftarrow{D}_\beta \tag{34}$$

where[10]

$$q_{\mu\nu\alpha\beta} = \delta_{\mu\nu}\ \delta_{\alpha\beta} + \eta^m_{\mu\nu}\eta^m_{\alpha\beta} = \frac{1}{2}\ \mathrm{tr}\ \left(\tau^+_\mu\ \tau_\nu\ \tau^+_\beta\ \tau_\alpha\right)$$

$1/D^4$ is the convolution of two isospin one scalar propagators. We find it convenient to work in regular gauge for part of the calculation; in this case, we have

$$- \frac{1}{D^2_{reg}} = \frac{\frac{1}{2}\ \mathrm{tr}\left(\tau_a\left(\tau^+_x\tau_y + \rho^2\right)\tau_b\left(\tau^+_y\ \tau_x + \rho^2\right)\right)}{4\pi^2\ (x - y)^2\ (x^2 + \rho^2)(y^2 + \rho^2)} \tag{35}$$

$1/D^4$ is thus given by

$$\frac{1}{D^4_{reg}} = \int d^4z\ \frac{\frac{1}{2}\ \mathrm{tr}\left(\tau_a\left(\tau^+_x\tau_z + \rho^2\right)\left(\tau^+_z\tau_y + \rho^2\right)\tau_b\left(\tau^+_y\tau_z + \rho^2\right)\left(\tau^+_z\tau_x + \rho^2\right)\right)}{\left(4\pi^2\right)^2\left(x^2 + \rho^2\right)\left(y^2 + \rho^2\right)\left(z^2 + \rho^2\right)^2 (x - z)^2\ (y - z)^2} \tag{36}$$

where we have used the relation $\frac{1}{2}\ \mathrm{tr}\ (A\tau_a)\ \frac{1}{2}\ \mathrm{tr}\ (\tau_a B) = \frac{1}{2}\ \mathrm{tr}\ (AB)$ provided either of the arbitrary 2 x 2 matrices A or B are traceless. (This follows from the more general relation that we will use several times, $\frac{1}{2}\,\mathrm{tr}\,(A\tau_\mu^+)\ \frac{1}{2}\,\mathrm{tr}\,(\tau_\mu B) = \frac{1}{2}\,\mathrm{tr}\,(AB)$ for any matrices A, B.)

The zero modes (in regular gauge) are as follows:

$$\begin{array}{llll} \text{global gauge:} & \phi_\mu^{a(b)} = 2\rho^2 \eta^a_{\sigma\mu}\,\bar\eta^b_{\sigma\nu}\ x^\nu/\left(x^2+\rho^2\right)^2 & \text{norm } 4\pi^2\rho^2 & \\ \text{dilatation:} & \phi_\mu^{a(0)} = 2\rho^2 \eta^a_{\mu\nu}\ x^\nu/\left(x^2+\rho^2\right)^2 & \text{norm } 4\pi^2\rho^2 & (37) \\ \text{translation:} & \phi_\mu^{a(\beta)} = 2\rho^3 \eta^a_{\mu\beta}\ /\left(x^2+\rho^2\right)^2 & \text{norm } 2\pi^2\rho^2 & \end{array}$$

We choose our constraints $\{f^z\}$ to be

$$\begin{array}{lll} \text{global gauge:} & f_\mu^{a(b)} = (2/\rho^2)\eta^a_{\sigma\mu}\bar\eta^b_{\sigma\nu}\ x^\nu\ f\ (x^2/\rho^2) & \\ \text{dilatation:} & f_\mu^{a(0)} = (2/\rho^2)\eta^a_{\mu\nu}\ x^\nu\ f(x^2/\rho^2) & (38) \\ \text{translation:} & f_\mu^{a(\beta)} = (2/\rho)\eta^a_{\mu\beta}\ (x^2/\rho^2)\ f\ (x^2/\rho^2) & \end{array}$$

The overlaps are given by

$$\int f^z\ \phi^{z'} = \delta^{zz'}\cdot 12\pi^2\rho^2\ Q$$

where

$$Q = \int_0^\infty dy\ f(y)/(1+1/y)^2$$

One easily finds that the sum of all normalized zero modes is given by

$$\hat\phi_\mu^z(x)\ \hat\phi_\nu^z(y) = \rho^2\,\frac{\frac{1}{2}\,\mathrm{tr}\left(\tau_a\ \tau_\mu^+\ \tau_\nu\tau_b\ \left(\tau_y^+\ \tau_x + 2\rho^2\right)\right)}{\pi^2\left(x^2+\rho^2\right)^2\left(y^2+\rho^2\right)^2} \qquad (39)$$

and similarly

$$-f_\mu^z(x)\,(\hat{\phi}f)^{-1}_{zz'}\,\hat{\phi}^{z'}_\nu = \frac{\frac{1}{2}\,\mathrm{tr}\left(\tau_a\tau_\mu^+\,\tau_\nu\tau_b\left(\tau_y^+\tau_x + x^2\right)\right)\,f\left(x^2/\rho^2\right)}{3\,\pi^2\left(y^2+\rho^2\right)^2\cdot Q\cdot\rho^2} \tag{40}$$

Unfortunately, the convolution integral (36) may not be easily evaluated analytically. However, for our purposes, it is sufficient to consider the two asymptotic limits, I: $x \gg y, \rho$ and II: $\rho \ll x,y$.

In the first limit, we find (see Appendix for details)

$$\frac{1}{D^4} = \frac{\frac{1}{2}\,\mathrm{tr}\left(\tau_a\tau_x^+\tau_y\tau_b\tau_y^+\tau_x\right)}{16\pi^2\,(x^2+\rho^2)(y^2+\rho^2)}\left[\ell n\left(\Omega^2/(x-y)^2\right) - \frac{\rho^2}{x^2}\;\ell n\;(x^2/\rho^2)\right]$$

$$+\frac{\frac{1}{2}\,\mathrm{tr}\left(\tau_a\left(\tau_x^+\tau_y+\rho^2\right)\tau_b\left(\tau_y^+\tau_x+\rho^2\right)\right)}{16\pi^2\,(x^2+\rho^2)(y^2+\rho^2)} + O\left(\frac{1}{x^2}\right) \tag{41}$$

Ω is a large distance cutoff needed to regulate the convolution. Thus the BCCL propagator in this limit is given by (we set $\xi = 1$ for simplicity)

$$G^{BCCL}_{\mu\nu,reg} = -q_{\mu\nu\alpha\beta}\,D_\alpha\,\frac{1}{D^4}\,D_\beta$$

$$\begin{aligned}
&= \delta_{\mu\nu}\,\tfrac{1}{2}\mathrm{tr}\left(\tau_a\tau_x^+\tau_y\tau_b\tau_y^+\tau_x\right)\Big/4\pi^2(x-y)^2(x^2+\rho^2)(y^2+\rho^2)\\
&\quad - \rho^2\,\tfrac{1}{2}\mathrm{tr}\left(\tau_a\tau_\mu^+\tau_\nu\tau_b\left(\tau_y^+\tau_x+\rho^2\right)\right)x^2/4\pi^2\left(x^2+\rho^2\right)^2\left(y^2+\rho^2\right)^2\\
&\quad + \rho^2\,\tfrac{1}{2}\mathrm{tr}\left(\tau_a\tau_x^+\tau_y\tau_\mu^+\tau_\nu\tau_b\tau_y^+\tau_x\right)/4\pi^2\left(x^2+\rho^2\right)^2\left(y^2+\rho^2\right)^2\\
&\quad + \rho^4\,\tfrac{1}{2}\mathrm{tr}\left(\tau_a\tau_\mu^+\tau_\nu\tau_b\tau_y^+\tau_x\right)\Big[3\;\ell n\left(\Omega^2/(x-y)^2\right)\\
&\quad + \;\ell n\left(x^2/\rho^2\right)\Big]\;/4\pi^2\left(x^2+\rho^2\right)^2\left(y^2+\rho^2\right)^2\\
&\qquad + O(1/x^3)
\end{aligned} \tag{42}$$

Here, the first term is just a gauge transform of the free propagator (up to the ρ^2 terms in the denominator). The second term contains the 1/r tail referred to previously and has the form $x^2\phi^z(x) \times \phi^z(y) \sim \rho^2/xy^3$. The last term contains the divergent amounts of global gauge and dilatation zero modes.[3]

To compute the behavior of the improved propagator in this limit, we need only consider the first two terms of (33) since the last two terms are automatically $O(1/x^3)$. Using the previously mentioned trace identities, one easily finds

$$\begin{aligned}
-G_\phi f(\phi f)^{-1}\phi = {} & \rho^2\, \tfrac{1}{2}\mathrm{tr}\left(\tau_a\tau_\mu^+\tau_\nu\tau_b\left(\tau_y^+\tau_x + \rho^2\right)\right)x^2 \,/4\pi^2\left(x^2+\rho^2\right)^2\left(y^2+\rho^2\right)^2 \\
& - \rho^4\, \frac{1}{2}\,\mathrm{tr}\left(\tau_a\tau_\mu^+\tau_\nu\tau_b\tau_y^+\tau_x\right)\Big[3\;\ln\left(\Omega^2/(x-y)^2\right) + \\
& + \ln\left(x^2/\rho^2\right)\Big]\,/4\pi^2\left(x^2+\rho^2\right)^2\left(y^2+\rho^2\right)^2 \\
& + O(1/x^3)
\end{aligned} \tag{43}$$

Thus the second and fourth terms of G^{BCCL} are canceled exactly, and for this limit of our improved propagator, we find (after transforming back to singular gauge)

$$\begin{aligned}
G^{ab}_{\mu\nu,\mathrm{sing}}(x,y) &= \frac{\delta_{\mu\nu}\;\delta^{ab}}{4\pi^2(x-y)^2\pi_x\pi_y} + \rho^2\,\frac{\tfrac{1}{2}\mathrm{tr}\left(\tau_a\tau_y\tau_\mu^+\tau_\nu\tau_y^+\tau_b\right)}{4\pi^2\;x^2y^2\;\pi_x^2\pi_y^2} + O(1/x^3) \\
&= \frac{\delta_{\mu\nu}\;\delta^{ab}}{4\pi^2(x-y)^2\pi_x\pi_y} + \rho^2\,\frac{\left(\delta_{\mu\nu}\;\delta^{ab}+\epsilon^{abc}\;\bar\eta^c_{\alpha\beta}I^y_{\alpha\mu}\;I^y_{\beta\nu}\right)}{4\pi^2\;x^2y^2\;\pi_x^2\;\pi_y^2} + O(1/x^3)
\end{aligned} \tag{44}$$

where $\pi_x = 1+\rho^2/x^2$, $\pi_y = 1+\rho^2/y^2$, and $I^y_{\alpha\beta} \equiv \delta_{\alpha\beta} - 2y^\alpha y^\beta/y^2$.

As expected, the divergent zero mode piece and the $1/r$ tail have been completely removed in constructing the improved propagator. The leading behavior of $G - G_0$ in this limit is $O(\rho^2/x^2y^2)$.

The second limit is more conveniently evaluated in singular gauge directly. In this case, we have

$$\frac{1}{D^4_{sing}} = \int d^4z\, \frac{1}{2}\mathrm{tr}\left[\tau_a\left(1+\frac{\rho^2}{x^2z^2}\,\tau_x\,\tau_z^+\right)\right.$$

$$\left(1+\frac{\rho^2}{y^2z^2}\,\tau_z\,\tau_y^+\right)\tau_b\left(1+\frac{\rho^2}{y^2z^2}\,\tau_y\,\tau_z^+\right)$$

$$\left.\left(1+\frac{\rho^2}{x^2z^2}\,\tau_z\tau_x^+\right)\right]\bigg/\left(\left(4\pi^2\right)^2\,\pi_x\pi_y\pi_z^2\,(x-z)^2\,(y-z)^2\right) \quad (45)$$

and one finds (see Appendix)

$$\frac{1}{D^4} = \frac{\delta^{ab}}{\left(16\pi^2\right)\pi_x\pi_y}\left[\ell n\left(\Omega^2/(x-y)^2\right) + 1 + \frac{\rho^2}{x^2}\,\ell n\left(y^2/(x-y)^2\right) + \right.$$

$$\left. + \frac{\rho^2}{y^2}\,\ell n\left(x^2/(x-y)^2\right)\right] + 2\,\frac{\rho^2}{4\pi^2}\,\varepsilon^{abc}\,\bar{\eta}^c_{xy}K + O(\rho^4) \quad (46)$$

where

$$K = \frac{\left(\vec{x}_\perp + \vec{y}_\perp\right)}{\left(x^2y^2 - (x\cdot y)^2\right)}\cdot\int d^4z\;\vec{z}/4\pi^2\; z^2\;(x-z)^2(y-z)^2$$

and

$$\vec{x}_\perp = \vec{x} - \frac{(x\cdot y)}{y^2}\,\vec{y},\quad \vec{y}_\perp = \vec{y} - \frac{(x\cdot y)}{x^2}\,\vec{x}$$

Unfortunately, K may not be easily evaluated analytically; however, one easily sees that $K \sim 1/x^2$ as $x \to \infty$.

The BCCL propagator is thus given in this limit by

$$G^{BCCL}_{\mu\nu,sing} = \frac{\delta_{\mu\nu}\,\delta^{ab}}{4\pi^2(x-y)^2} - \rho^2\,\delta^{ab}\,\frac{\frac{1}{2}\,tr\left(\tau^+_\mu\,\tau_\nu\tau^+_y\,\tau_x\right)}{4\pi^2\,x^2\,y^2}\left(\frac{1}{x^2}+\frac{1}{y^2}\right)$$

$$- \rho^2\varepsilon^{abc}\,\frac{\frac{1}{2}\,tr\left(\tau^+_\mu\tau_\nu\tau^+_\beta\tau_\alpha\right)}{4\pi^2\,(x-y)^2}\left[\bar{\eta}^c_{\alpha x}\,\frac{(x-y)^\beta}{x^4} + \bar{\eta}^c_{\beta y}\,\frac{(x-y)^\alpha}{y^4}\right]$$

$$+ \frac{2\rho^2}{4\pi^2}\,\varepsilon^{abc}\,\frac{1}{2}\,tr\left(\tau^+_\mu\tau_\nu\tau^+_\beta\tau_\alpha\right)\,\partial^x_\alpha\partial^y_\beta\;\bar{\eta}^c_{xy}\;K + O(\rho^4) \tag{47}$$

To calculate the correction terms needed to form the improved propagator (33) we must use the previous asymptotic limit of G^{BCCL}. Thus, we find (from (43))

$$-G_\phi f(\phi f)^{-1}\phi - \phi(f\phi)^{-1}fG_\phi = \rho^2\,\frac{\frac{1}{2}\,tr\left(\tau_a\tau_x\tau^+_\mu\tau_\nu\tau^+_y\tau_b\right)}{4\pi^2\,x^2y^2}\left(\frac{1}{x^2}+\frac{1}{y^2}\right) + O(\rho^4)$$

$$= \rho^2\delta^{ab}\,\frac{\frac{1}{2}\,tr\left(\tau^+_\mu\tau_\nu\tau^+_y\tau_x\right)}{4\pi^2\,x^2y^2}\left(\frac{1}{x^2}+\frac{1}{y^2}\right)$$

$$- \rho^2\varepsilon^{abc}\,\frac{1}{2}\,tr\,\frac{\left(\tau^+_\mu\tau_\nu\tau^+_\beta\tau_\alpha\right)}{4\pi^2\,x^2y^2}\left(\frac{\bar{\eta}^c_{\alpha x}y^\beta}{x^2} - \frac{\bar{\eta}^c_{\beta y}x^\alpha}{y^2}\right) + O(\rho^4) \tag{48}$$

and our improved propagator, in this limit, is given by

$$G^{ab}_{\mu\nu,sing}(x,y) = \frac{\delta_{\mu\nu}\,\delta^{ab}}{4\pi^2(x-y)^2} + \frac{2\rho^2}{4\pi^2}\,\varepsilon^{abc}\,q_{\mu\nu\alpha\beta}\,\partial^x_\alpha\partial^y_\beta\,\bar{\eta}^c_{xy}\,K + \frac{\rho^2}{4\pi^2}\,\varepsilon^{abc}\,q_{\mu\nu\alpha\beta}$$

$$\left[-\frac{\bar{\eta}^c_{\alpha x}x^\beta}{x^4(x-y)^2} + \frac{\bar{\eta}^c_{\beta y}y^\alpha}{y^4(x-y)^2} + \frac{\bar{\eta}^c_{\alpha x}y^\beta}{x^4}\left(\frac{1}{(x-y)^2}-\frac{1}{y^2}\right) - \frac{\bar{\eta}^c_{\beta y}x^\alpha}{y^4}\left(\frac{1}{(x-y)^2}-\frac{1}{x^2}\right)\right] + O(\rho^4) \tag{49}$$

In this limit, the leading correction to the free propagator is a traceless $1/r^2$ term. As before, the $1/r$ tail of the Brown et al. propagator is removed in forming the improved propagator. (One may similarly verify the cancellation of the $O(\rho^4 \ln \Omega^2)$ zero mode terms in this limit.)

Thus, we conclude that the difficulties with the BCCL propagator may be completely cured by constructing the improved propagator (33) based on localized collective coordinate constraints.

V. INSTANTON INTERACTIONS

As an application of our general formalism, we would like to consider the instanton interactions that appear as corrections in the instanton density n^z. First we examine the classical interactions present in the zero order density(8). These effects have been considered several places previously;[1] our purpose here is to show how the known results follow easily from our previous discussion.

The "classical" part of the instanton density (8), $\exp - \left(S(A^I) - \frac{1}{2} \mathscr{J} \hat{\mathscr{M}}^{-1} \mathscr{J} \right)$ requires us to evaluate the action of the multiinstanton field $A^I + \hat{\mathscr{M}}^{-1} \mathscr{J}$ (which, to this order, is the minimal action constrained solution). The action $S(A^I)$ may be expanded as shown in (22e). The lowest order term is of course just the sum of the single instanton actions. The various correction terms may be estimated by considering the magnitude of the integrand in the two important areas; the near regions $((x - x_i) \lesssim \rho$ for some $x_i)$ and the far region $((x - x_i) \gtrsim d$ for all $x_i)$.[11] (ρ is taken to refer to some typical instanton size, and d is a typical instanton separation. For our estimates, we will not bother to worry about which particular size or separation is appropriate.) The first correction term $\int \mathscr{F}^{(i)}_{\mu\nu} \mathscr{F}^{(j)}_{\mu\nu}$ is of order λ^4/d^4 since the near regions have integrands of size $1/d^4$ and volumes ρ^4, while the far region appears to contribute an integrand of order ρ^4/d^8 times a volume $\sim d^4$. The last three terms are similarly seen to be at most $O(\rho^6/d^6)$.[2b] By using (22c), the $\mathcal{J}\hat{\mathcal{M}}^{-1}\mathcal{J}$ term may also be shown to be of order ρ^6/d^6. To see this, we note that $\mathcal{J} = -D\cdot F$ is $O(1/d^3)$ in the near regions and $O(\rho^4/d^7)$ in the far region. The contribution to the double integral $\int d^4x \int d^4y\, \mathcal{J}(x)\, \hat{\mathcal{M}}^{-1}(x,y)\, \mathcal{J}(y)$ when both x and y are in the same near region is $O\left(\rho^8 \cdot \frac{1}{d^3}\,\frac{1}{\rho^2}\,\frac{1}{d^3}\right) = O(\rho^6/d^6)$. The contribution of all other regions is at most $O(\rho^8/d^8)$.

Thus the leading classical correction is simply given by the overlap of the Abelian terms

$$-\frac{1}{4}\sum_{i\neq j}\int \mathcal{F}^{(i)}_{\mu\nu}\,\mathcal{F}^{(j)}_{\mu\nu} \quad .$$

By integration by parts, the equivalent form

$$\frac{1}{2}\sum_{i\neq j}\int A^{(i)}_{\mu}\,\Box\,A^{(j)}_{\mu}$$

may be obtained.

Since, in the far region $\Box A^{(i)}_{\mu}$ is $O(\rho^4/x^7)$, not $O(\rho^2/x^5)$ as would naively be expected, the contribution of this region to $\int\mathcal{F}\mathcal{F}$ is at most $O(\rho^6/d^6)$. Thus, the leading contribution comes from the near region, where we have

$$\Delta S^{int} = -\sum_{i\neq j}\int_{\text{near } i} d^4x\, \partial_\mu A^{(i)}_{\nu}\, \mathcal{F}^{(j)}_{\mu\nu}$$

Near instanton (i), $\mathscr{F}^{(j)}$ is in its far region and coincides with the full field strength, thus

$$\mathscr{F}^{(j)}_{\mu\nu} = - \frac{4}{g} \rho^2 R^j_{ab} \eta^{ej}_{b\alpha\beta} I^{x-x_j}_{\alpha\mu} I^{x-x_j}_{\beta\nu} / (x - x_j)^4 + 0\left(\rho^4/(x - x_j)^6\right)$$

Since this is divergenceless, as required by the far region (linearized) equations of motion, the interaction may be reduced to a surface integral surrounding near region (i), and we find the established result[1]

$$\Delta S^{int} = - \sum_{i \neq j} \int_{S_i} d^3\Sigma_\mu A^{(i)}_\nu F^{(j)}_{\mu\nu}$$

$$= \sum_{i<j} \frac{8\pi^2}{g^2} \frac{\rho_i^2 \rho_j^2}{d^4} R^{(i)}_{ca} R^{(j)}_{cb} \eta^{e_i}_{a\alpha\beta} \eta^{e_j}_{b\mu\nu} I_{\alpha\mu} I_{\beta\nu} \tag{50}$$

We would now like to turn to the quantum instanton interactions induced through the dependence of the one-loop determinants in the instanton density (8) on the instanton positions and orientations. Using the multiple scattering expansions developed previously, the determinants in the density n^z_0 (except the collective coordinate Jacobian which will be discussed later) may be represented as

$$\left[\prod_i \det{}^{-1/2} \hat{\mathscr{M}}^{(i)} \det\left(\not{D}\left(A^{(i)}\right) + m\right) \det\left(- D^2\left(A^{(i)}\right)\right)\right] \cdot \exp - V_{int} \tag{51}$$

where V_{int}, the one-loop quantum instanton interaction, is given by

$$\tilde{V}_{int} = \sum_{i \neq j} - \mathrm{Tr} \left\{ \frac{1}{2} G_0 \sigma_v^{(i)} \; G_0 \sigma_v^{(j)} + \frac{1}{2} G_0 V_v^{(ij)} - S_0 \sigma_f^{(i)} \; S_0 \sigma_f^{(j)} \right.$$

$$\left. - \Delta_0 \sigma_s^{(i)} \Delta_0 \sigma_s^{(j)} - \Delta_0 V_s^{(ij)} + \ldots \right\} \tag{52}$$

The indicated terms comprise the first corrections to the DGA evaluation of the multiinstanton determinants. As mentioned previously, the evaluation of these loops is complicated by the need for regularization and renormalization. We will not attempt a complete calculation of these effects, but will merely try to find the dependence of the interaction on the instanton separations.

We begin with the scalar terms. Recalling the definition of $\sigma^{(i)}$, the first term we must consider is

$$\mathrm{Tr}\left(\Delta_0^{-1}\left(\Delta^{(i)} - \Delta_0\right)\Delta_0^{-1}\left(\Delta^{(j)} - \Delta_0\right)\right)$$

The isospin 1 scalar propagator in a background instanton field (located at the origin) is given by[3]

$$\Delta^{(i)} = \Delta_0 \frac{1}{2} \mathrm{tr}\left(\tau_a\left(1 + \frac{\rho^2}{x^2 y^2} \tau_x \tau_y^+\right) \tau_b\left(1 + \frac{\rho^2}{x^2 y^2} \tau_y \tau_x^+\right)\right) \Big/ \pi_x \pi_y$$

Note that when $\rho^2 << x^2, y^2$,

$$\Delta^{(i)} = \Delta_0 - \rho^2 \delta^{ab} \big/ x^2 y^2 + 2\rho^2 \varepsilon^{abc} \; \bar{\eta}^c_{xy} \big/ x^2 y^2 (x - y)^2 + 0(\rho^4) \quad .$$

We would now like to apply the same type of estimates used previously to the above trace. Explicitly, we must consider

$$\int d^4x d^4y \; \mathrm{tr}\left(\partial_x^2\left(\Delta^{(i)} \; (x,y) - \Delta_0\right) \partial_y^2 \; \left(\Delta^{(j)} \; (y,x) - \Delta_0\right)\right) \tag{53}$$

This integral is, of course, divergent due to the short distance $(x \to y)$ singularities. However, it seems clear that these singularities may be regulated without modifying the long distance dependence of the determinant on the instanton separations. (For example, by using point splitting techniques to regulate and remove the divergences, one sees that the finite contribution from the short distance $(x \approx y)$ region has the same long distance behavior that one would naively estimate.)

Thus, we may examine the behavior of the finite remainder of the loop (53) by essentially ignoring the presence of the divergences. By examining the integrand in the various important near and far regions, one may show that the integral is of $0(\rho^4/d^4)$. For example, to find the contribution when x and y are in the far region (i.e., $x, y, x-y \gtrsim d$), we note that both propagators (less the free propagators) are of order ρ^2/d^4, each derivative becomes a factor of $1/d$, so the total contribution is

$$0\left(d^8\left(\frac{1}{d^2}\ \frac{\rho^2}{d^4}\right)^2\right) = 0(\rho^4/d^4).$$

Similarly, for the contribution from $x, y \gtrsim d$; $x - y \lesssim \rho$, the volume is $0(d^4\rho^4)$ and the leading term of each propagator is the traceless piece which is of order ρ/d^3 since $\eta^c_{xy} \sim 0(\rho d)$. Taking all the derivatives to act on one of the propagators, we find that at least two of the derivatives must insert factors of $1/d$ since if three or more derivatives act on the $1/(x-y)^2$ factor, then two of them must form ∂^2 which collapses the $1/(x-y)^2$ to a δ-function. This causes the traceless term to vanish completely. Thus, the contribution of this region is $0(\rho^4/d^4)$. All other regions contribution at most $0(\rho^5/d^5)$.

The lowest order scalar loop involving the local vertex $V_s^{(ij)}$, explicitly $\int d^4x \operatorname{tr} \underset{\sim}{A}^{(i)}(x)\, \underset{\sim}{A}^{(j)}(x)\, \Delta_0(x,x)$, should be completely removed by the renormalization. Thus, the ghost contribution to the quantum instanton interaction induces a pairwise $O(\rho^4/d^4)$ potential.

The estimates for the vector contribution to V_{int} are identical to the scalar terms. This follows since in the long distance limit $\rho^2 << x^2, y^2$, the improved vector propagator has a behavior similar to the scalar propagator, namely, the $O(\rho^2)$ traceless terms proportional to $\varepsilon^{abc}\, \eta^c_{xy}$ have short distance singularities no worse than $1/(x-y)^2$. Thus the same estimates for the propagator (less the free propagator) are valid in the various regions. So the overall contribution of the vector terms in (52) is again of order ρ^4/d^4.

For the lowest order fermion contribution, we must consider the loop

$$\int d^4x \int d^4y \operatorname{tr} \left(\not{\partial}_x \left(S^{(i)}(x,y) - S_0\right) \not{\partial}_y \left(S^{(j)}(y,x) - S_0\right)\right) \qquad (54)$$

Now, for massive fermions, the single instanton propagator $S^{(i)}$ is not explicitly known. However, it is of course exponentially damped at large distances. This implies that the massive fermion contribution to V_{int} is $O(e^{-md})$.

For massless fermions, the analysis is modified due to the presence of fermionic zero modes. In the massless limit, one of the single instanton fermion eigenfunctions has an eigenvalue $\sim$ m. Thus, the DGA multiinstanton fermion determinant has a factor of $(m)^K$ which suppresses the zero order instanton density. The single instanton fermion propagator is given, for $m \to 0$, by

$$S^{(i)} = \frac{\psi_0 \psi_0^+}{m} + \tilde{S}^{(i)} + 0(m) \tag{55}$$

where

$$\psi_0\psi_0^+ = \frac{\rho^2 \tau_x \not{t}^+ \not{t} \tau_y^+}{8\pi^2 \left(x^2 + \rho^2\right)^{3/2} \left(y^2 + \rho^2\right)^{3/2}} \frac{(1 - \gamma_5)/2}{(x^2)^{\frac{1}{2}}(y^2)^{\frac{1}{2}}}$$

and

$$\tilde{S}^{(i)} = \not{\partial}\Delta^{(i)} \frac{(1 + \gamma_5)}{2} + \Delta^{(i)}\not{\partial} \frac{(1 - \gamma_5)}{2}$$

$$= S_0 \left(1 + \frac{\rho^2}{x^2 y^2} \tau_x \tau_y^+\right) / (\pi_x \pi_y)^{1/2}$$

$$-\Delta_0 \frac{\rho^2}{x^2 y^2} \tau_x \left[\frac{\not{t}^+ \tau_{x-y}}{(x^2+\rho^2)} \frac{(1+\gamma_5)}{2} + \frac{\tau_{x-y}^+ \not{t}}{(y^2+\rho^2)} \frac{(1-\gamma_5)}{2}\right] \tau_y^+ / (\pi_x \pi_y)^{1/2} \tag{56}$$

For antiinstantons, all $(1 \pm \gamma_5)/2$ projections reverse chirality.

In order to examine the $m \to 0$ limit of the fermion contribution, we must expand the exponential in $\exp\text{-}V_{int}$ so that the $1/m$ factors in the fermion propagators in V_{int} may be combined with the $(m)^K$ in the DGA result. Thus, a given term in the expansion of det $(\not{\partial} + m)$ now consists of any number of fermion loops. Each vertex in a loop consists of a zero mode piece $\psi_0^{(i)} \psi_0^{+(i)}/m$ and a non-zero mode piece $\tilde{S}^{(i)} + 0(m)$. As discussed by Mottola,[8] the only terms which survive in the $m \to 0$ limit are terms where the zero mode vertex for each instanton appears just once. (Terms with fewer zero mode vertices cannot completely cancel the $(m)^K$ of the DGA result and hence vanish; terms with too many zero mode vertices cancel among themselves due to the Fermi statistics.)

Thus, the first correction to the DGA result is given by the sum of all terms with only K zero mode vertices present. Each link of each loop in these terms consists of the factor

$$\frac{1}{m}\int \psi_0^{+(i)} \not{\partial} \psi_0^{(j)} \equiv \frac{1}{m} h_{ij} .$$

It is easy to see that the sum of all these terms is just $(m)^{-K}$ det h. Thus, the $(m)^K$ suppression factor of the DGA result is replaced by det h. This result is, of course, equivalent to performing degenerate perturbation theory in the space spanned by the different zero modes. Because the zero mode for an instanton has definite chirality, the matrix element h_{ij} is only non-zero between instantons and antiinstantons. Thus det h will be non-zero only if the configuration consists of an equal number of instantons and antiinstantons. This merely reflects the fact that exact fermion zero modes exist in any multiinstanton field with a net topological charge. Thus, in order to obtain a non-zero contribution to the functional integral, we will only consider "neutral" configurations, those with equal numbers of instantons and antiinstantons.

Each matrix element h_{ij} is of order $1/d^3$, so that if det h is represented as exp v, then we see that the massless fermion zero modes induce a logarithmic instanton-antiinstanton interaction potential.[1]

The next corrections to this result are given by those terms where we insert additional non-zero mode vertices $S_0^{-1}(\tilde{S}^{(i)} - S_0)S_0^{-1}$ into some of the loops of the lowest order terms. If we only insert a single new vertex, then the resulting contribution is

$$\sim \sum_j \int \psi_0^{+(i)} \not{\partial} \left(\tilde{S}^{(j)} - S_0\right) \not{\partial} \psi_0^{(k)} \cdot m_{ik} \tag{57}$$

where m_{ik} is the minor of h. This vanishes for (i) and (k) both instantons or antiinstantons, and provides a $1/d^2$ correction to the logarithmic instanton-antiinstanton potential. If we insert two non-zero mode vertices, then the leading contribution is given by

$$\sim \sum_{i \neq j} \int \delta\left(\tilde{S}^{(i)} - S_0\right) \delta\left(\tilde{S}^{(j)} - S_0\right) \cdot \det h$$

and provides an additional $1/d^4$ potential.

These results on corrections to the scalar, vector, and fermion determinants agree with previous analysis of one-loop instanton interactions based on the exact two-instanton solution (2b).

Finally, we must consider the dependence of the collective coordinate Jacobian on the instanton separations. We have

$$J_{zz'} = \int d^4x \, \tilde{f}_z(x) \, \widetilde{\frac{\partial}{\partial z'} A^I} \tag{58}$$

where $\widetilde{\partial_{z'} A^I}$

is the deformation $\partial_{z'} A^I$ placed in background gauge,

$$\widetilde{\partial_{z'} A^I} = \partial_{z'} A^I + D(A^I) \left(-D^2(A^I)\right)^{-1} \left(D(A^I) \, \partial_{z'} \, A^I\right)$$

In the DGA evaluation of J, the overlap of the constraints centered on one instanton with the deformations of other instantons are ignored. Also a deformation of instanton (i) is only placed in background gauge with respect to instanton (i), not in background gauge with respect to the total field. Thus, replacing the combined collective coordinate index z with the instanton number (i) and the single instanton coordinates, Ω_α , we see that the dilute gas Jacobian J^0 is given by

$$\tilde{J}^0_{\substack{(i)(j)\\ \alpha\ \ \beta}} = \delta^{ij} \int d^4x\, f_\alpha^{(i)} \widehat{\frac{\partial}{\partial\Omega_\beta} A^{(i)}}$$

where

$$\widehat{\partial_\beta A^{(i)}} = \partial_\beta A^{(i)} + D\left(A^{(i)}\right) \frac{1}{-D^2\left(A^{(i)}\right)} \left(D\left(A^{(i)}\right) \partial_\beta A^{(i)}\right) \tag{59}$$

So, the first order correction to the DGA Jacobian is given by

$$\delta(\ln \det J) = \operatorname*{tr}_{zz'} \left(J^0\right)^{-1} \left(J - J^0\right)$$

and we wish to estimate the size of the difference $(J - J^0)$. Since J^0 is diagonal in instanton number, only the diagonal elements of $(J - J^0)$ contribute to the trace. This correction is given by

$$\sum_j \left\{ \int f_\alpha^{(i)} D\left(A^{(i)}\right) \left(-D^2\left(A^{(i)}\right)\right)^{-1} \left[A^{(j)}, \widehat{\partial_\beta A^{(i)}}\right] + \int \left[\widehat{f_\alpha^{(i)}}, A^{(j)}\right] \left(-D^2\left(A^{(i)}\right)\right)^{-1} D\left(A^{(i)}\right) \partial_\beta A^{(i)} \right\} \tag{60}$$

These terms are easily seen to be of order $1/d^4$. (The apparent $1/d^3$ contribution from near region (i) vanishes due to the spherical symmetry after $A_\mu^{(j)} \sim \eta_{\mu d}/d^4$ is factored out.) This agrees with the previous partial considerations of C. Bernard.[2b]

Thus we see that, except for the previously known logarithmic instanton-antiinstanton interaction caused by massless fermions,[1] all corrections to the classical dipolar interaction (50) are of order ρ^4/d^4. No other interactions of longer range than the dipole force are found.

VI. HEAVY QUARK POTENTIAL

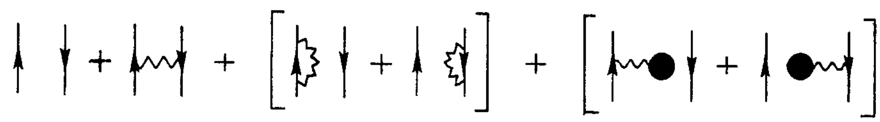
Consider the computation of quantum corrections to the potential between static, heavy quarks. We wish to examine the calculation for two reasons: first, to illustrate some of our formalism in a specific example and, second, to show explicitly that the long range limit of the heavy quark potential is, at least through $0(g^2)$, just a finitely renormalized Coulomb law.

As is well known, the static limit of the quark potential is obtained by evaluating the Wilson loop integral

$$\left\langle \exp\, ig \int A_\mu \, dy_\mu \right\rangle$$

for a Euclidean path of spatial separation R and time extent T. As $T \to \infty$, the Wilson loop equals $\exp -V(R)T$, where V(R) is the static potential. Following our general formalism, we now find a consistent expansion of V(R) by expanding the above expectation value in powers of $\bar{A} = A - A^I$ in a given sector about A^I. We must then sum over all configurations and identify the exponentiation in order to extract V(R).

Applying (18) with $\mathscr{O}$ as the Wilson loop operator, we can easily derive that the contribution from sector A_0^z is given, through $0(g^2)$ by

$W^z(R) \cdot \mathscr{T} =$ 

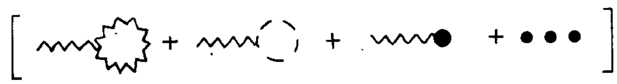
(a) (b) (c) (d)

(61)

where the static fermion propagators ( ) are defined to be ($\underset{\sim}{x}$ denotes a spatial 3-vector),

$$U^z\left(\underset{\sim}{x};\, t_1,\, t_2\right) = \exp - \int_{t_1}^{t_2} A_0^z(\underset{\sim}{x},\, t)\, dt,$$

and the (anti-) fermion-gluon vertex is (-) ig $\lambda^a/2$ where λ^a are the N^2-1 matrices of the adjoint representation of SU(N). The blob  represents the various terms in $\langle \bar{A} \rangle^z$, such as



(see Section II). $\mathcal{Z}$ is the partition function, $\int dz\, n^z$, which appears as a denominator for each term of W. If we restrict ourselves to the color singlet states, it is convenient at this stage to define $V^z(R)$ as the potential due to A_0^z and all configurations connected to A_0^z by global gauge transformations. When this is done, an overall color trace times a factor of $\frac{1}{N}$ is to be understood. For example, the one gluon exchange term (b) is explicitly given by

$$\lim_{T \to \infty} \frac{g^2}{N} \int_{-T/2}^{T/2} \left(dt_1 dt_2\right) G_{00}^{z,ab}(x_1,x_2)\ \mathrm{tr}\Big[U\left(\underset{\sim}{x}_1;-\infty,t_1\right)\frac{\lambda^a}{2}\, U\left(\underset{\sim}{x}_1;t_1,+\infty\right)$$
$$U\left(\underset{\sim}{x}_2;+\infty,\ t_2\right)\frac{\lambda^b}{2} U\left(\underset{\sim}{x}_2;t_2,-\infty\right)\Big] \tag{62}$$

We have taken the liberty of replacing the T's appearing in the fermion propagators by ∞ ; U always approaches its limiting form rapidly enough for this change to be irrelevant.

We again are concerned with multiinstanton configurations, and for simplicity, we restrict the discussion to the case of SU(2) with no light fermions and with the heavy quarks in a color singlet. The lowest order contribution is obtained by keeping only term (a) of Eq. (61) for $V^I(R)$. Clearly, the single instanton contribution is given by

$$W_0^{(1)}(R) = \frac{T}{2}\int \frac{d^3x_I d\rho}{\rho^5}\; n_0(\rho)\; \mathrm{Tr}\Big[U^{(1)}(\underset{\sim}{x}_1 - \underset{\sim}{x}_I;\, -\infty,\infty)\; U^{(1)}(\underset{\sim}{x}_2 - \underset{\sim}{x}_I;\, \infty,-\infty) - \mathbb{I}\Big]$$

where $n_0(\rho)$ is the density of instantons after global gauge averaging and

$$U^{(1)}(\underset{\sim}{x};t_1,t_2) = \exp i\,\tau\cdot(\underset{\sim}{x} - \underset{\sim}{x}_I)\rho^2\, f(\underset{\sim}{x}-\underset{\sim}{x}_I,\rho;\, t_1,t_2)$$

where

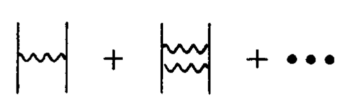
$$f(\underset{\sim}{x}_1\rho;t_1t_2) = -\frac{\partial}{\partial\rho^2}\left[\tan^{-1}\frac{t_1}{\sqrt{(\underset{\sim}{x}-\underset{\sim}{x}_I)^2+\rho^2}} - \tan^{-1}\frac{t_2}{\sqrt{(\underset{\sim}{x}-\underset{\sim}{x}_I)^2+\rho^2}}\right] \qquad (63)$$

It is easy to convince oneself that this single instanton result is exponentiated by multiinstanton configurations, with the final result that $V_0(R) = -2W^{(1)}(R)/T$, the factor of 2 coming from also considering anti-instantons. This result has been given before, in several places.[1] It now emerges as the lowest order of a systematic expansion, with the corrections given by terms b, c, d of Eq. (61).

Consider now the higher order contributions. In the vacuum sector, $U = 1$, $\langle \bar{A} \rangle = 0$ and we can use the free gluon propagator $G^{(0)} = \delta^{ab}\delta_{\mu\nu}/4\pi^2(x-y)^2$. $W^{vac}(R)$ is then given by

$$\frac{+3g^2}{8\,\pi^2}\iint \frac{dt_1\, dt_2}{R^2+(t_1-t_2)^2} \sim \frac{g^2T}{R}\;,$$

where terms independent of R have been dropped. This potential is exponentiated by the set of ladder diagrams



to give the usual, perturbative result, a Coulomb potential between the two heavy quarks. The one instanton sector is slightly more complicated. $\mathcal{U}^{(1)}$ was given in (63) and we must use $\langle \bar{A} \rangle^{(1)}$, the sector one point function, and G, the propagator, discussed in previous sections. This term contains an integral over instanton time t_I, as well as the times associated with fermion-gluon interactions, and will contain pieces proportional to both T and T^2. For example, consider the gluon exchange term. For very large average times

$$\frac{t_1 + t_2}{2} \to \pm \infty ,$$

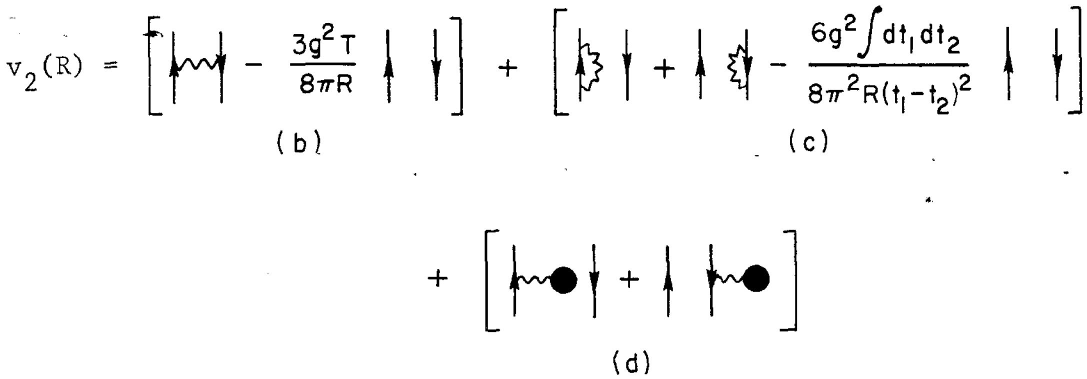
the $\mathcal{U}$'s approach their limiting form and can be factored out of the average time integral. Because one part of G is just G^0, which depends only on $(t_1 - t_2)$, and because there is still the t_I integral to do, there will clearly be a T^2 dependence. The T^2 piece is just part of the expansion of the exponential of the Coulomb piece added to the lowest order potential $V_0(R)$. In other words, building up the entire exponential of the Coulomb potential requires the inclusion of terms coming from the one (and also the multi) instanton sectors, and these pieces should be subtracted from the full sector contribution. Once this is done, we can define $W_2^{(1)}(R)$ as the $O(g^2T)$ term. It is again clear that this term will be exponentiated by multi-instanton configurations.

Our final expression is then

$$V(R) = V_{Coul}(R) - \int d^3x_I \frac{d\rho}{\rho^5} n(\rho) \left[v_0(R) + v_2(R) \right] \tag{64a}$$

$v_0(R) =$  (64b)

(a)



(64c)

Note that consistency requires that we now know $n(\rho)$, the density of instantons, to $O(g^2)$, i.e., to two loop order. Note also that to this order it is correct to include the entire set of ladder graphs in the vacuum sector and to only keep a single subtracted gluon exchange in the single instanton sector. This is because each additional rung leads to an extra power of g^2T in the vacuum while the localized nature of the instanton field configuration leads to each rung having an additional power of $g^2\rho$. The former are part of the $O(g^2)$ contribution to the potential $V(R)$, while the latter contribute terms to $V(R)$ suppressed by additional powers of g^2.

The problem of calculating $V(R)$ has now been reduced to calculations of $\langle \bar{A} \rangle$, G and $n(\rho)$ in the single instanton field, and to evaluating Eq. (64). This is a major undertaking, involving the evaluation of one and two loop Feynman diagrams in the presence of the external field. Rather than attempt the entire computation, we shall focus on the long range limit of $V(R)$, characterized by the approximation $R \gg \rho$ for all configurations that we consider. This has been attempted previously with somewhat ambiguous results[4] which have been traced to an inconsistent treatment of Eq. (64).[6]

We now define an integrated potential

$$\bar{v}(R, \rho) \equiv \int d^3x_I \left[v_0(R; x_I, \rho) + v_2(R; x_I, \rho) \right] \tag{65}$$

where all dependences have been explicitly shown. The dimension of $\bar{v}$ is $(\text{length})^{+3}$ and it therefore has a long distance expansion

$$\bar{v} \sim a\, \rho^2 R + b\, \rho^4/R \quad .$$

(We will ignore all logarithmic corrections and we will see that no odd powers of ρ appear.) Since the ρ integral in (64a) cannot introduce any additional R dependence, the asymptotic behavior of V(R) will be governed by V_{Coul} and by the asymptotic behavior of $\bar{v}$. Our aim is then to check which is the first non-vanishing term in the expansion of $\bar{v}$, or, simply stated, to see if $a \neq 0$ so that linear terms are present in V(R). The classical term $v_0(R)$ is known to have 1/R long range behavior[1]; we therefore restrict attention to v_2.

The simplest term to consider is the one corresponding to single gluon exchange. Substituting (64c (b)) into (65) yields

$$\begin{aligned}\bar{v} \sim g^2 \int d^3x_I dt_1 dt_2 \; \mathrm{Tr}\Big[& U^{(1)}(\underset{\sim}{x}_1 - \underset{\sim}{x}_I; -\infty, t_1) \frac{\tau^a}{2} \, U^{(1)}(\underset{\sim}{x}_1 - \underset{\sim}{x}_I; t_1, \infty) \times \\ & U^{(1)}(\underset{\sim}{x}_2 - \underset{\sim}{x}_I; \infty, t_2) \frac{\tau^b}{2} \, U^{(1)}(\underset{\sim}{x}_2 - \underset{\sim}{x}_I; t_2 - \infty) \; G^{ab}_{00}(x_1, x_2) \\ & - U^{(1)}(\underset{\sim}{x}_1 - \underset{\sim}{x}_I; -\infty, \infty) \, U^{(1)}(\underset{\sim}{x}_2 - \underset{\sim}{x}_I, \infty, -\infty) \; G^{(0)\,aa}_{00}(x_1, x_2) \Big]\end{aligned} \tag{66}$$

Imagine doing the x_I integral for fixed t_1, t_2. One region of integration is when x_I is close, on the scale of ρ, to either the quark or anti-quark. Since $R >> \rho$, G in this region is given by the form valid for $x >> y, \rho$, Eq. (44). It is easy to check that G is order ρ^0/R^2 and that the volume of the integration is proportional to ρ^4. Therefore, in our search for ρ^2 terms, we can safely assume that x_I is far from both fermion lines. From (63), it follows that

$$\cup^{(1)} = 1 + i\, \tau \cdot (\underset{\sim}{x} - \underset{\sim}{x}_I)\ \rho^2 f(x - x_I, \rho) + 0(\rho^4),$$

and the only possible ρ^2 terms arise from keeping the ρ^2 term in the expansion of any one particular $\cup$ or in G, the full propagator. The expansion of G in the asymptotic limit $x,y >> \rho$, Eq. (49), consists of the free propagator plus traceless $0(\rho^2)$ terms, and the simple formulas

$$\delta^{ab}\ \mathrm{Tr}(\tau^a\ \tau^b\ \underset{\sim}{\tau} \cdot \underset{\sim}{x}) = \varepsilon^{abc}\ \mathrm{Tr}(\tau^a\ \tau^b) = 0$$

suffice to show that all these terms vanish. The remaining integrations can never introduce compensating inverse factors of ρ^2. We have therefore shown that the exchange term does not give rise to a linear piece in V(R).

For the "loop" term, (64c(d)),we need to know $\langle \bar{A} \rangle^{(1)}$, the single instanton sector one point function. This is given by

$$\int G_{\mu\nu}\ (x,y)\ F_\nu(y)\ d^4y$$

where F is the blob defined graphically above. A careful and tedious study of all the various contributions to F [12] leads to the conclusion that F(y) vanishes asymptotically for large y as $1/y^5$. This means that the large x limit of $\langle \bar{A} \rangle$ can be derived by using the asymptotic form of G for large x, Eq. (49). Substituting that result into the definition of $\langle \bar{A} \rangle$ leads to $\langle \bar{A} \rangle = 0(\rho^2/x^3)$.

Returning to the final integrals in the loop term (over $\underset{\sim}{x}_I$, and t_1), phase space considerations similar to those which appeared in the exchange term allow us to restrict our attention to the region of integration $\underset{\sim}{x}_I - \underset{\sim}{x} \gg \rho$. (The leading contribution from the near regions only affects the single quark self energies.) Using the expansion of U given previously, it is again easy to show that the ρ^2 term vanishes upon taking the trace and again, there is no linear piece of the potential. Finally, the "self-energy" term (64c(c)) is treated in exactly the same fashion, taking care to subtract out the infinite Coulomb self energy. The conclusion is that none of the three terms can contribute a linear piece to $\bar{v}$ and that therefore $V(R) \sim \bar{g}^2/R$. It might be interesting to calculate the magnitude of the effective coupling $\bar{g}^2$ to check if the lowest order results are significantly modified, but this has not yet been done. Since g^2 is rather small, it would be very surprising if these higher order corrections had any important effects.

VII. SUMMARY

Up to now, a consistent formalism for handling higher order instanton effects has been lacking. In this paper, we have attempted to remedy this situation by developing a systematic formalism for expanding about multiple instanton configurations. As examples of our general method, we considered two problems, instanton interactions and the instanton contribution to the static quark potential. While the full analysis has not yet been carried through, the major result that we have found is that quantum corrections do not induce longer range potentials than the lowest order results and hence do not introduce qualitatively new effects.

There are many calculations that can be performed, at least in principle, using our formalism. However, as is already evident from the simple examples pursued here, these calculations are in general quite difficult largely due to the technically complicated form of the single instanton propagators. A number of computations are vital, however, in the development of a model of hadrons based on the multiinstanton approach. These include a calculation of the two loop corrections to the instanton density in order to relate our semiclassical coupling g to experimentally measurable quantities, and a detailed study of instanton interactions to further explore the statistical mechanics of the interacting instanton gas. This work is now in progress.

ACKNOWLEDGMENTS

We wish to thank Roger Dashen for many interesting conversations. One of us (H.L.) would like to thank the SLAC Theory Group for their hospitality.

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7. We follow the notation of reference 5. For example, the field A_μ is understood to be a traceless antihermitian matrix and $D_\mu(A)\Phi \equiv \partial_\mu\Phi + g[A_\mu,\Phi]$. Lorentz indices are suppressed whenever convenient.

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9. All determinants are understood to be normalized by the free field determinants.

10. We use the notation and many of the techniques described in Ref. 3. In particular,

$$\tau_\mu = (\vec{\tau}, i)$$

$$\tau_\mu^+ \tau_\nu = \left(\delta_{\mu\nu} + i\eta^a_{\mu\nu} \, \tau^a \right)$$

In addition, we use abbreviations such as τ_x in place of $\tau_\mu x_\mu$.

11. These estimates follow the approach of C. Bernard, reference 2b. The intermediate regions ($|x - x_i| \sim \rho^\alpha \, d^{1-\alpha}$, $0 < \alpha < 1$) may be shown to give subdominant contributions.

12. H. Levine, unpublished work. The function F is similar to, but not exactly the same as, the vacuum polarization defined in L. S. Brown, D. Creamer, "Vacuum Polarization about Instantons," U. of Wash. preprint (1978).

APPENDIX

We discuss here a few of the techniques and tricks used to evaluate $1/D^4$ and construct the improved propagator in the two limits: (i) $x \gg y,\rho$, and (ii) $\rho \ll x,y$.

In the first limit, one straightforward method to evaluate the convolution integral

$$1/D^4 = \int d^4z \frac{\frac{1}{2}\,\mathrm{tr}\left(\tau_a\left(\tau_x^+\tau_z+\rho^2\right)\left(\tau_z^+\tau_y+\rho^2\right)\tau_b\left(\tau_y^+\tau_z+\rho^2\right)\left(\tau_z^+\,\tau_x+\rho^2\right)\right)}{\left(4\pi^2\right)^2\left(x^2+\rho^2\right)\left(y^2+\rho^2\right)\left(z^2\,+\rho^2\right)^2(x-z)^2(y-z)^2} \tag{A.1}$$

is to split the integration region into an outside region, $\Lambda^2 \leq z^2 \leq \Omega^2$, where $\Omega^2 \gg x^2 \gg \Lambda^2 \gg y^2,\rho^2$, and an inside region $0 \leq z^2 \leq \Lambda^2$. In the outside region $1/(y-z)^2$ may be expanded as

$$\frac{1}{z^2}\left(1+\frac{2(y\cdot z)}{z^2}+\frac{4(y\cdot z)^2-y^2z^2}{z^4}+\ldots\right)$$

and the resulting integrals may be easily evaluated by performing the angular integrals directly and then integrating over z^2. In the inside region, $1/(x-z)^2$ may be expanded as

$$\frac{1}{x^2}\left(1+\frac{2x\cdot z}{x^2}+\frac{4(x\cdot z)^2-x^2z^2}{x^4}+\ldots\ldots\right)$$

and identical types of integrals are encountered. Keeping track of all terms through $O(\ln x^2/x^2)$ produces the result quoted (41).

Calculation of the covariant derivatives of $1/D^4$ needed to form the BCCL propagator (42) is greatly facilitated by noting that the isospin 1 covariant derivatives when acting on the traces may be converted to isospin 1/2 derivatives acting inside the trace. For example

$$\overleftarrow{D}^{1}_{\alpha} \frac{\frac{1}{2}\,\mathrm{tr}\left(\tau_a \tau^{+}_x \tau_y \tau_b \tau^{+}_y \tau_x\right)}{(x^2+\rho^2)(y^2+\rho^2)} \ell n \left(\Omega^2/(x-y)^2\right)$$

$$= \frac{\frac{1}{2}\,\mathrm{tr}\left(\tau_a \tau^{+}_x \tau_y \tau_b \tau^{+}_y \tau_x\right)}{(x^2+\rho^2)(y^2+\rho^2)} \partial_\alpha \ell n \left(\Omega^2/(x-y)^2\right)$$

$$+ \frac{1}{2}\,\mathrm{tr}\left(\tau_a \left[D^{\frac{1}{2}}_{\alpha} \frac{\tau^{+}_x \tau_y}{\left(x^2+\rho^2\right)^{1/2}}\right] \tau_b \frac{\tau^{+}_y \tau_x}{\left(x^2+\rho^2\right)^{1/2}} - \tau_a \frac{\tau^{+}_x \tau_y}{\left(x^2+\rho^2\right)^{1/2}}\right.$$

$$\left. \tau_b \left[\frac{\tau^{+}_y \tau_x}{\left(x^2+\rho^2\right)^{1/2}} \overleftarrow{D}^{\frac{1}{2}}_{\alpha}\right]\right) \frac{\ell n\left(\Omega^2/(x-y)^2\right)}{(y^2+\rho^2)} \tag{A.2}$$

Here

$$D^{1}_{\alpha} = \partial_\alpha \delta^{ab} + \varepsilon^{acb} A^{c}_{\alpha} = \partial_\alpha \delta^{ab} + 2\varepsilon^{acb} \frac{\eta^{c}_{\mu x}}{(x^2+\rho^2)} \qquad \text{(règular gauge)}$$

and

$$D^{\frac{1}{2}}_{\alpha} = \partial_\alpha - i\,\frac{\tau^a}{2} A^{a}_{\alpha} = \partial_\mu - \frac{\left(\tau^{+}_{\alpha}\tau_x - x_\alpha\right)}{(x^2+\rho^2)} = \partial_\mu + \frac{\left(\tau^{+}_x \tau_\alpha - x_\alpha\right)}{(x^2+\rho^2)}$$

Note that covariant derivatives acting backwards involve $-\overleftarrow{\partial}_\mu$.[3]

Using the first form of $D^{1/2}$, we have

$$D^{\frac{1}{2}}_{\alpha} \frac{\tau^{+}_x \tau_y}{\left(x^2+\rho^2\right)^{1/2}} = \frac{\left(\tau^{+}_{\alpha}\tau_y (x^2+\rho^2) - \tau^{+}_{\alpha}\tau_x \tau^{+}_x \tau_y\right)}{\left(x^2+\rho^2\right)^{3/2}} = \rho^2 \frac{\tau^{+}_{\alpha}\tau_y}{\left(x^2+\rho^2\right)^{3/2}} \tag{A.3}$$

while the second form is useful for evaluating

$$\frac{\tau^{+}_y \tau_x}{\left(x^2+\rho^2\right)^{1/2}} \overleftarrow{D}^{1/2}_{\alpha} = -\rho^2 \frac{\tau^{+}_y \tau_\alpha}{\left(x^2+\rho^2\right)^{3/2}} \quad .$$

Note that the derivatives on the $(x^2 + \rho^2)$ denominators are always canceled by the x_α piece of the field term.

As mentioned previously, the tensor manipulations needed for dotting $q_{\mu\nu\alpha\beta}$ with $D_\alpha 1/D^4\, D_\beta$, and for evaluating the correction terms $\sim G^{BCCL}\, f\times\phi$ are vastly simplified by using the trace formula

$$\frac{1}{2}\,\mathrm{tr}\left(A\tau_\mu^+\right)\frac{1}{2}\,\mathrm{tr}(\tau_\mu B) = \frac{1}{2}\,\mathrm{tr}\,(AB).$$

This allows all the tensor structure to be combined into a single trace which may be easily manipulated by using the anticommutation relations $\left\{\tau_\mu^+, \tau_\nu\right\} = 2\delta_{\mu\nu}$, the cyclic trace identify, and the fact that $\mathrm{tr}(A) = \mathrm{tr}(A^+)$ if $\mathrm{tr}(A)$ is real (as is true for all our traces).

Finally, the integrations involved in computing $G^{BCCL}\, f\times\phi$ are all trivial since, to the order considered, all of the integrands are spherically symmetric. Since the constraints f are localized, the appropriate asymptotic expansion of G^{BCCL} needed for computing the correction terms is always limit I, $x^2 >> y^2$ (or vice versa).

To evaluate the convolution integral in the second limit $\rho^2 << x^2, y^2$, we proceed from the singular gauge expression (45) directly. Splitting the integrand into an inside region $0 \leq z \leq \rho$ and an outside region $\rho \leq z \leq \Omega$, one immediately sees that the contribution of the inside region is $O(\rho^4)$ and will be dropped. In the outside region, one may blindly expand the integrand in powers of ρ^2. For the order ρ^2 terms, one finds

$$-\rho^2\int d^4z\,\frac{\delta^{ab}\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{2}{z^2}\right) - \delta^{ab}\left(\frac{2x\cdot z}{x^2z^2}+\frac{2y\cdot z}{y^2z^2}\right) - 2\epsilon^{abc}\left(\frac{\bar{\eta}^c_{xz}}{x^2z^2} - \frac{\bar{\eta}^c_{yz}}{y^2z^2}\right)}{(4\pi^2)^2\,(x-z)^2\,(y-z)^2} \tag{A.4}$$

The trace term combines to form

$$-\rho^2 \int d^4z \, \frac{\delta^{ab}}{(4\pi^2)^2} \left(\frac{1}{x^2\, z^2\, (y-z)^2} + \frac{1}{y^2\, z^2\, (x-z)^2} \right)$$

which may be easily evaluated; however, the traceless piece

$$2\rho^2 \epsilon^{abc} \int \frac{d^4z}{(4\pi^2)^2} \left(\frac{1}{x^2}\, \bar{\eta}^c_{xz} - \frac{1}{y^2}\, \bar{\eta}^c_{yz} \right) \Big/ z^2 (x-z)^2 (y-z)^2 \equiv \frac{2\rho^2}{4\pi^2}\, \epsilon^{abc}\, \bar{\eta}^c_{xy} K$$

may not be easily evaluated analytically.

Since only the trace part of $1/D^4$ is easy to evaluate in this limit, the various trace manipulations mentioned previously are not as useful as before. However, the covariant derivatives may simply be blindly evaluated through $O(\rho^2)$ yielding the quoted result (47). The correction terms are evaluated exactly as before, and their tensor structure may be separated into trace and traceless pieces as shown in (48).