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# ABOUT COUPLING CONSTANT SINGULARITIES IN QUANTUM ELECTRODYNAMICS*

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## ABSTRACT

The contribution of the multibubble insertions in the photon propagator to the anomalous magnetic moment of the electron is reexamined. Using the asymptotic form of the photon propagator one finds a softer singularity than that found in a recent analysis.

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It has been known since Dyson that the perturbation expansion in the quantum electrodynamics is divergent, which implies an existence of a singularity at the origin in the coupling constant plane. The reason for the occurrence of this singularity is the instability of the standard vacuum against the spontaneous production of $e^+e^-$ pairs for the negative coupling constant. Further analysis shows [1,2], that this singularity is of a Borel summable type. Recently [3,4] it has been noted, that the renormalization causes an occurrence of a non-Borel summable singularity at the origin. Whereas the divergent, but Borel summable series are a rather natural phenomenon in the field theory, the non-Borel summability indicates an incompleteness of the theory. In the quantum electrodynamics the source of the singularity are diagrams of the type shown on Fig. 1 contributing to the anomalous magnetic moment of the electron. Following Lautrup [4] one writes (Fig. 2)

$$a(\alpha) = \frac{\alpha}{\pi} \int_0^1 dx\ (1-x) \left[-\pi\left(-\frac{x^2}{1-x}\, m^2\right)\right] \tag{1}$$

where $\pi(k^2)$ is connected with the photon propagator by

$$D_{\mu\nu}(k^2) = \left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}\right) \frac{\pi(k^2)}{k^2} \tag{2}$$

For an n-bubble insertion (Fig. 1) that contributes to the n-th order of the expansion in $\alpha$, $[-\pi(k^2)] = [-\pi_2(k^2)]^n$ where $\pi_2$ is the second order contribution. In the large n-limit one can use the saddle point method for the evaluation of the integral. Using the expression for $\pi_2$ calculated in the perturbation theory one finds that the expansion coefficients behave like n! and the series is therefore non-Borel summable. This non-Borel summability is connected with the break-down of the perturbation theory for large momenta and the appearance of a spurious pole in $\pi(k^2)$. One can find, however, examples where $a(\alpha)$ has a non-Borel summable expansion, even though $\pi(k^2)$ has no unphysical pole [6]. In the following I shall analyse the above singularity from the point of view of the spectral representation and the renormalization group.

For $-k^2 >> m^2$ the renormalization group arguments [7] tell one that

$$\pi(k^2) = \frac{1}{\alpha} F\left(\phi\left(q(\alpha)\right) + \ell n \frac{-k^2}{m^2}\right) \tag{3}$$

where F and $\phi$ are mutually inverse functions, i.e. $F\left(\phi(q)\right) \doteq q$. Perturbation theory [8] gives

$$q(\alpha) = \alpha\left(1 + \frac{5}{9\pi}\alpha + \ldots\right)$$
$$\phi(x) = -\frac{3\pi}{x} + \ldots \qquad 0 < x << 1 \tag{4}$$

and consequently

$$F(x) = -\frac{3\pi}{x} \tag{5}$$

for $x < 0$ and $|x| >> 1$

Spectral representation for $\pi(k^2)$ requires that F is a monotonically increasing function. This is a consequence of the positivity of the spectral function. The actual behavior of F has been found by Anselm [9] to be

$$F_{as}(x) = e^{\nu x} P(x) \tag{6}$$

where

$$\nu = 1 - \lim_{k^2 \to -\infty} \frac{k^2 \int_0^\infty \frac{dM^2 \sigma(M^2)}{(k^2-M^2)^2}}{\int_0^\infty \frac{dM^2 \sigma(M^2)}{k^2-M^2}} \qquad 0 \leqslant \nu \leqslant 1 \tag{7}$$

Function P has the property

$$\lim_{x \to \infty} \frac{P(x+y)}{P(x)} = 1 \tag{8}$$

This means, that the function $F_{as}$ behaves essentially like an exponential for very large x. However, in case of the finite renormalization, $\int dM^2\sigma(M^2) < \infty$, $\nu=0$ and $F_{as}(x) = P(x)$ goes to a constant when $x \to \infty$.

After the insertion of (3) and the change of variables $x=1-e^{-t}$ equation (1) takes the form

$$a(\alpha) = \frac{1}{\pi} \int_0^\infty dt\ e^{-2t}\ F\left(-\frac{3\pi}{\alpha} + t\right) \tag{9}$$

One observes here that $a(+0) \neq a(-0)$ as a consequence of the monotonic behavior of F.

The behavior of the coefficients $a_n$ in the expansion $a(\alpha) = \sum_n a_n \alpha^n$ can be found by using Lipatov's method [12]. One considers a double integral

$$a_n = \frac{1}{2\pi i} \int_0^\infty dt \oint_0 \frac{d\alpha}{\alpha^{n+1}}\ F\left(-\frac{3\pi}{\alpha} + t\right) e^{-2t} \tag{10}$$

and looks for a saddle point in $(\alpha,t)$ variables. The saddle point is determined by equations:

$$2 - f'\left(-\frac{3\pi}{\alpha} + t\right) = 0$$

$$\frac{n}{\alpha} - \frac{3\pi}{\alpha^2} f'\left(-\frac{3\pi}{\alpha} + t\right) = 0 \tag{11}$$

where $f'(x) \equiv \frac{dF}{dx}\Big/F(x)$. The solution is $\alpha \doteq \frac{6\pi}{n}$ and $t = \frac{n}{2} + \lambda$. $\lambda$ is an n-independent number such that $f'(\lambda) = 2$. One sees immediately that

$$a_n \sim \frac{n!}{(6\pi)^n} \quad , \text{ n large} \tag{12}$$

which is the result of Ref. [4] . Therefore, if one can find a solution to the saddle point equations, i.e. if f' can achieve the value of 2, (9) would give a non-Borel summable contribution. The function F, however, satisfies the spectral representation

$$F(\phi+t) = 1+e^t \int_0^\infty \frac{\sigma(s,\phi)}{e^t+s}\, ds \tag{13}$$

and in consequence, as one can easily convince oneself, the values of f' have to lie in the interval $[0,1]$. Thus the existence of a real saddle point, which would imply the non-Borel summability of the perturbation expansion, is prevented by the spectral representation. On the other hand, it is difficult to say anything definite about the contribution, that a complex saddle point would make, if it existed. It would have to lie in the region (Re $e^t$ <0), where one does not trust the above approximation anyway.

One can insert now into (9) an identify $F(x) = \left[F(x) - F_{as}(x)\right] + F_{as}(x)$. $F-F_{as}$ coincides with $F_{pert}$ for $\nu \neq 0$ for sufficiently small x, say $x<-\lambda_1$, $\lambda_1>0$, and is vanishingly small for large $x>\lambda_2$. The integral over the perturbative region can be performed and the result is expressed in terms of the exponential-integral functions. It has an essential singularity at $\alpha=0$ and vanishes as $\alpha \to 0_+$.

One finds that this contribution takes form

$$a_{pert}(\alpha) = \sum_{n=0}^{\infty} c_n(\alpha)\; n!\left(\frac{\alpha}{6\pi}\right)^{n+1} \tag{14}$$

where

$$c_n(\alpha) = 1 - e^{-\frac{6\pi}{\alpha}\Lambda} \sum_{k=0}^{n} \left(\frac{\alpha}{6\pi\Lambda}\right)^{-k} \frac{1}{k!}$$

and $\Lambda=1-\frac{\alpha\lambda_1}{3\pi}$. $c_n(\alpha)$ are of order 1 for $n \lesssim \frac{6\pi}{\alpha}$ and decrease rapidly above this value. Therefore the series (14) mimics the results of Ref. [3,4] up to this order.

The integral over the intermediate region of $\left[F-F_{as}\right]$ gives simply

$$e^{-\frac{6\pi}{\alpha}} \int_{-\lambda_1}^{\lambda_2} dt\; e^{-2t}\left(F-F_{as}\right)(t) = e^{-\frac{6\pi}{\alpha}}\;\text{const.} \tag{15}$$

Finally, from the asymptotic contribution alone

$$a_{as} = \int_0^{\infty} dt\; e^{-2t}\; F_{as}\left(-\frac{3\pi}{\alpha}+t\right)$$

assuming that $P(x)$ behaves like a polynomial, one obtains the result

$$a_{as} = e^{-\frac{3\pi}{\alpha}\nu} \left[P\left(\frac{d}{ds}\right) \frac{e^{-\frac{3\pi}{\alpha}s}}{2-\nu-s}\right]_{s=0} \tag{16}$$

In case of the finite renormalization when $\nu=0$, one can approximate $F_{as}$ by a $\theta$-function. Then $F-F_{as}$ still can be approximated by $F_{pert}$ for $x<-\lambda_1$. One finds in this case that

$$a_{as} = \text{const} \cdot e^{-\frac{6\pi}{\alpha}} \tag{17}$$

As one sees, $a(\alpha)$ contains non-perturbative terms of the type $e^{-\frac{1}{\alpha}}$.

We notice that the non-perturbative contribution is consistent with the (weak) Borel summability [10,11] of the perturbation series. The terms of the type $e^{-\frac{1}{\alpha}}$ can always show up after the resummation of an asymptotic series as they vanish identically in the expansion in $\alpha$. Of course, whether other classes of diagrams, besides those on Fig. 1, 2, can contribute to a non-Borel summable singularity is still an open question.

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## FIGURE CAPTIONS

1. n-bubble insertion.
2. General blob insertion.

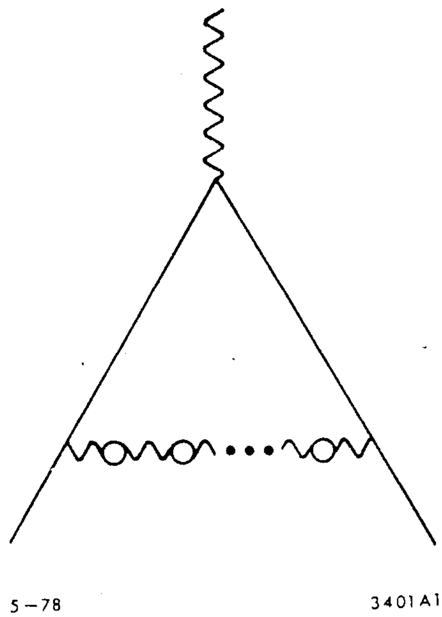


Fig. 1

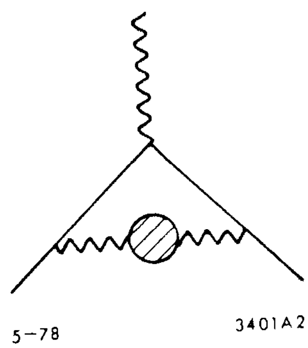


Fig. 2