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EVALUATION OF BEAM DISTRIBUTION PARAMETERS IN AN ELECTRON STORAGE RING*

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ABSTRACT

The motion of a charged particle in a linear electromagnetic device can often be analyzed by using the transport matrices. For an electron storage ring, this technique has been applied to yield fruitful results such as the trajectory of the particle distribution center and the beam sizes and shapes in phase space. Coupling effects among the horizontal, vertical and longitudinal motions are included in a straightforward manner.

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I. INTRODUCTION

Ignoring nonlinear perturbations, the equilibrium particle distribution in an electron storage ring is gaussian in the phase space with canonical coordinates $X=(x,x',y,y',z,\delta)$, where x, y, and z are the horizontal, vertical, and longitudinal displacements relative to the center of particle distribution; x' and y' are the corresponding conjugate momenta; and $\delta = \frac{\Delta E}{E_0}$ is the relative energy deviation.[1,2,3,4] To describe this particle distribution, it is necessary and sufficient to specify the distribution parameters which include the 6 first-moments $\langle x_i \rangle$ and the 21 second-moments $\langle x_i x_j \rangle$, $i,j = 1,2,\ldots 6$ with x_i's denoting the canonical coordinates of the state vector X. Various storage ring quantities, such as the closed orbit trajectory of the beam, the beam width, beam height, bunch length, tilting angle in the x-y plane, beam energy spread, etc., are directly given by those distribution parameters. Since the closed orbit and the beam size and shape must fulfill the requirements set by the design goals of storage ring, we must be able to obtain the distribution parameters for any storage ring lattice being designed. The matrix formalism presented in this paper will serve as a straightforward method of obtaining the beam distribution parameters under a wide variety of storage ring operating conditions.

II. BACKGROUND

As an electron circulates in the guiding magnetic field, it emitts photons due to synchrotron radiation. The energy loss of an electron due to emitting a photon excites the transverse and longitudinal occillations of the electron in its subsequent motion. In general, this effect of quantum excitations causes the beam size to diffuse in all three dimensions. As the beam emittances grow due to diffusion, the effect of radiation damping[2,3,5] becomes significant and an equilibrium beam distribution is reached as the quantum diffusion and radiation damping balance each other.

Assuming an ideal storage ring and ignoring any coupling between x, y and z dimensions, most beam distribution parameters simply vanish and the remaining parameters can be obtained by relatively elementary considerations[2,3]. However, in practice, some coupling effects may not be negligible. For example, x- and z-motions are coupled if the rf cavities are located at positions with finite energy dispersion;[6] x- and y-motions are coupled if there are skew quadrupole fields in the storage ring. One possible way to include coupling in the evaluation of distribution parameters is to assume that the coupling effects can be approximately described by a set of coupling coefficients which specify the coupling strength averaged over one revolution of the storage ring.[4] Distribution parameters are then obtained by solving the corresponding Fokker-Planck diffusion equation.[1,2] Another method ignores couplings to the longitudinal z dimension and the relevant distribution parameters are obtained by a careful analysis of the balance between quantum diffusion and radiation damping.[7,8] In the above theories, it has been necessary to introduce the horizontal and vertical energy dispersion functions[9] in order to carry out the analysis.

In the following we describe a general framework which determines the distribution parameters in a coherent manner and with the above-mentioned coupling effects fully taken into account. In this method, each linear element in the storage ring lattice is represented by a 6×6

TRANSPORT matrix[10] which transforms the state vector X as an electron passes through the element. Nonlinear elements are approximated by linearizing their electromagnetic fields around the center of particle distribution. Knowing the TRANSPORT matrix transformations around the storage ring, the distribution parameters can be obtained from the eigenvalues and eigenvectors of some proper matrices yet to be described. In particular, no knowledge of the energy dispersion functions is needed.

III. TRAJECTORY OF THE BEAM CENTER

Relative to the design orbit, the motion of an electron can be decomposed into two components, X_0 and X, with X_0 describing the trajectory of the beam distribution center and X being the oscillatory deviation from X_0. The fact that the trajectory of the beam distribution center does not coincide with the design orbit (i.e. $X_0 \neq \vec{0}$) could be due to several causes: (i) A quadrupole magnet misaligned in its transverse position is the same as superimposing a dipole magnetic field to the magnet. This accidental dipole field kicks the beam orbit transversely; (ii) Additional dipole magnets may exist in the lattice for orbit correction or beam injection purposes; (iii) The energy loss of an electron due to synchrotron radiation is compensated by the acceleration provided at the rf cavities. Since the rf cavities are usually not located at places where synchrotron radiation occurs, the beam distribution center can deviate from the design trajectory as a result.[11,12]

Existing methods of obtaining X_0 treat each orbit distortion mechanism individually and ignore the possibility of couplings among x, y, and z motions. In the following of this section, we present a method which includes various orbit distortion mechanisms simultaneously and taking into account the coupling effects.

We first form a 7-dimensional vector V by adding to X_0 a seventh component which is always given by unity.[13] The vector V_f at the exit of a given lattice element is linearly related to the vector V_i at the entrance by $V_f = MV_i$. The 7×7 transformation matrix M can be obtained from the electromagnetic fields in the lattice element. For simplicity, we assume that all lattice elements aside from the drift spaces are short compared with the focal lengths so that a thin-lens approximation applies. Generalization to the thick-lens case is straightforward. Under this condition, the diagonal elements of M are always equal to unity and off-diagonal elements except those listed in Table 1 vanish. In Table 1, we have defined ℓ=length of the lattice element, $B\rho$ = particle rigidity, $2\pi R$ = circumference of the storage ring, E_0 = design particle energy, $C_\gamma = \frac{4\pi}{3} r_e/(m_e c^2)^3$ with r_e the classical electron radius and m_e the electron mass; and for the rf cavities,[3] $\hat{V}$ = peak applied voltage, h = harmonic number, ϕ_s = synchronous phase. The upper left 6×6 corners of these matrices are the thin-lens version of the usual TRANSPORT matrices.[10]

The trajectory of the particle distribution center, described by $V(s) = (X_0(s), 1) = (x_0, x_0', y_0, y_0', z_0, \delta_0, 1)$, with s indicating the position around the storage ring, must satisfy

$$x_0, x_0', y_0, y_0' (s+2\pi R) = x_0, x_0', y_0, y_0' (s),$$

$$\text{net energy change per turn} = 0 \qquad (1)$$

and

$$\text{net path length travelled per turn} = \text{exactly } 2\pi R,$$

where the last condition is imposed by the fact that the rf cavity frequency is accurately fixed. To find V(s), we first look for the transformation matrix for one revolution, W(s), by multiplying the M-matrices of all

TABLE 1.

Nonvanishing Off-diagonal Elements of the Matrices M

Lattice Element	Matrix Elements
drift space (DRF)	$m_{12} = m_{34} = \ell$
horizontal bending magnet (HB)	$m_{26} = -m_{51} = \frac{\ell}{B\rho} B_y$, $m_{67} = -C_\gamma E_0^3 (\frac{\ell}{B\rho} B_y)^2/2\pi\ell$, $m_{27} = \frac{1}{2} (\frac{\ell}{B\rho} B_y) m_{67}$
vertical bending magnet (VB)	$-m_{46} = m_{53} = \frac{\ell}{B\rho} B_x$, $m_{67} = -C_\gamma E_0^3 (\frac{\ell}{B\rho} B_x)^2/2\pi\ell$, $m_{47} = -\frac{1}{2} (\frac{\ell}{B\rho} B_x) m_{67}$
quadrupole magnet (QUA)	$-m_{21} = m_{43} = \frac{\ell}{B\rho} \frac{\partial B_y}{\partial x}$
skew quadrupole magnet (SKQ)	$-m_{23} = -m_{41} = \frac{\ell}{B\rho} \frac{\partial B_y}{\partial y}$
rf cavity (CAV)	$m_{65} = e\hat{V}h \cos\phi_s / RE_0$, $m_{67} = e\hat{V} \sin\phi_s / E_0$
horizontal dipole kicker (HK)	$m_{26} = -m_{51} = -m_{27} = \frac{\ell}{B\rho} B_y$, $m_{67} = -C_\gamma E_0^3 (\frac{\ell}{B\rho} B_y)^2/2\pi\ell$, $m_{27} = \frac{1}{2} (\frac{\ell}{B\rho} B_y) m_{67}$
vertical dipole kicker (VK)	$-m_{46} = m_{53} = m_{47} = \frac{\ell}{B\rho} B_x$, $m_{67} = -C_\gamma E_0^3 (\frac{\ell}{B\rho} B_x)^2/2\pi\ell$, $m_{47} = -\frac{1}{2} (\frac{\ell}{B\rho} B_x) m_{67}$

lattice elements successively from s to s + $2\pi R$. V(s) is then simply given by the eigenvector of W(s) with eigenvalue 1, i.e.,

$$W(s)\ V(s) = V(s) \quad . \tag{2}$$

That one of the eigenvalues of W(s) is equal to 1 can be easily proved. The trajectory of the particle distribution center obtained here, $X_0(s)$, will also be referred to as the closed orbit of the beam.

Effects on closed orbit due to nonlinear elements can only be treated by iteration. A sextupole magnet is thus represented by a horizontal and a vertical kicker with the nontrivial transformation matrix elements

$$\text{SEX:}\ m_{26} = -\ m_{51} = -m_{27} = \frac{\lambda}{2}\ (x_0^2-y_0^2) \quad , \tag{3}$$

$$-m_{46} = m_{53} = m_{47} = \lambda\ x_0 y_0 \quad ,$$

where $\lambda = \frac{\ell}{B\rho}\ \partial^2 B_y \Big/ \partial x^2$ is the strength of the sextupole field. Nonlinear effects due to radiation energy losses in quadrupole, skew quadrupole and sextupole magnets can also be included by inserting the following additional matrix elements:

$$\begin{aligned}
\text{SEX:}\ m_{67} &= -C_\gamma E_0^3 \lambda^2 (x_0^2 + y_0^2)^2/8\pi\ell \\
\text{SKQ:}\ m_{67} &= -C_\gamma E_0^3 \left(\frac{\ell}{B\rho}\ \frac{\partial B_y}{\partial y}\right)^2\ (x_0^2 + y_0^2)/2\pi\ell \\
\text{QUA:}\ m_{67} &= -C_\gamma E_0^3 \left(\frac{\ell}{B\rho}\ \frac{\partial B_y}{\partial x}\right)^2\ (x_0^2 + y_0^2)/2\pi\ell\ .
\end{aligned} \tag{4}$$

The iteration steps are: (i) calculate V without nonlinearities, (ii) evaluate transformation matrices for nonlinear elements according to Eqs. (3) and (4), (iii) Recalculate V with the new matrices, (iv) iterate. A few iterations should be enough for convergence. The total energy loss per turn due to synchrotron radiation can be obtained from

$$U_0 = \sum_{CAV} e\hat{V}\left(\sin\phi_s + \frac{h}{R} z_0\cos\phi_s\right) \quad . \tag{5}$$

IV. RADIATION DAMPING CONSTANTS[2,3,5,7,8]

Once the closed orbit is obtained, a particle's motion can be described by the state vector $X=(x, x', y, y', z, \delta)$, which is the deviation from closed orbit. The transformation matrices for X are given by the upper left 6×6 corner for lattice elements listed in Table 1. The transformation for sextupoles, linearized around the closed orbit, is a combination of HK, VK, SKQ and QUA;

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\lambda x_0 & 1 & \lambda y_0 & 0 & 0 & \frac{\lambda}{2}(x_0^2-y_0^2) \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \lambda y_0 & 0 & \lambda x_0 & 1 & 0 & -\lambda x_0 y_0 \\ -\frac{\lambda}{2}(x_0^2-y_0^2) & 0 & \lambda x_0 y_0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

From here on, we will concentrate on these 6×6 TRANSPORT transformations. By successively multiplying all M-matrices from s to $s+2\pi R$, one gets the transformation for one revolution, T(s), which will be extensively used in the next section. To obtain the radiation damping constants, however, further modifications on the matrix elements are necessary.

As an electron passes through an rf cavity, it receives a boost in its longitudinal momentum, while keeping the transverse momentum unchanged. Consequently, the slopes x' and y' of the electron's trajectory is reduced in magnitude by a factor of $1 - \frac{u}{E_0}$, where $u = e\hat{V}\left(\sin\phi_s + \frac{h}{R} z_0 \cos\phi_s\right)$ is the energy gain of the electron at the rf cavity. This effect provides the transverse radiation damping to the motion of an electron. In the matrix formalism, it is taken into account by replacing the following diagonal element in the CAV matrices:

$$m_{22} = m_{44} = 1 - \frac{e\hat{V}}{E_0}\left(\sin\phi_s + \frac{h}{R} z_0 \cos\phi_s\right) \quad . \tag{7}$$

The longitudinal radiation damping comes from the fact that an electron with higher energy tends to lose energy by radiating more in a magnetic field. This results in the following changes:

$$\begin{aligned} &\text{HB, VB, HK, VK: } m_{66} = 1 - C_\gamma E_0^3\left(\frac{\ell}{B\rho} B_{x,y}\right)^2 / \pi\ell \\ &\text{QUA, SKQ: } m_{66} = 1 - \Lambda_q (x_0^2 + y_0^2) \\ &\text{SEX: } m_{66} = 1 - \frac{1}{2}\Lambda_s (x_0^2 + y_0^2) \quad , \end{aligned} \tag{8}$$

where $\Lambda_q = C_\gamma E_0^3 \left(\frac{\ell}{B\rho} \frac{\partial B_y}{\partial x,y}\right)^2 / \pi\ell$ and $\Lambda_s = C_\gamma E_0^3 \lambda^2 (x_0^2 + y_0^2)/2\pi\ell$. In addition, energy radiated in quadrupole, skew quadrupole and sextupole magnets also depends on the transverse displacements of an electron. This contributes to the mixing among the transverse and longitudinal damping effects. Upon linearization, it yields

$$\begin{aligned} &\text{QUA, SKQ: } m_{61} = -\Lambda_q x_0, \; m_{63} = -\Lambda_q y_0 \\ &\text{SEX: } m_{61} = -\Lambda_s x_0, \; m_{63} = -\Lambda_s y_0 \quad . \end{aligned} \tag{9}$$

After making the modifications (7), (8) and (9), one calculates once again the transformation matrix for one revolution, D(s), by multiplying M-matrices from s to s+2πR. The eigenvalues of D(s) are expressed as $\exp(-\alpha_k \pm i2\pi\nu_k)$, k=I, II, III, where k is the mode index, ν_k is the tune[9] of mode k and α_k's are the radiation damping constants. In particular, if any one of the α_k's is negative, the motion will be unstable. By evaluating the eigenvalues of D(s), we have thus derived a method of calculating the radiation damping constants in the presence of various coupling effects. Furthermore, it follows from the property of eigenvalues that

$$e^{-2(\alpha_I + \alpha_{II} + \alpha_{III})} = \det(D(s)) \quad . \tag{10}$$

For weak damping with $|\alpha_k| << 1$, it reduces to the usual sum rule[5]

$$\alpha_I + \alpha_{II} + \alpha_{III} = 2U_0/E_0 \quad . \tag{11}$$

In the special case of no closed orbit distortion and no coupling, the damping constants are given by

$$\alpha_x = \alpha_y = U_0/2E_0 \quad , \quad \alpha_z = U_0/E_0 \quad . \tag{12}$$

V. BEAM SIZES AND SHAPES

Our method can be used to determine quantities like the horizontal and vertical beam sizes, the tilt angle in the x-y plane, the natural bunch length and energy spread, etc. To do this, let us consider a photon being emitted at position s_0 with energy deviating from the mean value by a random amount δE. Let $T(s_0)$ be the coupled 6×6 transformation matrix for one revolution obtained in the previous section without radiation damping. The eigenvalues, λ_k, and eigenvectors, $E_k(s_0)$, of $T(s_0)$ with k = ±I, ±II, ±III are defined by

$$T(s_0)\ E_k(s_0) = \lambda_k E_k(s_0)$$

$$E_k^*(s_0) = E_{-k}(s_0) \tag{13}$$

$$\lambda_{\pm k} = e^{\pm i2\pi\nu_k}$$

Following the photon emission, the subsequent motion of the electron is described by

$$X(s) = \sum_k A_k E_k(s) \quad , \quad s \geq s_0 \tag{14}$$

where $E_k(s)$ is the eigenvector of $T(s)$ obtained from $E_k(s_0)$ by the matrix transformation from s_0 to s. Equation (14) satisfies the initial condition

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\delta E/E_0 \end{bmatrix} = \sum_k A_k E_k(s_0) \tag{15}$$

where the left-hand side is the impulse perturbation to electron state due to the photon emission event. It should be pointed out that the transverse dimensions of the electron are not excited at the instance of photon emission and consequently no knowledge of the energy dispersion functions is needed.

From the symplecticity condition[9] of $T(s)$, i.e.,

$$\tilde{T}ST = S \quad , \tag{16}$$

where a tilde means taking the transpose of a matrix and

$$S = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} ,$$

one can prove that

$$\tilde{E}_j S E_i = 0 \quad \text{unless } j = -i \quad . \tag{17}$$

We will normalize the eigenvectors so that

$$\tilde{E}_k^* S E_k = i \quad , \quad k = \text{I, II, III} \quad . \tag{18}$$

This normalization condition is preserved as a function of s due to the symplecticity of T(s). Using Eqs. (17) and (18), Eq. (15) yields

$$A_k = -i \frac{\delta E}{E_0} E_{k5}^*(s_0) \tag{19}$$

where E_{ki} means the i^{th} component of the vector E_k. Assuming all photon emission events are uncorrelated, one obtains the quantum diffusion rate of $|A_k|^2$ by averaging Eq. (19) around the storage ring:

$$\frac{d}{dt} \langle |A_k|^2 \rangle = \frac{1}{2\pi R} \oint ds \, \langle \dot{N} \frac{\delta E^2}{E_0{}^2} \rangle |E_{k5}(s)|^2 \quad . \tag{20}$$

In Eq. (20), $\dot{N}$ is the number of photons emitted per unit time and[2,3]

$$\langle \dot{N} \frac{\delta E^2}{E_0{}^2} \rangle = 2C_L \gamma^5 / |\rho(s)|^3 \quad , \tag{21}$$

where $C_L = \frac{55}{48\sqrt{3}} r_e \hbar / m_e$ with $\hbar$ the reduced Planck's constant, γ is the relativistic factor and $\rho(s)$ is the bending radius.

So far we have ignored the radiation damping which, when taken into account, gives an additional contribution

$$\frac{d}{dt} <|A_k|^2> = -\frac{2\alpha_k}{T_0} <|A_k|^2> \quad , \tag{22}$$

with T_0 the revolution time and α_k the radiation damping constants found in the previous section. The equilibrium values of $<|A_k|^2>$ is given by a balance between quantum diffusion and radiation damping, which gives

$$<|A_k|^2> = <|A_{-k}|^2> = C_L \frac{\gamma^5}{c\alpha_k} \oint ds \frac{|E_{k5}(s)|^2}{|\rho(s)|^3} \quad . \tag{23}$$

It follows from Eq. (14) that the particle distribution parameters at position s are given by

$$<x_i x_j>(s) = 2 \sum_{k=I,II,III} <|A_k|^2> \mathrm{Re}\left[E_{ki}(s)\; E^*_{kj}(s)\right] \quad . \tag{24}$$

Equations (23) and (24) are our final expressions. The tilt angle θ of the x-y beam profile relative to the horizontal axis can be found from

$$\tan 2\theta = \frac{2<xy>}{<x^2>-<y^2>} \tag{25}$$

and the transverse beam area (for luminosity calculation) is given by

$$A = \pi \sqrt{<x^2><y^2>-<xy>^2} \quad . \tag{26}$$

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