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On the Question of Gauge Ambiguity

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Abstract

It is pointed out that the ambiguity which characterizes gauge conditions of the type $\partial^\mu \beta_{\mu\nu} A^a(x)^\nu = C_1^a(x)$ for nonAbelian gauge theories is also characteristic of the so-called axial-like gauge conditions $n \cdot A^a = C_2^a(x)$, where, here, A_μ^a is the nonAbelian gauge potential, $\beta_{\mu\nu} = g_{\mu\nu}$ or $\beta_{\mu\nu} = g_{\mu\nu} - \delta_{\mu o}\delta_{\nu o}$, n_μ is a four-vector such that $n^2 = 0, 1,$ or -1, and $C_{1,2}^a$ are usually A_μ^a-independent functions of x; $g_{\mu\nu}$ is the Minkowski metric and δ_{ij} is the Kronecker delta.

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As has been emphasized by Mandelstam and Gribov, the specification of the gauge in nonAbelian gauge theory is a delicate matter in the Coulomb gauge and is, in fact, ambiguous in this gauge.[1] It appears to be generally accepted[2,3] that the Mandelstam-Gribov problem is also present in gauges which have gauge conditions

$$\partial_\mu A^{a\mu}(x) = C_1^a(x) \quad , \tag{1}$$

where A_μ^a is the Yang-Mills field and $C_1^a(x)$ is an A_μ^a-independent function of the space-time coordinate x. However, it does not seem[2,4] to be general knowledge that the ambiguity of Mandelstam and Gribov is also present in gauges of the type

$$n \cdot A^a = C_2^a(x) \tag{2}$$

where n^μ is a four-vector such that $n^2 = 0$, 1, or -1, and $C_2^a(x)$ is an A_μ^a-independent function of x. We should like to clarify this particular point in this note.[†]

Specifically, we use a matrix realization of the Yang-Mills theory so that we introduce

$$A_\mu = A_\mu^a t^a$$

where the hermitian matrices t^a carry the adjoined representation of the respective gauge group $\mathscr{G}$:

$$[t^a, t^b] = i f_{abc} t^c \tag{3}$$

where f_{abc} are the real structure constants of $\mathscr{G}$ and

$$(t^a)_{bc} = -i f_{abc} \quad . \tag{4}$$

We normalize t^a according to

[†]Recently, A. Balachandran et al., Phys. Rev. Lett. 40, 988 (1978), have discussed the ambiguities associated with the U-gauge.

$$\mathrm{tr}(t^a t^b) = N\,\delta_{ab} \quad . \tag{5}$$

Then, (2) reads

$$n\cdot A = C_2 \tag{6}$$

where

$$C_2 = c_2^a t^a \quad . \tag{7}$$

Now, under a gauge transformation $U \equiv \exp[-i\vec{\omega}\cdot\vec{t}]$, A transforms by

$$A_\mu \rightarrow U A_\mu U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1} \tag{8}$$

when we take the Yang-Mills field strength tensor $F_{\mu\nu}$ to be

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] \quad , \tag{9}$$

where g is the gauge coupling constant.† Thus, the gauge condition (6) becomes

$$n^\mu \left(U A_\mu U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1}\right) = C_2 \tag{10}$$

or

$$U C_2 U^{-1} + \frac{i}{g}(n\cdot\partial U)U^{-1} = C_2 \quad . \tag{11}$$

Thus, it has been commonly accepted that, for $C_2 \equiv 0$, for example, the only ambiguity in (6) is an uninteresting A_μ-independent U(x) which only depends on $\vec{x}_\perp$, where $\vec{x}_\perp$ are the <u>three</u> coordinates orthogonal to n^μ. For, if $C_2 \equiv 0$, then, (11) reads

$$\frac{i}{g}(n\cdot\partial U)U^{-1} = 0 \tag{12}$$

so that (see footnote (4))

$$n\cdot\partial U = 0 \quad . \tag{13}$$

However, the last equation, Eq. (13), for a simple compact Lie group, has other solutions than $U(x) = U(\vec{x}_\perp)$, since

$$n\cdot\partial U(\vec{\omega}(x)) = \frac{\partial U}{\partial\vec{\omega}}\cdot n\cdot\partial\vec{\omega} \quad . \tag{14}$$

Hence, Eq. (13) can be satisfied if

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†The Yang-Mills Lagrangian is $\mathcal{L}_{YM} = -\frac{\mathrm{tr}}{4N} F_{\mu\nu} F^{\mu\nu}$ in our convention.

(i) $n\cdot\partial\vec{\omega}|_{\vec{\omega}=\vec{\omega}_o} \equiv \vec{0}$ and $\left\|\frac{\partial U}{\partial\vec{\omega}}\Big|_{\vec{\omega}=\vec{\omega}_o}\right\| \neq \infty$, or (15a)

(ii) $\frac{\partial U}{\partial\vec{\omega}}\Big|_{\vec{\omega}=\vec{\omega}_o} = \vec{\tau} \neq \vec{0}$ and $n\cdot\partial\vec{\omega}\Big|_{\vec{\omega}=\vec{\omega}_o} = \vec{\sigma} \neq \vec{0}$ (15b)

but $\vec{\tau}\cdot\vec{\sigma} = 0$.

Here $\|\vec{a}\|$ denotes the norm of the array $\vec{a}_{ij}$ as

$$\|\vec{a}\| = \max\{(\sum_i(\sum_j a^k_{ij}\lambda_j)^2)^{\frac{1}{2}} | (\sum_j \lambda_j^2)^{\frac{1}{2}} = 1\} . \quad (16)$$

Clearly, case (i) is the well-discussed case

$$\vec{\omega} = \vec{\omega}_o(\vec{x}_\perp) \rightarrow U(x) = U(\vec{x}_\perp) . \quad (17)$$

Notice that (17) could still be interesting because we may have, for example,

$$\vec{\omega}_o(\vec{x}_\perp) = \vec{V}(\int_{-\infty}^{\infty} \alpha^{\mu\nu}(x)\, A_\nu(x)\, dn,\ \vec{x}_\perp) \quad (18)$$

for some weighting field $\alpha^{\mu\nu}(x)$ and some functional $\vec{V}$. Here dn is the line element along n^μ. Thus, U may be A_ν-dependent.

Case (ii) involves those functions $\vec{\omega}(x)$ such that the gradient matrix of $U(\vec{\omega})$ is orthogonal to $n\cdot\partial\vec{\omega}$. Thus, to be specific, we take $\mathscr{G}$ = SU(2) so that

$$(t^a)_{kj} = i\epsilon_{kaj} = -i\epsilon_{akj} \quad (19)$$

where ϵ_{akj} is the totally antisymmetric symbol in three dimensions so that $\epsilon_{123} = 1$. Then,

$$U(\vec{\omega}) = I - (\hat{\omega}\cdot\vec{t})^2\ (1 - \cos\omega) - i(\hat{\omega}\cdot\vec{t})\sin\omega\ , \quad (20)$$

where

$$\omega = |\vec{\omega}|\ ,\quad \hat{\omega} = \vec{\omega}/|\vec{\omega}|\ . \quad (21)$$

Therefore

$$\frac{\partial U}{\partial \vec{\omega}} = -\left\{\frac{\vec{t}\omega^2 - \vec{t}\cdot\vec{\omega}\vec{\omega}}{\omega^3}, \hat{\omega}\cdot\vec{t}\right\}(1 - \cos\omega)$$

$$- (\hat{\omega}\cdot\vec{t})^2 \hat{\omega} \sin\omega$$

$$-i\left[\frac{\vec{t}\omega^2 - \vec{t}\cdot\vec{\omega}\vec{\omega}}{\omega^3}\right]\sin\omega$$

$$-i \vec{t}\cdot\hat{\omega} \hat{\omega} \cos\omega \quad , \tag{22}$$

where $\{\ ,\ \}$ denotes anti-commutation and $\vec{t} = (t^1, t^2, t^3)$. There are obviously cases in which $\partial U/\partial\vec{\omega}$ is orthogonal to $n\cdot\partial\vec{\omega}$. For example, suppose $\cos\omega = 1$, $\sin\omega = 0$. Then,

$$\left.\frac{\partial U}{\partial \vec{\omega}}\right|_{\cos\omega = 1} = -i \vec{t}\cdot\hat{\omega} \hat{\omega} \tag{23}$$

will be orthogonal to $n\cdot\partial\vec{\omega}$ if

$$\hat{\omega}\cdot(n\cdot\partial)\vec{\omega} = 0$$

$$\Longrightarrow n\cdot\partial(\omega^2) = 0$$

$$\Longrightarrow \omega^2 \equiv \Lambda(\vec{x}_\perp) \ . \tag{24}$$

Thus,

$$\vec{\omega} = (f_1(x),\ f_2(x),\ \pm\sqrt{\Lambda(\vec{x}_\perp) - f_1^2(x) - f_2^2(x)}) \tag{25}$$

such that†

$$\Lambda^{1/2}(\vec{x}_\perp) = \sum_n (2n\pi)\ \theta(x_\perp^2 - a_n),\ a_n > 0,\ n \text{ integral} \tag{26}$$

will satisfy the condition (15b) for real functions $f_1(x)$, $f_2(x)$. Hence, f_1 and f_2 may be real functions of $A_\nu(x)$ also:

$$f_i = f_i(A_\nu(x), x) \quad , \quad i = 1, 2 \ . \tag{27}$$

In practice, we expect that $n \le 2$ in (26) and that $f_{1,2}$ must be such that $\vec{\omega}$

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†Here, $\theta(s) = 0$ for $s < 0$ and $\theta(s) = 1$ for $s \ge 0$.

is a real vector-valued function. Evidently, this question of gauge ambiguity is quite involved.

Notes added:

(1) Note that the condition

$$\left.\frac{\partial U(\vec{\omega})}{\partial \vec{\omega}}\right|_{\vec{\omega} = \vec{\omega}_0} = \vec{0}$$

cannot be satisfied for any real $\vec{\omega}_0$ since the t^a are linearly independent. Thus, it is only necessary to consider cases (i) and (ii) in discussing the meaning of Eq. (13).

(2) R. Jackiw, I. Muzinich, and C. Rebbi (Brookhaven preprint, 1977) have shown that Mandelstam-Gribov ambiguities exist in the Coulomb gauge even if one imposes the condition (here, $r = |\vec{x}|$)

$$\lim_{r\to\infty} r\vec{A} = 0$$

on transverse potentials. These authors show that, under this condition, a single potential in the $A_0 = 0$ gauge corresponds to several transverse Coulomb gauge potentials when the potentials are large enough - in agreement with Gribov and Mandelstam. However, these authors do not address the question of ambiguities in the temporal gauge itself.

(3) As pointed-out by Jackiw in reference 4, I. Singer (unpublished), working in S^4, has shown that Mandelstam-Gribov ambiguities must exist in all gauges for gauge theories defined in S^4. However, as emphasized by Jackiw, S^4 is not 4-dimensional Minkowski space, not even up to homeomorphisms.

(4) In the theory of functions of several variables (see, for example, L. M. Graves, <u>Theory of Functions of Several Variables</u>, (McGraw-Hill Book Company, Inc., New York, 1956) pp. 76-78) the theorem that, at $x_\alpha = \bar{x}_\alpha$, $\alpha = 1,\ldots,4$,

$$dU = \sum_{\alpha=1}^{4} (\partial U/\partial x_\alpha) dx_\alpha$$

if and only if $\partial U/\partial x_\alpha$, $\alpha = 1, \ldots, 4$, are all finite at $\bar{x}$ is not true. There is the additional requirement that all $\partial U/\partial x_\alpha$ must be continuous in a neighborhood of $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_4)$. The example of Eq. (15b) given in Eqs. (19)-(27) does not even have all $\partial U/\partial x_i$ finite. Thus, for this example

$$dU \neq \sum_{\alpha=1}^{4} (\partial U/\partial x_\alpha)\, dx_\alpha$$

so that $\partial U/\partial x_4 = 0$ does not imply that $\partial U/\partial x_j$, $j = 1, 2, 3$, are independent of x_4. Here, dU is the change in $U(\bar{x})$ due to the changes $\bar{x}_\alpha \to \bar{x}_\alpha + dx_\alpha$.

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