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A CONSTRUCTIVE METHOD FOR THE LATTICE GAUGE THEORY*

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ABSTRACT

A method is given for defining gauge-invariant actions for the lattice theory. The basic idea is to first define the theory without respecting gauge-invariance, and <u>then</u> study its gauge-invariant projection. The constraints that the lattice theory exhibit free-field behavior for weak-coupling and disorder for strong-coupling limits us to a special set of possible actions. We study the simplest of these non-trivial theories, which involves a nonlinear nearest-neighbor scalar coupling. This theory has a phase transition which separates the weak and strong coupling sectors. For dimension near 4, the phase transition from the strong coupling phase to the weak coupling phase is shown to be a continuous (second-order) phase transition.

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I. The Action

The lattice gauge theory[1] as defined by Wilson takes as its starting point the continuum Yang-Mills action. In the lattice theory, the Yang-Mills continuum action is understood as describing a classical non-Abelian gauge field. To quantize the theory, a lattice (cut-off) is introduced; the quantum action is then defined. The Yang-Mills theory is recovered in the classical (weak field) limit. In the lattice theory, the salient feature of the quantum action is exact local gauge invariance for the cut-off field theory. The way this was obtained by Wilson[1] was to discretize the classical theory, and then to directly generalize the classical action to the appropriate quantum action. What he then obtained was an action for the lattice theory which involved the local coupling of four gauge field degrees of freedom. This action is fairly complicated and is particularly formidable in the weak coupling sector. We hence attempt to redefine the lattice action so that we achieve a simpler theory with essentially the same physics. We will fail in this attempt due to a phase transition.

The essential idea is that gauge-invariance is produced by the interaction of the gauge-field with an unobservable scalar quantum field. The coupling of the gauge-field to the unobservable field is via a (lattice) gauge-transformation. It is this interaction which is responsible for gauge-invariance when one looks at only the gauge-field sector after having summed over all possible interactions of the gauge-field with the underlying medium. In essence, this means defining the gauge-invariant action via a path integral. To quantify these ideas consider a d-dimensional Euclidean lattice; let $U_{n\mu}$ be the local space-time gauge-field degree of freedom at the lattice point n. $U_{n\mu}$ is a finite group element of the gauge group, which is SU(N). (See Wilson[1] for the connection of $U_{n\mu}$ with the continuum quantum field.) Let V_n be a lattice scalar quantum field, and a finite element of the gauge group. Recall that the gauge-transformation is defined for the lattice theory by

$$U_{n\mu} \to U_{n\mu}(V) = V_n U_{n\mu} V^+_{n+\mu} \tag{1.1}$$

Let $A[U]$ be an arbitrary functional of the gauge-field $\{U_{n\mu}\}$. Note that

$$Z = \prod_{n\mu} \int dU_{n\mu} \; e^{A[U]} \equiv \langle e^{A[U]} \rangle \tag{1.2'}$$

We choose $A[U]$ such that it needs no gauge-fixing for weak coupling calculations. We then <u>define</u> the gauge field action functional by

$$e^{A_{GF}[U]} \equiv \prod_n \int dV_n e^{A[U(V)]} \tag{1.2}$$

(where dV_n is the invariant group measure and the integration runs over the group space). It is clear that $A_{GF}[U]$ is manifestly gauge-invariant by construction.

We discuss the gauge-invariant sector for the gauge-field. Let $K[U] = K[U(V)]$ be an arbitrary gauge-invariant functional of the gauge-field. Then

$$\langle K[U] \, e^{A_{GF}[U]} \rangle = \int dV \langle K[U] \, e^{A[U(V)]} \rangle$$

$$= \{\int dV\} \cdot \langle K[U] \, e^{A[U]} \rangle \tag{1.3'}$$

$$= \langle K[U] \, e^{A[U]} \rangle \tag{1.3}$$

where we used a gauge-transformation to obtain (1.3) from (1.3') and the fact that $K[U]$ is gauge-invariant. In particular

$$\langle e^{A_{GF}[U]} \rangle = \langle e^{A[U]} \rangle = Z \tag{1.4}$$

Hence, as far as the gauge-invariant sector is concerned, removing the interaction with the underlying field $\{V_n\}$ is equivalent to choosing a particular gauge for $A_{GF}[U]$. One can view $A[U]$ as the result obtained by performing gauge-fixing on $A_{GF}[U]$. There is, however, a significant difference between this approach and the approach which starts with $A_{GF}[U]$. In our case, $A[U]$ will always be chosen as local, and $A_{GF}[U]$ will usually be non-local, whereas in the other approach, $A_{GF}[U]$ is chosen to be local and the gauge-fixed action turns out to be non-local (at least for those cases where one needs ghosts).

From (1.4), we see that the phase transitions for the gauge theory given by A_{GF} is the same as that of the non-gauge-invariant action A, since they both have the same Z.

Consider the case of nearest-neighbor interaction and without any coupling of the vector indices, i.e. an essentially scalar nearest-neighbor interaction

$$A[U] = \frac{1}{g^2} \sum_{n\mu\nu} \mathrm{Tr}\,(U_{n\mu} U^{+}_{n+\hat{\nu},\mu} + \mathrm{h.c.}) \tag{1.5}$$

Let

$\delta_\mu f_n = f_{n+\mu} - f_n$. Then

$$A = -\frac{1}{g^2} \sum_{n\mu\nu} \mathrm{Tr}(\delta_\nu U_{n\mu}\, \delta_\nu\, U^{+}_{n\mu}) \tag{1.6}$$

which is the SU(N) non-linear model and has a SU(N) × SU(N) global symmetry. This action has been studied[2] for its phase diagram. We will study this action in the rest of this paper.

II. Strong and Weak Coupling Limits

Consider the coupling constant g to be very large. We can then expand the exponential of $A[U(V)]$ into a power series (since the V_n

integrations are compact). The path integral $\int dV$ is then reduced to the integration over a finite number of variables. Note

$$A[U(V)] = \frac{1}{g^2} \sum_{n\mu\nu} \mathrm{Tr}(V_n U_{n\mu} V^+_{n+\mu} V_{n+\mu+\nu} U^+_{n+\hat{\nu},\mu} V^+_{n+\nu} + \mathrm{h.c.}) \tag{2.1}$$

From equation (1.2) we have

$$e^{A_{GF}[U]} = \int dV \; \{1 + A[U(V)] + \frac{1}{2!} A^2[U(V)] + \ldots \} \tag{2.2}$$

The V_n integrations are performed, giving (upto a constant $K \sim 0(1)$)

$$A_{GF}[U] \simeq K(\frac{1}{g^2})^2 \sum_{n\mu\nu} \mathrm{Tr}(U_{n\mu} U_{n+\hat{\mu},\nu} U^+_{n+\hat{\nu},\mu} U^+_{n\nu}) + 0\left(\frac{1}{g^6}\right) \tag{2.3}$$

Note from (2.3) that A_{GF} is simply the Wilson action but with a coupling constant renormalization. This action confines quarks for large enough g and is manifestly gauge-invariant.

To study the system for its weak coupling, we consider the limit $g^2 \to 0$. We represent $U_{n\mu} = \exp(i B^\alpha_{n\mu} X^\alpha)$, where X^α are the generators of SU(N) and $B^\alpha_{n\mu}$ the spacetime quantum field. Then, as shown in Ref. 2,

$$A \simeq -\frac{1}{2g^2} \sum_{n\mu\nu} [(\delta_\nu B_{n\mu})^2 - \frac{1}{12} (B_{n\mu} \times B_{n+\hat{\nu},\mu})^2$$

$$- \frac{1}{48} \{ \frac{2}{N} (\delta_\nu B_{n\mu})^2 (\delta_\nu B_{n\mu})^2 + (\delta_\nu B_{n\mu} * \delta_\nu B_{n\mu})^2 \}] + 0(B^6/g^2) \tag{2.4}$$

where

$$a \cdot b = a^\alpha b^\alpha, \quad (a \times b)^\alpha = C^{\alpha\beta\gamma}\phi^\beta b^\gamma, \quad (a * b)^\alpha = d^{\alpha\beta\gamma} a^\beta b^\gamma .$$

$C^{\alpha\beta\gamma}$, $d^{\alpha\beta\gamma}$ are completely anti-symmetric, symmetric tensors. We want to compute $A_{GF}[B]$ for $g^2 \simeq 0$. Hence, we have to perform a gauge-transformation on $U_{n\mu}$ and integrate over all possible gauge-transformations. From Ref. 3, in vector notation, the gauge-transformed field for $V_n = \exp(i X^\alpha \phi^\alpha_n)$, from (1.1), is

$$B_{n\mu}(\phi) = B_{n\mu} - \delta_\mu \phi_n - \frac{1}{2} (\phi_n + \phi_{n+\mu}) \times B_{n\mu}$$

$$+ \frac{1}{2}\,\phi_n \times \phi_{n+\mu} + 0(B\phi^2,\ \phi^3) \tag{2.5}$$

where $\{\phi_n\}$ specify the gauge-transformation. Then from (1.2), using $dV_n = \mu(\phi_n)\pi_\alpha\, d\,\phi_n^\alpha$, we have for $g^2 \simeq 0$,

$$e^{A_{GF}[B]} \simeq \pi_{n\alpha} \int_{-\infty}^{+\infty} d\phi_n^\alpha\ \mu(\phi_n) e^{A[B(\phi)]} \tag{2.6}$$

A straightforward Feynman perturbation, using eqns. (2.4), (2.5) and (2.6) gives

$$\begin{aligned} A_{GF} = & -\frac{1}{4g^2} \sum_{n\mu\nu} (\delta_\mu B_{n\nu} - \delta_\nu B_{n\mu})^2 \\ & + \frac{1}{g^2} \sum_{\substack{nm\ell \\ \lambda\pi\sigma}} f_{\lambda\pi\sigma}(n,m,\ell) B_{n\lambda} \cdot B_{m\sigma} \times B_{\ell\pi} + 0(B^2, B^4/g^2) \end{aligned} \tag{2.7}$$

We see that to lowest order, A_{GF} is simply the abelian sector of the Yang-Mills lagrangian. However, A_{GF} has cubic and quartic plus higher order terms, and these interacting pieces of the action A_{GF}, unlike the Yang-Mills action, are non-local. Since the weak-coupling limit of A_{GF} is non-local, we may expect it to be separated from the Yang-Mills theory by a phase transition. We will in fact show in the next section that A_{GF} has a phase transition at finite coupling, whereas the Yang-Mills theory is known to have no such phase transition for d=4.

III. The Phase Transition

A criterion (order parameter) for assessing the phases of the gauge field is the expectation value of the Wilson loop integral. Consider the contour Γ of a closed square loop given by Fig. 1. Then

$$\begin{aligned} e^W &= \langle Tr(\Pi_\Gamma U_{m\lambda}) e^{A_{GF}[U]} \rangle / Z \\ &= \langle Tr(\Pi_\Gamma U_{m\lambda}) e^{A[U]} \rangle / Z \end{aligned} \tag{3.1}$$

where we have used (1.3) to obtain (3.1). We use the fact that the action A does not couple the vector indices to obtain (see Fig. 2)

$$e^W = (e^{W'})^2 \tag{3.2}$$

where, dropping the prime on W' and the vector indices on $U_{n\mu}$ gives

$$e^W = \frac{1}{Z}\prod_n \int dU_n \prod_{\Gamma'} U_m \exp\left\{\frac{1}{g^2}\sum_{n\mu} \mathrm{Tr}(U_n U^+_{n+\mu} + \mathrm{h.c.})\right\} \tag{3.3}$$

The set of points Γ' is shown in Fig. 3 and consists of two parallel lines of length L separated by a distance L.

For large g^2 , we can do the strong coupling expansion, and obtain

$$e^W \sim \left(\frac{1}{g^2}\right)^{L^2} \tag{3.4}$$

which is the expected area-law confinement.

For weak coupling, we use lowest order Feynman perturbation theory (which is not justified in Yang-Mills due to infrared divergences) to obtain

$$e^W \sim e^{-g^2 L} \tag{3.5}$$

We see (naively) that there is a phase transition in the theory, since there is a change in the behavior of W.

The action

$$A = \frac{1}{g^2}\sum_{n\mu} \mathrm{Tr}(U_n U^+_{n+\mu} + \mathrm{h.c.}) \tag{3.6}$$

has been studied for its phase transition in Ref. 2. We will study the theory for d=4. The critical coupling constant of the lattice theory for d=4 is given by $g_c^2 = 1.086$ (d=4). Let Ng_ℓ^2 be the effective coupling constant for distance on the lattice of size 2^ℓ. Then the phase diagram is given by Fig. 4; note $Ng_0^2 = Ng^2$ is the initial (bare) coupling constant. The arrows on the lines in Fig. 4 indicate the direction of the change in Ng_ℓ^2 as ℓ is increased.

We want to approach the phase transition at g_c^2 from the strong coupling phase. However, the calculation performed to deduce the existence of the phase transition was done on the assumption that $g^2 \simeq 0$. The question is, can we use this weak coupling result to say anything about the strong coupling phase? We assume that near $Ng^2 \simeq 1$, there is an intermediate domain where both the weak and strong coupling expansions are valid (see Fig. 5). Since $g_c^2 = 1.086$, we assume we are in the intermediate domain.

We consider $Ng^2 > g_c^2$, i.e. we are in the strong coupling phase. In studying the phase diagram of the nonlinear scalar action A, a one-loop calculation was done for its coupling constant renormalization. The one-loop renormalized action is given by Ref. 2 as

$$A = -\frac{1}{2g^2}\sum_{n\mu}(\delta_\mu B_n)^2 - \frac{\alpha}{24g^2}\sum_{nm\mu} B_n \times B_{n+\mu}\Gamma(n-m)B_m \times B_{m+\mu} + \dots \quad (3.7)$$

and where, for $|n| >> 1$

$$\Gamma(n) \simeq \int_{-\pi}^{+\pi} \frac{d^d p}{(2\pi)^d} \frac{e^{ipn}}{r^2 + p^{d-2}} \quad (3.8)$$

$$r^2 = c(Ng^2/g_c^2 - 1) \quad (3.9)$$

[α and c are constants and are given in Ref. 2.] $\Gamma(n)$ is well defined for all n and is finite for n=0.

For the case of d=4

$$\Gamma(n) \simeq (\text{const.}) \frac{r}{n} K_1(rn) \quad (3.10)$$

where $K_1(x)$ is a modified Bessel function of the second kind (see Fig. 6). It gives the two asymptotic expansions

$$\lim_{n\to\infty} \Gamma(n) \sim \begin{cases} \frac{1}{n^2}, & r^2=0 \\ \frac{r^{1/2}}{n^{3/2}} e^{-rn}, & r^2>0 \end{cases} \tag{3.11}$$

Hence we see that for $r^2>0$, the vertex function has a mass-like term e^{-rn} which cuts-off the interaction range and makes it (short-ranged) finite. For $r^2=0$, the interaction becomes long-ranged and scale-invariant, and the limit $r^2\to 0$ approaches the phase transition at g_c^2 from the strong-coupling (massive) phase. In the ordered (weak coupling) phase characterized by the weak coupling perturbation theory, there are no masses. We therefore conclude that in the ordered phase, $r^2=0$. In terms of the effective mass $r \sim \sqrt{Ng^2/g_c^2-1}$, we hence have the phase diagram given by Fig. 7, and where the mass term r vanishes for $Ng^2=g_c$.

We have not computed r^2 for $Ng^2<g_c$ and shown that it is zero. The one-loop summation that was carried out to obtain the renormalized action becomes divergent for $r^2<0$, so this summation is no longer permissible. We, however, still have the (order by order) well-defined Feynman perturbation theory in the $g^2\approx 0$ neighborhood with which to calculate r for $Ng^2<g_c^2$.

The critical properties of the model under consideration are very analogous to that of the N component spinor interaction $G(\sum_{i=1}^{N}\bar{\psi}_i\psi_i)^2$ studied by Wilson (Ref. 4).

To complete the picture we evaluate e^W at the critical point. Although we calculated the critical action using the expansion about $g^2 \simeq 0$, we assume that since $g_c^2 \simeq 1.086$, the strong coupling expansion is also valid for the critical action. This in effect means that we can freely exponentiate the linear variables B_n^α into the non-linear variables $U_n = \exp(iB_n^\alpha X^\alpha)$.

Hence, we exponentiate the action given by (3.7) as follows:

$$A \simeq -\frac{1}{2g^2}\sum_{n\mu}(\delta_\mu B_n)^2 - \frac{\alpha}{24g^2}\sum_{nm\mu} B_n\times B_{n+\mu}\cdot\Gamma(n-m)B_m\times B_{m+\mu} \tag{3.7}$$

$$\simeq \frac{1}{g^2}\sum_{n\mu}\mathrm{Tr}(U_n U^+_{n+\hat{\mu}} + \mathrm{h.c.})$$

$$+\frac{\alpha}{6g^2}\sum_{nm\mu\alpha}\mathrm{Tr}(X^\alpha U_n U^+_{n+\mu} U^+_n U_{n+\mu})\Gamma(n-m)\mathrm{Tr}(U_{m+\mu} U^+_m U^+_{m+\mu} U_m X^\alpha) \tag{3.12}$$

The new term is purely non-abelian. For $r^2>0$ the new term in (3.12) gives short ranged coupling and is, for large g^2, an irrelevant perturbation on the initial nearest-neighbor coupling action.

Doing a strong coupling expansion using (3.12) gives for the propagator

$$D_n = \frac{1}{Z}\langle\mathrm{Tr}(U_n^+ U_o)e^A\rangle \sim \left(\frac{1}{g^2}\right)^n + \frac{\alpha}{6g^2}\Gamma(n). \tag{3.13}$$

For $r^2=0$, i.e. at the phase transition, we have using (3.11) and (3.13)

$$D_n \sim \frac{1}{n^2} + 0(e^{-n}) \tag{3.14}$$

From (3.14) we find that the <u>compact</u> variables U_n exhibit scale invariance at g_c^2, and is a reflection of the $\sim 1/n^2$ interaction term between the compact variables in the critical action.

Computing e^W using the strong coupling expansion gives, ignoring certain constant matrices and using (3.13)

$$e^W \simeq (D_L)^L \tag{3.15'}$$

or,

$$e^W \simeq \left(\frac{1}{g^2}\right)^{L^2} + \left\{\frac{\alpha}{6g}\Gamma(L)\right\}^L + \text{lower order} \tag{3.15}$$

For $L>>1$, using (3.11) gives, for $r^2>0$

$$e^W \simeq \left(\frac{1}{g}\right)^{L^2} + (\text{const.})\, e^{-rL^2} \tag{3.16}$$

Hence, $r^2>0$ is still the strong-coupling phase with the area dependence for W. Note that the strong coupling has two expansion parameters i.e. $\frac{1}{g^2}$ and $e^{-r} \sim e^{-g}$. If we ignore the $1/g^2$ term, then each link-variable $U_{n\mu}$ in strong coupling carries a mass e^{-g}.

However, for $r^2 = 0$ i.e. $Ng^2 = g_c^2$, we have

$$e^W \sim (\frac{1}{g^2})^{L^2} + (\frac{1}{L^2})^L \sim e^{-L\ell nL} + 0(e^{-L^2}) \tag{3.17}$$

Therefore at the critical point we have,

$$W \sim -L\ell nL \ , \ Ng^2 = g_c^2 \tag{3.18}$$

To obtain this result using weak-coupling perturbation theory would entail summing an infinite set of Feynman diagrams. To summarize our results, the order parameter W has the following behavior

$$W \sim \begin{cases} -L^2 & Ng^2 > g_c^2 \\ -L\ell nL & Ng^2 = g_c^2 \\ -L & Ng^2 < g_c^2 \end{cases} \tag{3.19}$$

We have the expected change in the analytic behavior of W as the system goes from one phase to the other via the phase transition point g_c^2. The phase of the system at g_c^2 is yet a third distinct phase of the system. The system at g_c^2 is scale-invariant.

We see that the essential difference between the strong and weak coupling phases is the presence or absence of a mass-scale. For d near 4, the theory spontaneously generates a mass parameter r for the strong coupling phase, and this makes the correlation length finite - which in turn gives the confinement (linear) potential.

The analysis we have done is valid for $d = 4 \pm \varepsilon$. For d = 3, the vertex function Γ (n), for $r^2>0$ and large n, has no exponential damping but instead has oscillations of the type $\sin(r^2n)$. Hence, for d=3 to approach the phase transition from strong coupling as done here would not be useful.

IV. Discussion

The main purpose of this paper was to postulate the concept of gauge-invariance in an explicit manner, so as to study its properties. This way of defining the lattice gauge theory allowed us to reduce the problem to that of studying a non-linear scalar field with nearest neighbor coupling. We essentially evaluated the loop integral e^W using a scalar quantum field.

We found that this theory, as a function of its bare coupling constant, describes a system (for $d \simeq 4$) with a second order phase transition from the free-field (weak-coupling) non-confining phase to the disordered strong coupling confining phase. The scale-invariant phase ($Ng^2 = g_c^2$) is a common boundary of the weak and strong coupling phase. What links the weak and strong coupling phases to each other continuously is that they both have the same phase transition at $Ng^2 = g_c^2$, giving the second-order phase transition. If they had different phase transitions at $Ng^2 = g_c^2$, we would instead have had a first-order phase transition separating the weak and strong coupling phases. Each of the three phases, i.e. the weak, strong and scale-invariant phases, can each by themselves define a renormalized continuum field theory (see Ref. 2,4).

We derived that for the scale-invariant phase, $E \sim \ell nL$, where E is the energy between a charge and anti-charge separated by a distance L. The single-particle wave functions with a potential of ℓnL has an eigen-spectrum of only bound states. However, the ℓnL potential may allow for macroscopic size separation of the quarks while they are still in their bound state. Here "macroscopic" means sizes much larger than the length of confinement given by the strong coupling phase.

The gauge theory studied here is asymptotically free for $d=2$, and has a phase transition for $d>2$. Hence, the critical dimension for this theory is $d_c=2$ as opposed to the Yang-Mills theory which has $d_c=4$. The reason for this difference is the non-locality of our gauge-invariant action in the weak coupling domain.

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References

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2. B. E. Baaquie, Phys. Rev D (1978).
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Figure Captions

1. The square contour Γ defining the loop integral.
2. The scalar decomposition of Γ into Γ'.
3. The set of points constituting Γ'.
4. The phase diagram of the nonlinear scalar action together with the renormalization group flow diagram.
5. The three domains of coupling constant, where it is assumed that an intermediate domain exists where both weak and strong approximations are both valid.
6. The plot of the function $e^x K_1(x)$.
7. The phase diagram using the effective mass r as the order parameter.

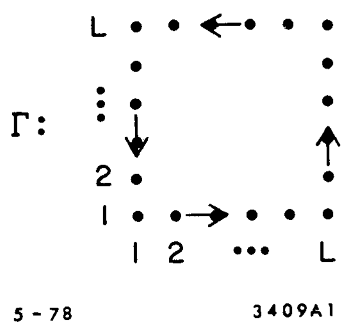

Fig. 1

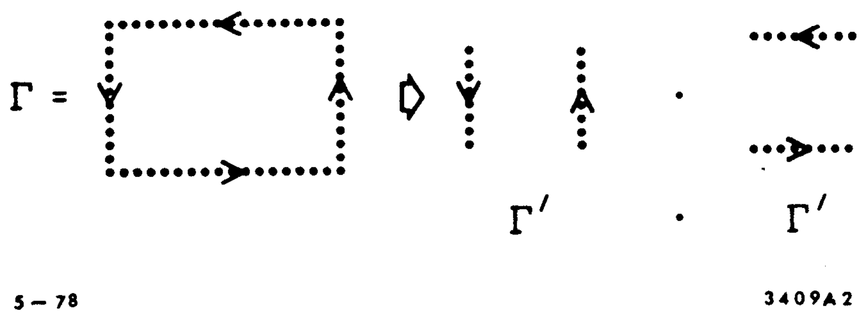

Fig. 2

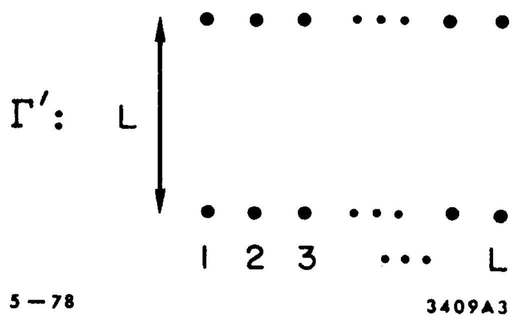

Fig. 3

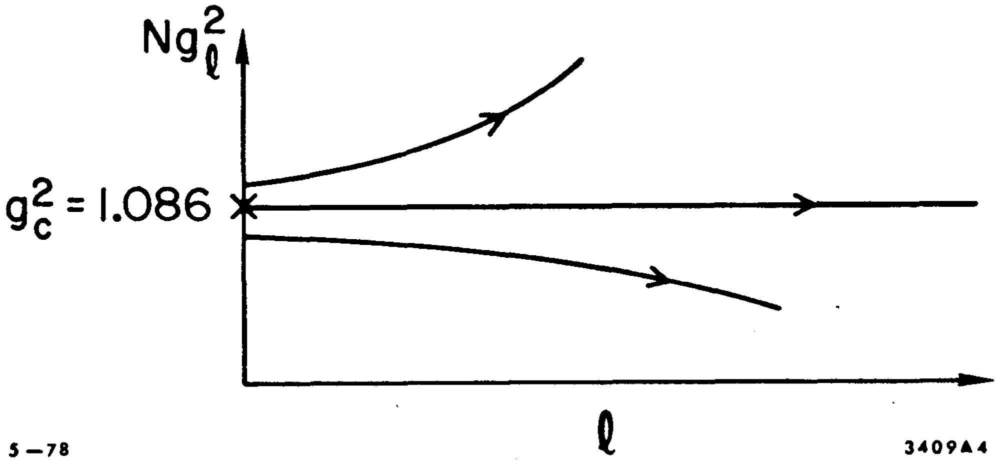

Fig. 4

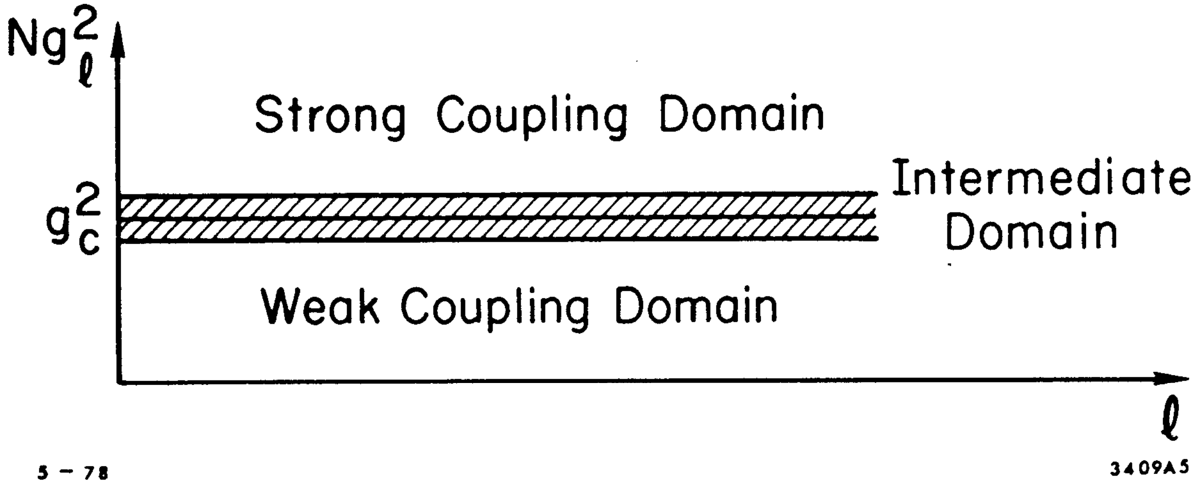
Ng_ℓ^2
Strong Coupling Domain
g_c^2
Intermediate Domain
Weak Coupling Domain
ℓ
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Fig. 5

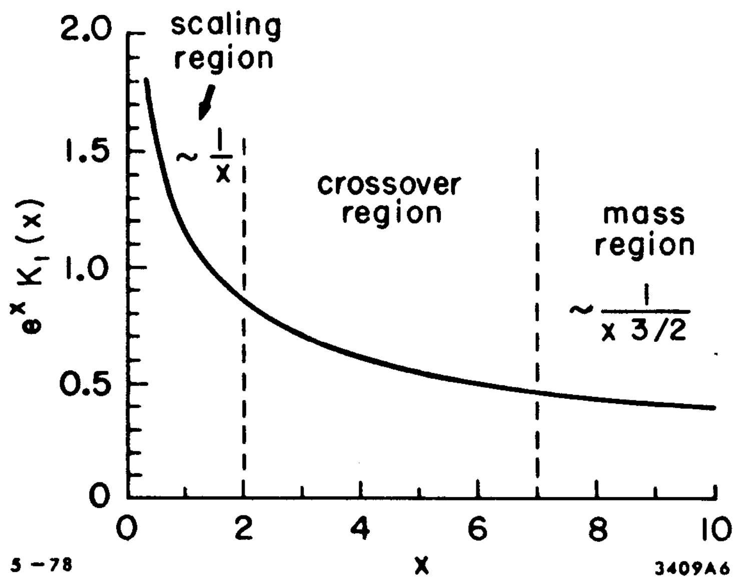

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Fig. 6

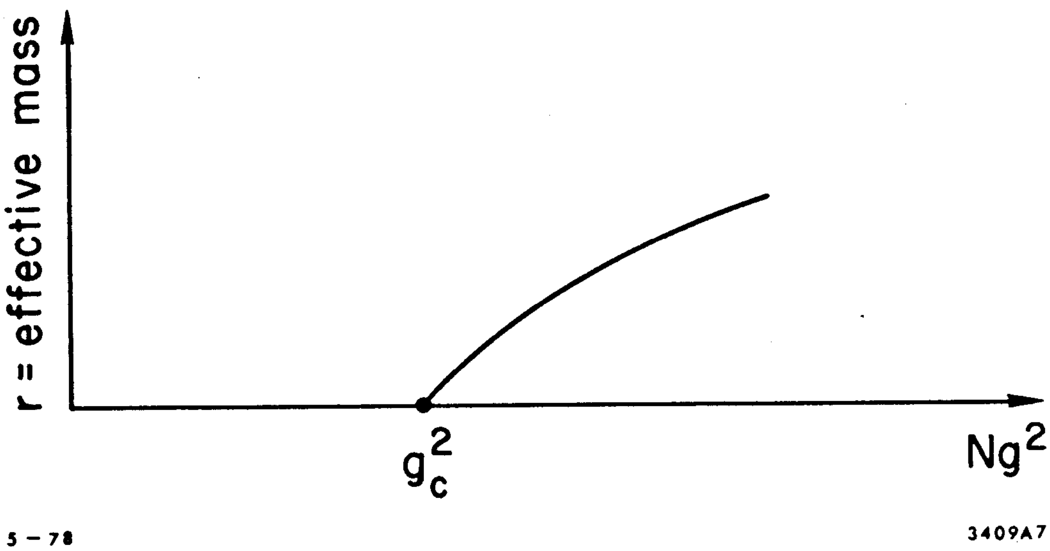

Fig. 7