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RENORMALIZATION APPROACH TO THE TWO-DIMENSIONAL QUANTUM ISING MODEL*

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ABSTRACT

The two-dimensional Quantum Ising Model is studied using a Variational Renormalization Group method. The critical paremeters are obtained for the square, hexagonal and triangular lattices.

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I. INTRODUCTION

The Ising Model (IM) in a transverse magnetic field (or Quantum Ising Model) is represented by the Hamiltonian

$$H = -\frac{\varepsilon}{2}\sum_{i}\sigma_z(i) - \Delta \sum_{n.n.}\sigma_x(i)\sigma_x(j) \qquad (1)$$

$$K \equiv \varepsilon/\Delta$$

where σ_x and σ_z are the usual Pauli matrices. The i index runs throughout the sites of the lattice, and n.n. picks only once every i and j indices located in a nearest neighbor position.

This spin lattice system corresponds to the pseudospin formulation of phase transition problems and has been used to study order-disorder ferroelectrics with a tunnelling effect[1] or the magnetic ordering in materials with a singlet crystal field ground state.[2] The equivalence of the critical behavior of Quantum I.M. at zero temperature and d dimensions to that of a d+1 Classical I.M. is also well known. For the quantum system, the transverse field rather than temperature is the disordering agent.

The properties of the Hamiltonian (1) have been obtained exactly, in the one-dimensional case by Pfeuty,[3] and the correspondence between classical and quantum systems is justified using the transfer matrix technique.[4]

From the exact solution of the one-dimensional case is known that, at zero temperature, a second order transition occurs for K=2. The behavior of the order parameter or magnetization in this model is given by $\langle\sigma_x\rangle = \left[1 - \left(\frac{K}{2}\right)^2\right]^{.125}$ for K<2 (ferromagnetic phase) and $\langle\sigma_x\rangle = 0$ for K>2 (paramagnetic phase). All the other critical exponents are identical to the ones obtained by Onsager for the I.M. in two dimensions.

For d>1 no exact solution exists for these models although all the qualitative and quantitative estimations of the criticality of Classical I.M. are applicable here, because of the correspondence we have commented before.

In this paper we will study the Quantum I.M. in two dimensions which constitutes a prediction for the critical exponents of the three-dimensional I.M. The model can be located in any two-dimensional lattice; we will consider the three usual cases, i.e. square, hexagonal and triangular lattices.

Some magnitudes like ground state energy density and critical constant are different in each case. However, if universality is accepted, the critical exponents should be the same because in principle they only depend on the dimension of the lattice.

Our working method will be the lattice Variational Renormalization Group (V.R.G.) approach developed at SLAC by S. Drell, M. Weinstein and collaborators.[5] This is a method derived to extract information about any quantum field theory without implementing a renormalization program based in an expansion in Feynman graphs. As a finite spin approximation to the ϕ^4 theory in 1x-1t dimensions, in the strong coupling limit, the SLAC authors have studied[6] the one-dimension Quantum I.M. obtaining good results for all the properties of that theory.

In a subsequent paper I have proposed,[7] for that case, a modification in the choice of the block hamiltonian used to implement the R.G. iterations. The difference consists in omitting one of the site energy terms that in principle exists inside the block. For that model, with this omission, the block hamiltonian has the same number of site and link terms, which

is a general property of that theory. The remarkable benefit of that "self-dual?"blockspinning is the obtaining of the exact critical coupling constant and also the exact thermal eigenvalue of the model.

Here we will study the Quantum I.M. in two dimensions, using both procedures. In this case a similar omission does not leave the block hamiltonian with the same proportion of site to link terms of the model; but it takes a step in that direction. So the comparison we present here is very interesting as a testing ground for the procedure beyond the one-dimensional case. The good or bad performance will be judged by comparing our results with existing data for the I.M. in three dimensions obtained with the high temperature expansion.

The paper is organized as follows. In Section II we present in detail the steps one has to do to apply the V.R.G. technique to any sort of lattice and the way of obtaining some physical information from the method. The differences, where they exist, between both procedures will be especially remarked. Our results and conclusions appear in Section III. And finally in the Appendix we present a Mean Field derivation of the critical coupling constant for this kind of model.

II. VARIATIONAL RENORMALIZATION GROUP METHOD

This approach we use[6,8] is variational, so that it is based in the Rayleigh-Ritz[9] procedure and simultaneously is a R.G. method, so that it consists in a progressive thinning of degrees of freedom.[10] It is devised to construct a wave function for the ground state wave function of the model under consideration. To apply the V.R.G. method to any sort of lattice we can distinguish four steps: (a) block defnition, (b) choice of the variational block hamiltonian, (c) truncation of states, and (d) representation of original hamiltonian in the truncated basis.

After doing these four steps we will have some R.G. equations for the model, from which we will be able to extract useful information about the different magnitudes of the theory.

A. Block Definition

Given the lattice one wants to solve, the first step is to define the elementary blocks. All the blocks should be identical; each of them will contain a few spins and no spin will remain out of a block. Furthermore, the interaction between blocks has to be identical in form to the original interaction between spins. This is necessary because after the thinning of degrees of freedom all the spins inside the block will be reduced to a new effective spin which must posses the same dynamical properties of the original system. If this requirement is not fulfilled after a R.G. transformation, new kinds of couplings in the Hamiltonian will appear and, even in the best case, the problem will acquire new difficulties.

In Fig. 1a we present the smallest block we can define in the square lattice. Here the block contains four spins and it is quite apparent that each block is connected with its four closest neighbors as dictated by the Ising interaction in the square lattice. Another sort of block definition for this lattice is presented in Fig. 1b. Now each block has five spins and as before each of them interacts only and symmetrically with its four nearest neighbors.[11] Obviously there is no problem in defining big blocks, in fact the larger the blocks are, the better results (in principle) one expects to obtain. However, as eventually the block hamiltonian has to be diagonalized, the calculations will be possible if the block size is reasonably small.

In Fig. 2 we present the smallest block definition of the hexagonal lattice; there each 4 spin block interacts with its three nearest neighbors as it should. In Figs. 3a and 3b, 3 spin and 7 spin blocks are defined for the triangular lattice. From these illustrations we see that with the Ising-like interaction it is trivial to define blocks.

B. Choice of the Variational Block Hamiltonian

Once the block is defined, we know the number of degrees of freedom that will appear in each step of the R.G. process. Now it is necessary to choose the hamiltonian of the block. As our ultimate goal is the construction of a reasonably good ground state of the Hamiltonian of the full theory, we will keep in each step if the process only a set of the lowest eigenstates of the block hamiltonian (in fact, for simplicity we will always maintain only the two lowest ones); the rest of them will be dropped by truncating the Hilbert space. How we choose the truncated basis, which is the real heart of the variational method, is directly connected with the choice of the block hamiltonian itself.

In principle it may seem that the most logical choice is that the block hamiltonian consists of all the site terms and all the link terms included in the individual block. This would correspond in the case of Fig. 1a to choosing

$$h = -\frac{\varepsilon}{2}\left[\sigma_z(1) + \sigma_z(2) + \sigma_z(3) + \sigma_z(4)\right] - \Delta\left[\sigma_x(1)\sigma_x(2) + \sigma_x(2)\sigma_x(3) + \right.$$
$$\left. + \sigma_x(3)\sigma_x(4) + \sigma_x(4)\sigma_x(1)\right] \qquad (2)$$

In fact this was the original version of the method[6] and we will use it here too. There is not however any compelling reason to do so. Indeed all we need is a hermitian operator defined in the degrees of freedom

existing in the block, so that we can retain some of its eigenstates. Actually the scheme is not so loose because not all the choices one can guess are proper. For success of this procedure it is essential that the truncation iteration procedure leads to a new Hamiltonian qualitatively identical to the original: i.e. one of the same form but with different coefficients on the individual terms. In practical calculations this general condition usually unfolds in two subconditions, i) the expectation values of all the intrablock terms of the general Hamiltonian in the two retained states per block, must be different. This must be so because otherwise in a renormalization step we would lose the single site terms of the Hamiltonian and all the scheme would collapse. ii) the representation of the interblock terms in the trial basis must lead to an isotropic (in all the directions of the lattice) interaction between the blocks. Obviously it must be so because otherwise the renormalization method would drive us to quite a different dynamics without any usefulness for us. These two conditions ban some choices of the block hamiltonian as we will see later in some examples.

We will also use an alternative way to (2) the following block hamiltonian

$$h = -\frac{\varepsilon}{2}\left[\sigma_z(1) + \sigma_z(2) + \sigma_z(4)\right] - \Delta\left[\sigma_x(1)\sigma_x(2) + \sigma_x(2)\sigma_x(3) + \right. \\ \left. + \sigma_x(3)\sigma_x(4) + \sigma_x(4)\sigma_x(1)\right] \tag{3}$$

i.e. we omit one of the site terms of (2). This is inspired in our work with the one-dimensional case.[7] There, as we have commented in the Introduction, it was a dynamical system with the same number of link and site terms, so that a site omission as in (3) makes the block hamiltonian

have that general property of the theory, and some of the results of the method are dramatically improved, even made exact. The omission trick had the technical effect of making the two lowest eigenstates of the block hamiltonian degenerate, in opposition to the original analysis where they were nondegenerate.

In the two-dimensional case, an omission, like in (3) again produces a degeneracy for the two lowest eigenstates of the block, not only for the square lattice but also in the others (independently of the block size one chooses). What we obviously do not get is a block hamiltonian with the same proportion of site-link terms of the general theory. Then there is a trivial question. Why not proceeding any further with the omission trick until getting the same proportion? The answer is very easy: that is not possible, because a second omission would already violate the i) condition commented before. We would lose the site terms of the Hamiltonian and the R.G. implemented so would be meaningless.

So that the two basic options we will use in all cases will be the corresponding generalization of (2) and (3), i.e. either all the intra-block terms form the block hamiltonian or all except one site. These two possibilities are extensively illustrated for all the lattices in Figs. 4, 5, and 6. The block hamiltonian of Fig. 6a cannot have any partner with omission because it would violate the ii) condition of isotropy. The same would happen to the block of Fig. 4d if the omission were not central but in one of the external positions. Although they have been drawn as a possible block hamiltonian of the triangular lattice, the ones of Figs. 6b and 6c, they will not be used in actual calculations because they are too expensive (the effective matrix to diagonalize in both cases is a 64×64 one).

C. Truncation of States

After the choice of the block hamiltonian, we have to diagonalize it in order to know which are the two lowest eigenstates, which will be all we retain in the truncation process. To clarify that, we will use the square lattice with the smallest block size (Figures 1, 4a and 4b). In the other cases to do the analogous is also straightforward.

The representation of (2) and (3) in the space spanned by

$$\begin{aligned} &|\downarrow\downarrow\downarrow\downarrow\rangle, |\uparrow\uparrow\downarrow\downarrow\rangle, |\uparrow\downarrow\uparrow\downarrow\rangle, |\uparrow\downarrow\downarrow\uparrow\rangle \\ &|\downarrow\uparrow\uparrow\uparrow\rangle, |\downarrow\uparrow\downarrow\uparrow\rangle, |\uparrow\uparrow\uparrow\uparrow\rangle, |\downarrow\uparrow\uparrow\downarrow\rangle \\ &|\downarrow\downarrow\uparrow\uparrow\rangle, |\uparrow\downarrow\uparrow\uparrow\rangle, |\uparrow\downarrow\downarrow\uparrow\rangle, |\uparrow\uparrow\downarrow\uparrow\rangle \\ &|\downarrow\uparrow\uparrow\uparrow\rangle, |\downarrow\uparrow\downarrow\downarrow\rangle, |\uparrow\uparrow\downarrow\downarrow\rangle, |\downarrow\downarrow\downarrow\uparrow\rangle \end{aligned} \tag{4}$$

is decoupled in two 8 × 8 matrices. Explicitly we have

$$\langle h \rangle = \begin{pmatrix} \mathscr{B} & 0 \\ 0 & \mathscr{V} \end{pmatrix} \tag{5}$$

where

$$\mathscr{B} = \begin{pmatrix} -\frac{\varepsilon}{2}(-3-x) & -\Delta & 0 & -\Delta & 0 & -\Delta & 0 & -\Delta \\ -\Delta & -\frac{\varepsilon}{2}(1-x) & -\Delta & 0 & -\Delta & 0 & -\Delta & 0 \\ 0 & -\Delta & -\frac{\varepsilon}{2}(-1+x) & -\Delta & 0 & -\Delta & 0 & -\Delta \\ -\Delta & 0 & -\Delta & -\frac{\varepsilon}{2}(1-x) & -\Delta & 0 & -\Delta & 0 \\ 0 & -\Delta & 0 & -\Delta & -\frac{\varepsilon}{2}(1-x) & -\Delta & 0 & -\Delta \\ -\Delta & 0 & -\Delta & 0 & -\Delta & -\frac{\varepsilon}{2}(-1+x) & -\Delta & 0 \\ 0 & -\Delta & 0 & -\Delta & 0 & -\Delta & -\frac{\varepsilon}{2}(3+x) & -\Delta \\ -\Delta & 0 & -\Delta & 0 & -\Delta & 0 & -\Delta & -\frac{\varepsilon}{2}(-1+x) \end{pmatrix} \tag{6}$$

and

$$\mathcal{V}\begin{pmatrix}
-\frac{\varepsilon}{2}(-3+x) & -\Delta & 0 & -\Delta & 0 & -\Delta & 0 & -\Delta \\
-\Delta & -\frac{\varepsilon}{2}(1+x) & -\Delta & 0 & -\Delta & 0 & -\Delta & 0 \\
0 & -\Delta & -\frac{\varepsilon}{2}(-1-x) & -\Delta & 0 & -\Delta & 0 & -\Delta \\
-\Delta & 0 & -\Delta & -\frac{\varepsilon}{2}(1+x) & -\Delta & 0 & -\Delta & 0 \\
0 & -\Delta & 0 & -\Delta & -\frac{\varepsilon}{2}(1+x) & -\Delta & 0 & -\Delta \\
-\Delta & 0 & -\Delta & 0 & -\Delta & -\frac{\varepsilon}{2}(-1-x) & -\Delta & 0 \\
0 & -\Delta & 0 & -\Delta & 0 & -\Delta & -\frac{\varepsilon}{2}(3-x) & -\Delta \\
-\Delta & 0 & -\Delta & 0 & -\Delta & 0 & -\Delta & -\frac{\varepsilon}{2}(-1-x)
\end{pmatrix}$$

This decoupling of the 16 × 16 matrix is due to the fact that h is invariant under a rotation about the z axis of π, hence h only connects among themselves the first eight and the last eight states of (4). In the construction of (6) and (7), precisely the order of states consigned in (5) has been used. Notice that the h given in (2) corresponds to x=1, and the h given in (3) is obtained with x=0. If x=1 both matrices are different and so their eigenvalues are also different. On the contrary, if x=0, $\mathcal{B}=\mathcal{V}$ and then the lowest eigenstate of $\mathcal{B}$: $|b\rangle$ and the lowest

eigenstate of $\mathcal{V}$: $|v\rangle$ will be degenerate. This is the degeneracy we announced previously that happens when the omission trick is used.

In general for arbitrary ε and Δ parameters it is not possible to obtain analytic expression for the eigenvalues and eigenvectors of $\mathcal{B}$ and $\mathcal{V}$ and it is necessary to do it numerically for each situation.

So let us suppose that we have solved the problem and the symbolic solution for the two lowest eigenstates of the block hamiltonian matrix is

$$\begin{aligned} |b\rangle &= b_1|\downarrow\downarrow\downarrow\downarrow\rangle + b_2|\uparrow\downarrow\downarrow\downarrow\rangle + b_3|\downarrow\uparrow\downarrow\downarrow\rangle + b_4|\downarrow\downarrow\uparrow\downarrow\rangle + \\ &\quad + b_5|\downarrow\downarrow\downarrow\uparrow\rangle + b_7|\downarrow\downarrow\uparrow\uparrow\rangle + b_8|\uparrow\uparrow\uparrow\uparrow\rangle + b_8|\downarrow\uparrow\uparrow\downarrow\rangle \\ |v\rangle &= v_1|\downarrow\downarrow\uparrow\downarrow\rangle + v_2|\uparrow\downarrow\uparrow\uparrow\rangle + \sqrt{3}|\uparrow\downarrow\downarrow\downarrow\rangle + v_4|\uparrow\uparrow\downarrow\downarrow\rangle + \\ &\quad + v_5|\downarrow\uparrow\uparrow\uparrow\rangle + v_6|\downarrow\uparrow\downarrow\downarrow\rangle + \sqrt{7}|\uparrow\uparrow\downarrow\uparrow\rangle + v_8|\downarrow\downarrow\uparrow\uparrow\rangle \end{aligned} \tag{8}$$

Remember that if $x=0$, $b_i=v_i$. As all the others eigenstates have been dropped, $|b\rangle$ and $|v\rangle$ are now the two situations of an effective block spin. In the next section the original operators of (1) will be expressed as operators acting in the truncated basis formed with all the $|b\rangle_s$ and $|v\rangle_s$ of the lattice. That transition constitutes a R.G. transformation.

D. Renormalization Transformation

In the previous section we have reduced the Hilbert space of (1) because from 16 states per block we have passed to 2, the two lowest ones, which, very plausibly will be fundamental ingredients of the actual ground state of (1).

The representation of H in the new truncated basis is very easily attained. In (1) there are two kinds of terms, those that operate only on the degrees of freedom that are inside a block, and those that operate simultaneously on variables existing in two blocks.

Let us separate both kinds of terms

$$H = H_0 + V \tag{9}$$

H_0 is obviously

$$H_0 = \sum_I h_0(I) \tag{10}$$

h_0 being precisely the h used in (2).

To calculate the new single site block terms we will have to calculate

$$\begin{pmatrix} \langle b|h_0|b\rangle & \langle b|h_0|v\rangle \\ \langle v|h_0|b\rangle & \langle v|h_0|v\rangle \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \frac{(\lambda_1+\lambda_2)}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \frac{(\lambda_1-\lambda_2)}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so that we see that

$$\varepsilon \to \varepsilon' = \frac{\lambda_1-\lambda_2}{2} \tag{12}$$

is the R.G. equation for that coefficient. $\left(\frac{\lambda_1+\lambda_2}{2}\right)$ is a global energy term per block that we store in each step to calculate the ground state energy density.

Looking now at the terms that compose V, each of them is like this

$$-\Delta\ \sigma_x(i,J)\ \sigma_x(i',J') \tag{13}$$

where i is here an index describing the position of a spin in the block, while J is the block index.

To obtain Δ', we have to calculate the representation of $\sigma_x(i,J)$ in the truncated block space and analogously for $\sigma_x(i',J')$. Let us suppose for example that i=1.

$$\begin{pmatrix} \langle b|\sigma_x(1)|b\rangle\langle b|\sigma_x(1)|v\rangle \\ \langle v|\sigma_x(1)|b\rangle\langle v|\sigma_x(1)|v\rangle \end{pmatrix} = Z_1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{14}$$

That is each $\sigma_x(i)$ provides a Z factor and the σ_x operator in the $|b>, |v>$ basis. The Z factors are trivial expressions of b_i and v_i.

For the case we are describing in detail (Figs. 1, 4a, 4b)

$$\Delta \to \Delta' = (Z_1 Z_4 + Z_3 Z_2) \tag{15}$$

Our previous warning about the isotropy of the block interaction, specifically means that the Δ' obtained in any direction has to be identical.

Noting that the hamiltonian depends only on $K \equiv \frac{\epsilon}{\Delta}$ up to a scale factor and a global energy term, one can easily obtain a R.G. equation for that relevant coefficient. Using (12) and (15) we have

$$K \to K' = \frac{(\lambda_1 - \lambda_2)/2}{\Delta(Z_1 Z_4 + Z_2 Z_3)} \tag{16}$$

This R.G. equation has two trivial fixed points for K=0 and K=∞, and also a nontrivial K=K*, K* finite, that represents the critical coupling constant of the model because in that situation the Hamiltonian reproduces itself up to a scale factor, and so the physics going on at different length scales is essentially the same.

If the process we have outlined is repeated for the new effective Hamiltonian, the coupling parameters will change again and the successive iteration drives to any of the trivial fixed points (0 or ∞) depending on the position of the initial K with respect to the K*.

The properties of the R.G. transformation (16) near the critical point are determined by the derivative λ_T

$$\lambda_T \equiv \frac{dK'}{dK}\Big|_{K=K^*} \tag{17}$$

known as "temperature-like" eigenvalue. From which we obtain the thermal exponent

$$y_T = \frac{\ell n(\lambda_T)}{\ell n(n)} \tag{18}$$

n being the number of spins per block used in the method.

Another critical index is the magnetic eigenvalue λ_H,[12] defined as the response to a perturbation of the form

$$\ell \sum_i \sigma_x(i) \tag{19}$$

due to the existence of a longitudinal magnetic field, of the fixed point hamiltonian. We determine this number in our scheme by representing this perturbation in the space of the truncated basis and divide by (Δ'/Δ) in order to have the coefficient of $\left[\sum_{n.n.} \sigma_x(i)\sigma_x(j)\right]$ always normalized to (-1).

$$\ell \rightarrow \ell' = \ell(Z_1+Z_2+Z_3+Z_4) \tag{20}$$

and so

$$\lambda_H = \frac{Z_1+Z_2+Z_3+Z_4}{Z_1Z_4+Z_2Z_3} \tag{21}$$

Analogously with the thermal case, the magnetic exponent is

$$y_H = \frac{\ell n(\lambda_H)}{\ell n(n)} \tag{22}$$

In the next section, along with the critical coupling constants in the different variational options, and these two critical exponents y_T and y_H, we present also the results for the β exponent, and also for the ground energy density. In order to calculate these two physical observables directly from the iterative procedure, we follow the method described in detail in the Reference 6.

The magnetization function is found iterating until convergence the product

$$\mathcal{M} = \prod_q \left[\frac{1}{4}\ (Z_1+Z_2+Z_3+Z_4)\right] \tag{23}$$

Let us recall that we are always using as working example the square lattice with the block hamiltonian depicted in (2) or (3). β is found by fitting the numerical points obtained in the iteration of (23) with the function $\left[1 - \left(\frac{K}{K^*}\right)^2\right]^\beta$ at points K very near to K*.

To get the ground state energy density it is necessary to calculate the sum

$$\mathcal{E}_0 = \sum_q \frac{1}{4^q} \left[\left(\frac{\lambda_1+\lambda_2}{2}\right)_q\right] \tag{24}$$

for a q large enough to obtain satisfactory results. We see that this formula collects the accessive coefficients of the $\mathbf{1}$ matrix by the volume of the block in each step.

III. RESULTS AND CONCLUSIONS

Our results for critical constants and exponents are collected in Table 1, and the latter compared with the high temperature expansions.[13] For the critical constants there is no way of knowing if they are good or bad because there is no alternative precise calculation for them. If we grant a higher credit to the results obtained doing the omission, then, it is curious the proximity of them to the coordination number (number of lines that emerge from each site) of the lattice. Notice that in the one-dimensional case the exact result was K*, that is supposed to be correct for a dimension higher than four, is twice the coordination number (see Appendix).

As can be seen in Table 1, the values of the critical exponents calculated on doing the omission trick are systematically better than the others, and our best results correspond to the hexagonal lattice.

In Fig. 7 we present a complete display of the results obtained for magnetization of the square lattice, using the four spins block with and without the omission trick. In Fig. 8, for the same lattice, in the same approximation, our results for the ground state energy density appear. Here we observe a similar behavior to the one of the chain,[7] i.e. on doing the omission trick the results are higher, specially in the critical area. But in spite of this drawback, it has the remarkable advantage of showing the expected peak in its second derivative (see Fig. 9). We have observed these two qualitative features not only here but also for the hexagonal lattice.

As a general conclusion of the paper, we would say that for the Quantum I.M., except for the ground energy density where it works worse, the omission trick shows itself as an easy way of improving quantitatively the Variational R.G. method and even making it able to expose qualitative features which otherwise would remain buried. Therefore we plan to apply it to other Field Theories previously studied[6] (like the U(1) scalar theory in 1x-1t dimensions, the Thirring model, etc.) to know if there we find any discrepancy between both methods.

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APPENDIX

In this appendix we present a Mean Field Theory (M.F.T.) computation of the critical constant of any Quantum I.M.

Let us suppose that any single spin of the lattice under consideration is surrounded by c nearest neighbors (c is the coordination number). In a M.F.T. approach the hamiltonian of any spin of the lattice is

$$h = -\frac{\varepsilon}{2}\sigma_z - c\Delta s\,\sigma_x \tag{A1}$$

where $s \equiv \langle\sigma_x\rangle$ is the effective magnetic field acting on each spin due to the presence of each one of its nearest neighbors.

The ground state of the lattice is simply the tensorial product of all the individual ground spin states. So we have to find the ground state of h in (A1). It is

$$|0\rangle = \frac{1}{\sqrt{1+\left(\frac{a+\lambda}{b}\right)^2}}\left[|\uparrow\rangle + \left(\frac{a+\lambda}{b}\right)|\downarrow\rangle\right] \tag{A2}$$

corresponding to the eigenvalue

$$\lambda = -\sqrt{a^2+b^2}$$

$$a \equiv \frac{\varepsilon}{2} \tag{A3}$$

$$b \equiv c\Delta s$$

Knowing $|0\rangle$ we can calculate $\langle 0|\sigma_x|0\rangle$ and impose the consistency condition $s = \langle 0|\sigma_x|0\rangle$. So that we have

$$s = \frac{-2\left(\frac{a+\lambda}{b}\right)}{1+\left(\frac{a+\lambda}{b}\right)^2} \tag{A4}$$

And from (A4), bearing in mind that the criticality corresponds to the vanishing of the order parameter s we easily obtain $K^*=2c$, i.e. in this approach the critical constant is just twice the coordination number.

As it is well known M.F.T. provides exact results for a dimension higher than four.

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8. Z. Friedman, Phys. Rev. Lett. 36, 1326 (1976) and K. Subbarao, Phys. Rev. Lett. 37, 1712 (1976) have developed a Renormalization Group method a la Niemeijer and Van-Leeuwen to compute the critical parameters in Quantum I.M. They do the calculations in a triangular lattice with the minimal block size (three sites per block), up to second order in perturbation theory. Their results are roughly better than ours for the magnetic index and worse for the electric one.

9. In the usual Rayleigh-Ritz variational method in Quantum Mechanics, the ground state wave function is guessed depending on variable parameter, which is fixed so that it minimizes the energy of the trial wave function. In the method we use here for a dynamical system with infinite degrees of freedom, we do not use any similar parameter. In the original approach to the one-dimensional Quantum-Ising model,[6]

Drell, Weinstein and Yankielowicz did use that, which increased remarkably the power of the method, but also increased considerably its expensiveness. For the two-dimension case, the use of similar variable parameters would be a really cumbersome task. We therefore prefer to rely in the naive and simple method of retaining the two lowest eigenstates of each block, improved with the omission trick that so good results provided in one dimension. In spite of not using variable parameters, the method is variational, because it is based on guessing an adequate ground state wave function, in opposition to other approaches that construct it perturbatively.[8]

10. The Renormalization Group ideas were introduced in Statistical Mechanics by L. P. Kadanoff in Physics 2, 263 (1966) and developed later on by K. G. Wilson and others. A complete list of references can be found in K. G. Wilson and J. Kogut, Phys. Rep. 12, 75 (1974).

11. In an unrelated work P. Turner has used five spin blocks similar to ours to study the Classical ϕ^4 Theory in two dimensions. (Forthcoming SLAC preprint.)

12. See for example Th. Niemeijer and J. M. J. Van-Leeuwen in C. Domb and M. S. Green, Vol. 6, Academic Press 1976.

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TABLE 1

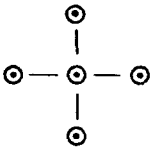
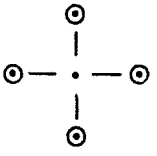
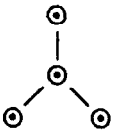
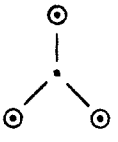
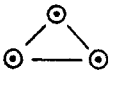
		K^*	y_T	y_H	β
High Temperature Expansion			.7999	1.2499	.312
Square Lattice		6.5601	.4175	.8798	.94
		4.2009	.8408	.8731	.42
		5.2595	.4471	.8772	.91
		3.8942	.7636	.9366	.48
Exagonal Lattice		3.9369	.5524	1.0138	.66
		3.0873	.8602	1.0770	.36
Triangular Lattice		10.4946	.3232	.7793	1.3

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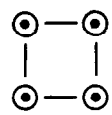
FIGURE CAPTIONS

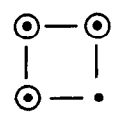
1. Blocks on a square lattice, (a) four spins per block and (b) five spins per block.
2. Blocks with four spins in a hexagonal lattice.
3. Blocks on a triangular lattice, (a) three spins per block and (b) seven spins per block.
4. (a), (b), (c) and (d), variational options for the block hamiltonian of the square lattice.

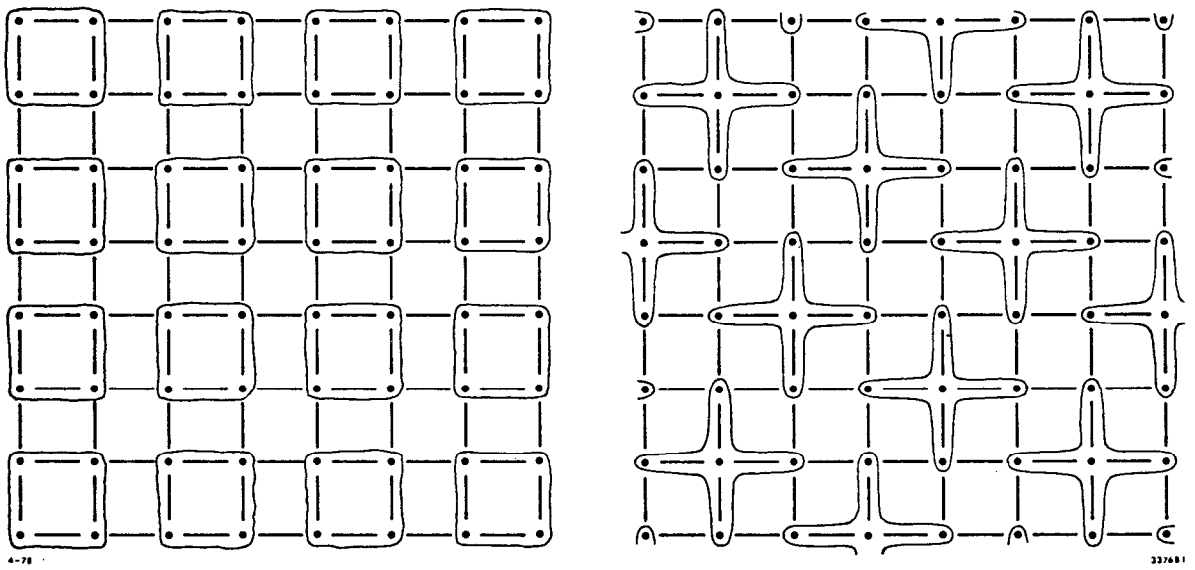
   ● stands for the spin degree of freedom.

   o stands for the site term of the Hamiltonian.

   —— stands for the link term of the Hamiltonian.
5. (a) and (b), variational options for the block hamiltonian of the hexagonal lattice.
6. (a), (b), and (d), variational options for the block hamiltonian of the triangular lattice.
7. Comparison of the order parameter $\mathcal{M}$ (K) versus K.

   oooo corresponds to 

   •••• corresponds to 
8. Ground energy density as a function of K. with the same conventions as Fig. 7.
9. Second derivative of the ground energy density, with the same conventions as Fig. 7.



(a) (b)

Fig. 1

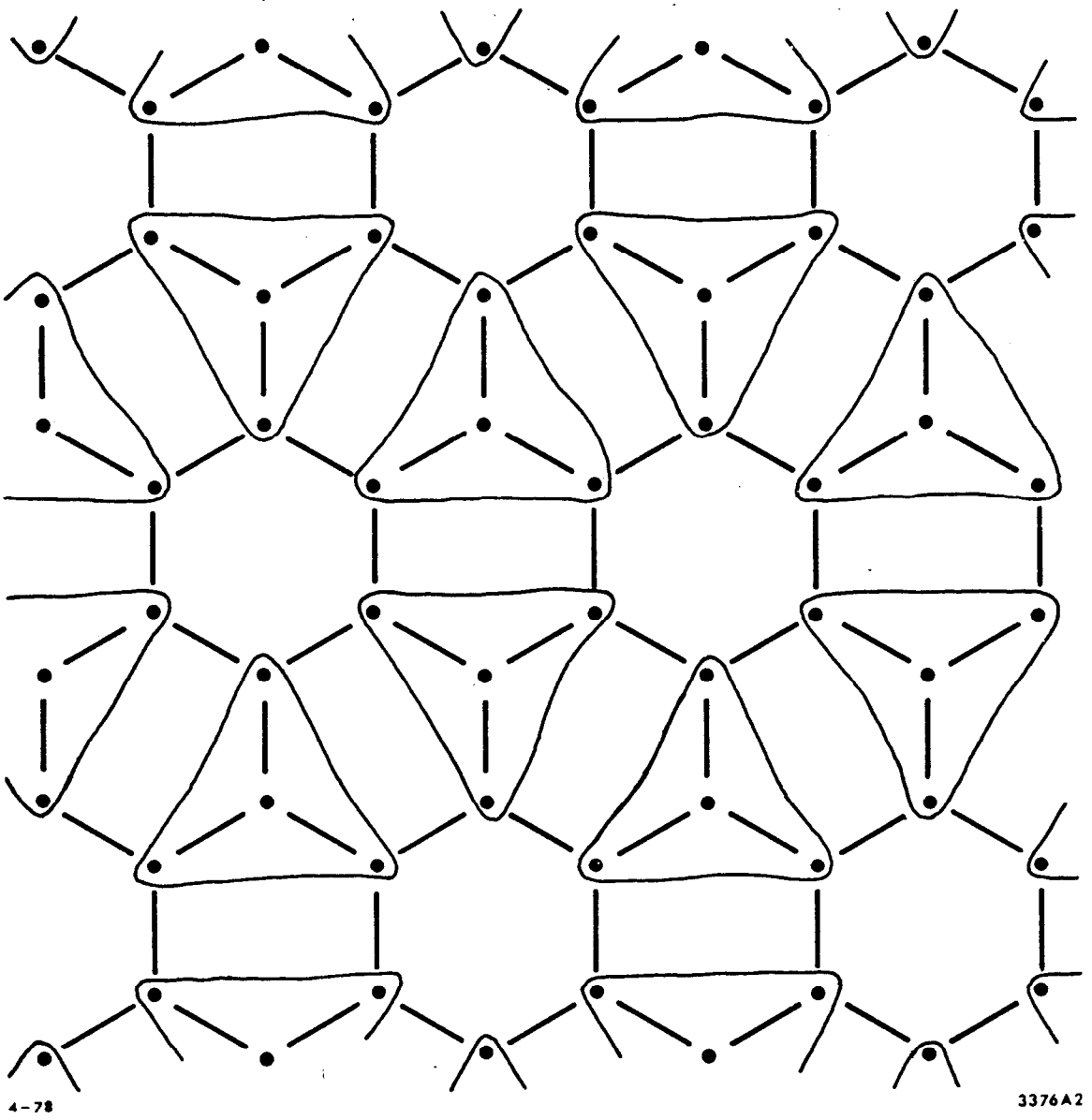


Fig. 2

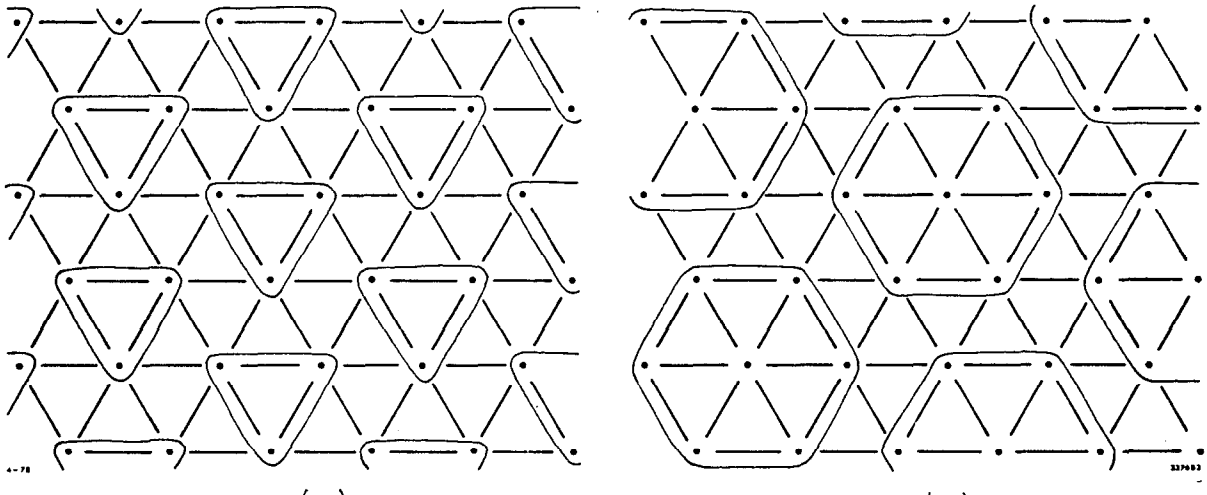


Fig. 3

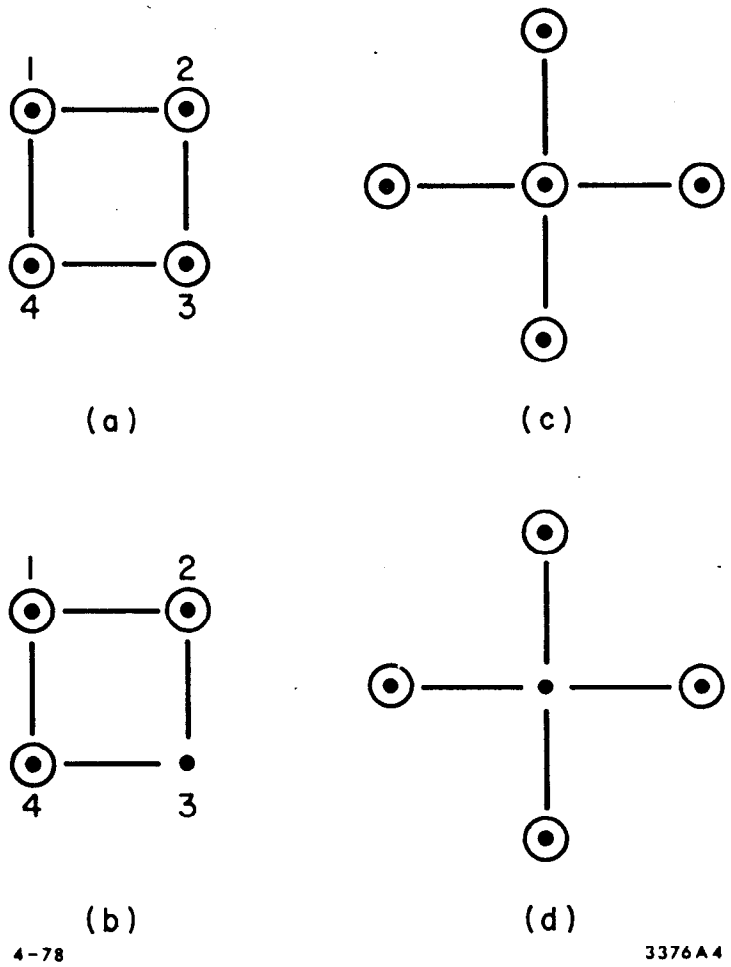


Fig. 4

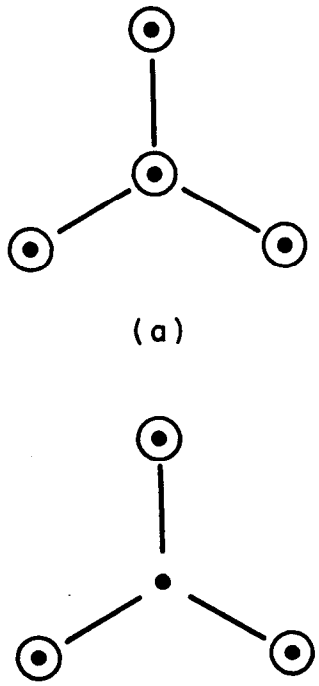


Fig. 5

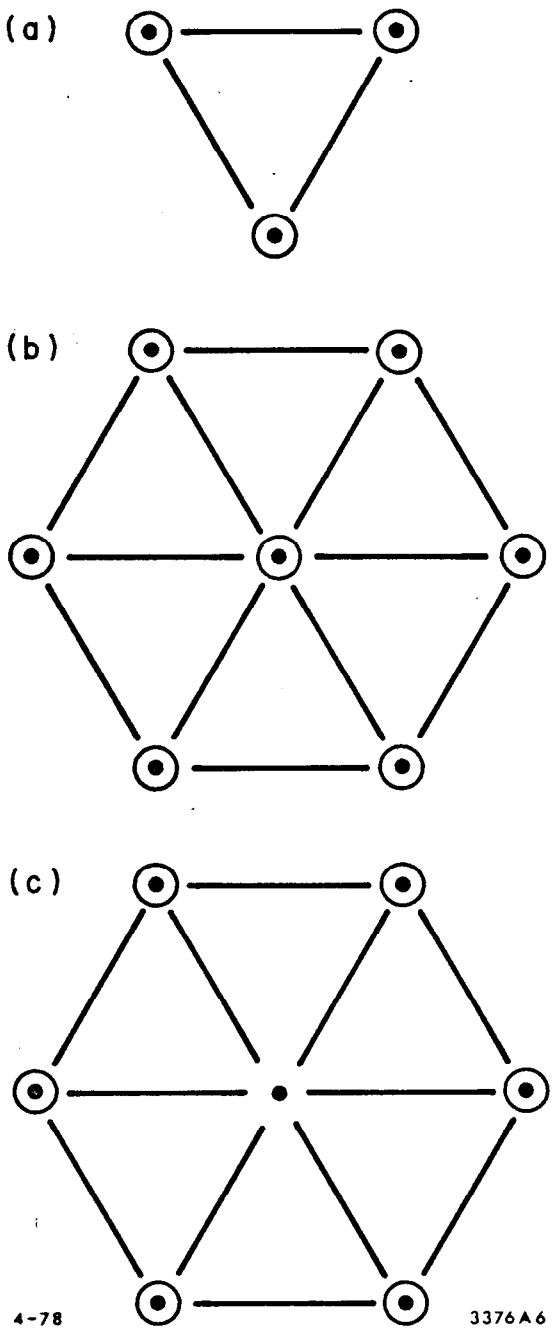


Fig. 6

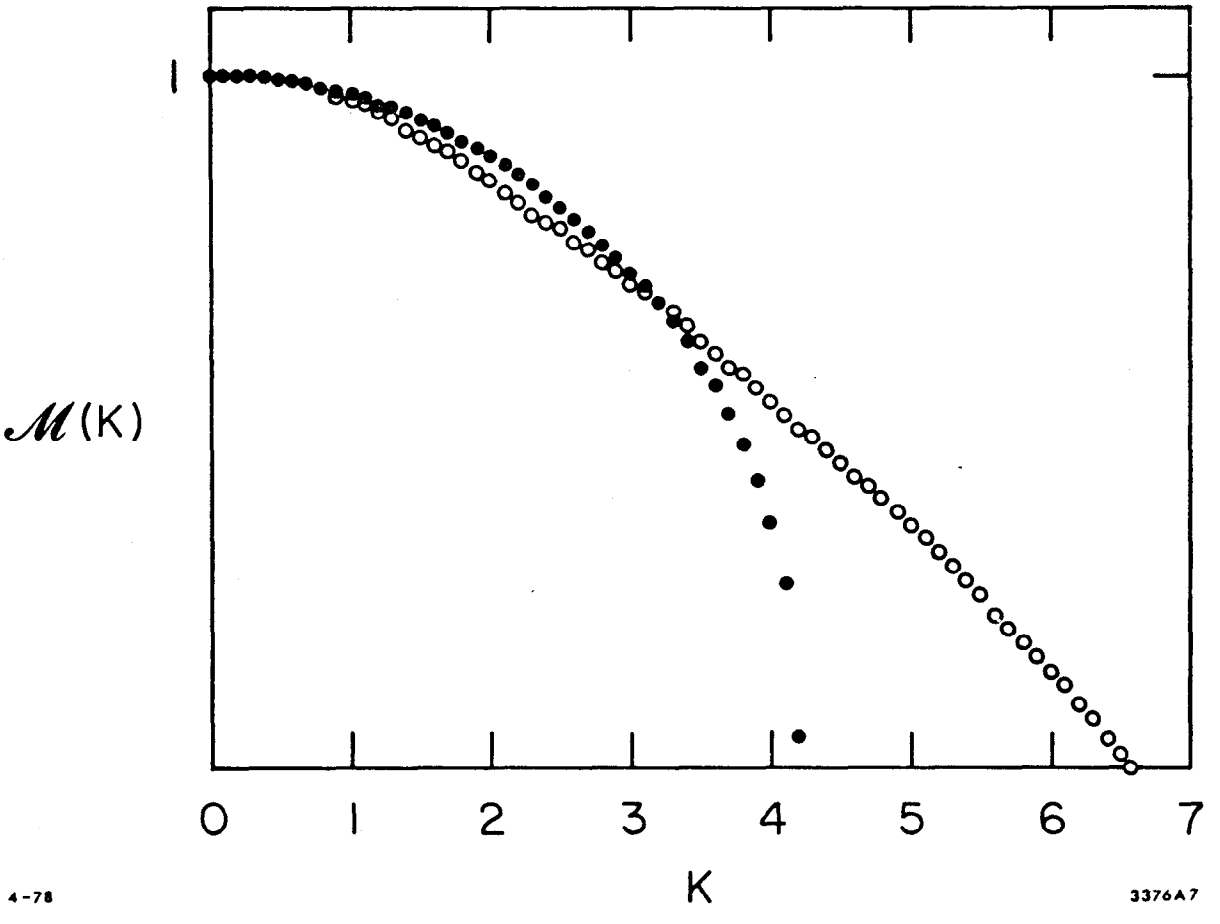


Fig. 7

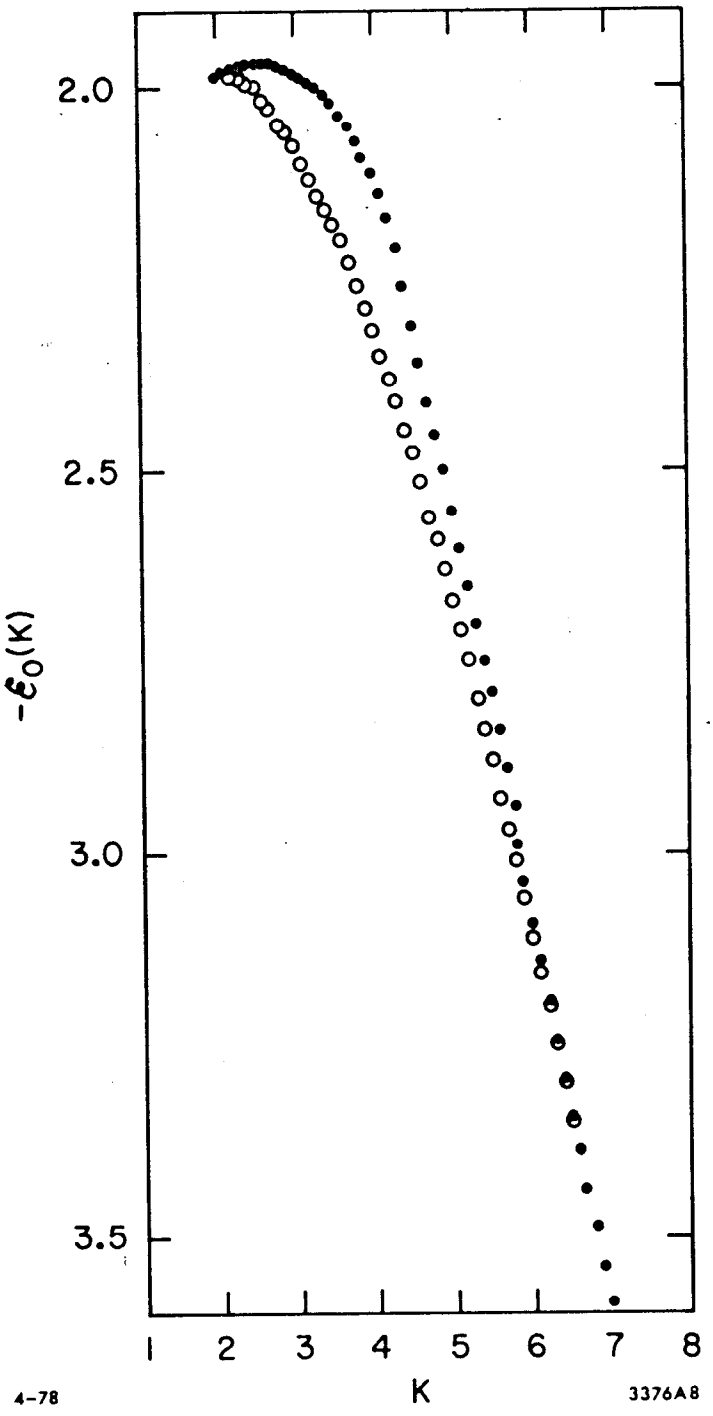


Fig. 8

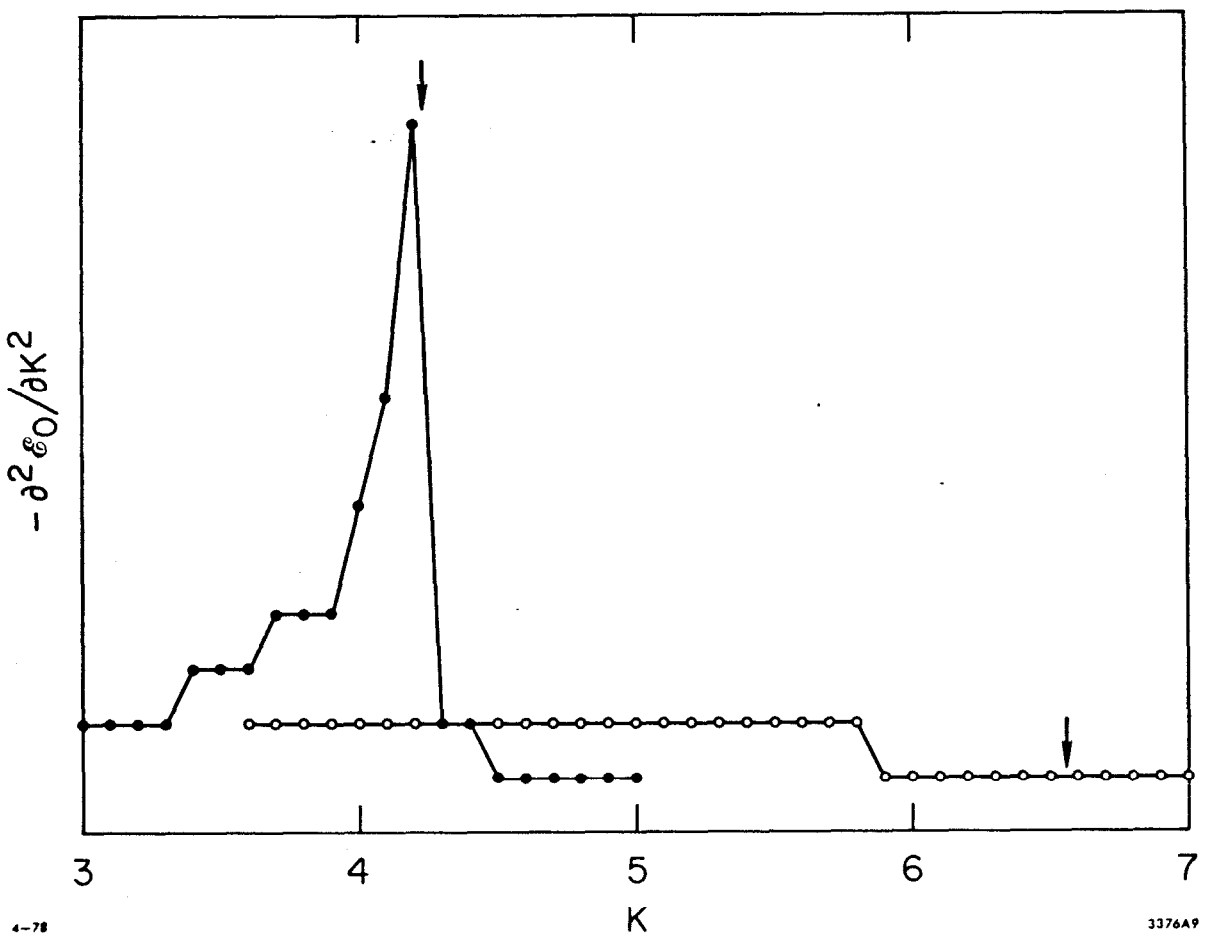


Fig. 9