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ZERO RANGE THREE-PARTICLE EQUATIONS*

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ABSTRACT

In order to separate the entire effect of two-particle on-shell scatterings in three-particle systems from the effects of hidden mesonic degrees of freedom (off-shell effects and three-body forces) we take the zero range limit of the Karlsson-Zeiger equations. Although the Faddeev equations are ambiguous in this limit, the KZ equations remain well defined. Using only two-particle phase shifts, binding energies, and reduced widths, these zero-range equations uniquely predict the three-particle observables which would occur in the absence of hidden mesonic degrees of freedom. The three-particle amplitudes possess all requisite physical symmetry properties, and can be proved to be unitary if the spectator basis is orthonormal and complete. Possible extensions of the scheme for the analysis of three-particle final states, to zero range four-particle equations, and to relativistic systems are conjectured.

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Ever since Wick[1] showed that the finite range of nuclear forces in Yukawa's meson theory[2] arises naturally from the coupling of the uncertainty principle with special relativity, and more particularly since the experimental discovery of the pion, we have known that nuclei must, in some sense, contain pions as well as protons and neutrons. Yet most of nuclear physics has been developed using phenomenological nuclear potentials which do not take explicit account of these mesonic degrees of freedom. The main counter-example is the study of the two nucleon problem, where much effort has gone into the construction of the so-called "meson-theoretic potentials" using in most cases either a combination of field theory and dispersion theory or one boson exchange models roughly correlated with the empirical boson mass spectrum. Although such models can, after considerable effort and a generous use of empirical parameters, provide a reasonably quantitative description of two nucleon elastic scattering, this is no guarantee that these models provide a correct description even of the nucleonic degrees of freedom at short distances, due to the fact that there are always an infinite number of ways to describe this short range behavior which lead to identical elastic scattering amplitudes.

Thus the first stringent test of the nuclear force models comes from confronting them with three nucleon data. Here, as we have known for some time, the test fails. The "realistic potentials" underbind the triton by 1 to 1.5 MeV, and predict electrostatic form factors with a first minimum at higher energy and a second maximum of much smaller magnitude than given by the experimental results of e-He^3 elastic scattering. Thus the mesonic degrees of freedom do matter in a quantitative sense, whether due to the fact that they give short range behavior quite different from (though phase equivalent to) the "realistic potentials"—in the jargon of the trade these are called "off-shell effects"—or due to the fact that they give rise to short range three-body forces, or most likely both.

When I started work on the three body problem in 1965 these general considerations led me to anticipate that it would be important to separate the long range or on-shell effects which arise in the region where the particles are outside the range of forces from the more complicated short range effects. By 1969 I had succeeded in making a formal separation between these exterior and interior regions [3], but still did not have sufficient understanding of some of the subtleties in the three-body problem to reduce this formalism to a practical analytic tool.

The key to the successful solution of the problem came from quite other considerations connected with the interpretation of quantum mechanics, as I discussed in my talk on "Three-Body Forces" at UCLA in 1972 [4]. If we formulate the scattering problem directly in terms of the on-shell scatterings between free particles, as can be done consistently in a descriptive sense [5], then we can for example introduce mesonic effects into the three nucleon problem by assuming that this system consists of three nucleons and a pion and solving this four-body problem. The scheme I conjectured which would make this possible was general in that I envisaged that it would be possible to discuss any n-particle system in terms of the n-1 particle on-shell scatterings and an intrinsic n-particle process to be determined empirically. The basic difficulty is that in this type of zero range theory, which has no interaction Hamiltonian, the task of guaranteeing unitarity is left to the form of the equations themselves, and must be reinvestigated separately for each n.

By 1975 I thought I had abstracted a viable, though ad hoc, set of three-body equations from the Faddeev equations. I presented these on-shell equations at Liblice, [6] only to have them shot down by Lambrecht Kok. One basic difficulty in starting from the Faddeev equations is that the two-particle on-shell amplitude $\tau^{\pm}(q^2) = e^{\pm i\delta_q}\sin\delta_q/\sqrt{q^2 \pm i0}$ occurs in the kernels with the energy argument $z - \tilde{p}^2$ and must be known in the nonphysical region. Worse, since we wish to use on-shell amplitudes with a "left-hand cut" representing meson exchanges, the integrals run over the cut and the equations become ambiguous. I tried to avoid this by simply using the on-shell amplitude in the physical region, but this did not lead to consistent equations. Subsequent efforts to avoid the cuts by using the n/d separation of dispersion theory also ran into difficulties.

Meanwhile one of us (EMZ) in collaboration with Bengt Karlsson of Göteborg was developing a consistent set of equations using spectator wave functions as a basis (i.e. scattering and bound state wave functions for the pair times a plane wave for the third particle)—the tool I had earlier used to construct the exterior-interior separation [3]. These equations were published in 1975 [8] and indeed depend only on half-on-shell wave functions and t-matrices with physical energy values, as anticipated [9]. The KZ equations for two-particle systems that scatter only in a finite number of partial waves have the further advantage that the kernels are real and energy independent except for the usual three-particle Green's function.

The KZ equations are fully equivalent to the Faddeev equations as can be seen by starting from the Low equation or completeness relation which allows us to construct fully off-shell t-matrices from half-off-shell t-matrices, namely [10]

$$t(q,q_0;z) = t(q;\tilde{q}_0^2 + i0)$$

$$+ \frac{2}{\pi}\int_0^\infty k^2 dk\ t(q;\tilde{k}^2)t^*(q_0;\tilde{k}^2)\left[\frac{1}{k^2-2\mu z} - \frac{1}{k^2-q_0^2-i0}\right] \tag{1}$$

$$= t(q_0;\tilde{q}^2 + i0) + \frac{2}{\pi}\int_0^\infty k^2 dk\ t(q;\tilde{k}^2)t^*(q_0;\tilde{k}^2)\left[\frac{1}{k^2-2\mu z} - \frac{1}{k^2-q^2-i0}\right]$$

By using time reversal invariance, i.e. $t(q,q_0;z) = t(q_0,q;z)$, we find that this puts a constraint on the half-off-shell t-matrix $t(k;q^2 \pm i0)$ which may be written as

$$\int_0^\infty q^2 dq\ \psi_q^{\pm}(k)t(k';\tilde{q}^2 \mp i0) = \int_0^\infty q^2 dq\ t(k;\tilde{q}^2 \pm i0)\Psi_q^{\mp}(k') \tag{2}$$

where

$$\psi_q^{\pm}(k) = \frac{\delta(k-q)}{kq} + \frac{2t(k;\tilde{q}^2 \pm i0)}{\pi(k^2-q^2 \mp i0)} \tag{3}$$

This constraint in a somewhat different form has been investigated by Baranger, et al. [11]. When the fully-off-shell t-matrix can be constructed in this way, it is easy to show that the existence of the construction is both a necessary and a sufficient condition for the KZ equations and the Faddeev equations to define the same theory.

Last spring I finally realized that the KZ equation remains well defined in the zero range limit, contains only phase shifts, binding energies and reduced widths, and hence defines precisely the theory I had been looking for - provided it also predicts physically meaningful amplitudes. This is not so obvious. In the on-shell limit

$$t^{\pm}(k;\tilde{q}^2) \to \tau^{\pm}(q^2) = \frac{e^{\pm i\delta}\sin\delta}{\sqrt{q^2 \pm i0}} \tag{4}$$

the condition becomes

$$\lim_{t \to \tau} t(q,q_0;z) = \tau^{\pm}(q^2) + \frac{2}{\pi}\int_0^{\infty} \sin^2\delta_k dk \left[\frac{1}{k^2-2\mu z} - \frac{1}{k^2-q\mp i0}\right] \tag{5}$$

or using the usual dispersion-theoretic representation for τ

$$= \frac{2}{\pi}\int_0^{\infty} \sin^2\delta_k \; dk \frac{1}{k^2-2\mu z} + \int_{-\infty}^{-m_x^2} \frac{\rho(k^2)dk^2}{k^2-q^2\mp i0} \tag{6}$$

This condition can only be satisfied if the integral over the left-hand cut vanishes for any needed value of q^2, a condition which cannot in general be satisfied when the left-hand cut arises from meson exchanges. For the special case $q \operatorname{ctn} \delta = \text{const.}$ the condition is met.

I wish to stress the point, which I should have realized from the start but have only now fully appreciated, that the limit I am about to define takes us outside the framework of conventional theories. This is often a barrier for theorists who try to understand what I have done. They realize that the KZ and Faddeev theories are fully equivalent before the limit is taken, but are not prepared for the possibility that the KZ equation remain valid in the zero range limit while the Faddeev equations do not. Of course, we have known ever since Thomas pointed it out in 1935 [12] that the Schroedinger equation is also not well defined in this limit.

Provided the wave function inside the range of forces R is non-singular when transformed into momentum space and goes smoothly to zero as R goes to zero, we can take this limit directly in the KZ equations simply by replacing $t^{\pm}(k;\tilde{q}^2)$ by $\tau^*(q^2)$ and $\psi_q^{\pm}(k)$ by

$$\psi_q^{\pm}(k) = e^{\pm i\delta}\left[\frac{\delta(k-q)}{kq}\cos\delta + \frac{\mathscr{P}}{\pi}\frac{\sin\delta}{k^2-q^2}\right] \tag{7}$$

Assuming for simplicity only s-wave scatterings between the pairs and total three-particle angular momentum zero, we then find that the KZ amplitudes are separable, and determined by one-variable integral equations. For the 3-3 amplitude $\mathscr{T}_{\beta\alpha}$ this representation is

$$\mathscr{T}^{\pm}_{\beta\alpha}(p_\beta,q_\beta;p_\alpha^{(o)},q_\alpha^{(o)};W\pm i0) \times \pi^2\mu_\alpha\mu_\beta$$

$$= \tau_\beta^{\pm}(q_\beta^2)\mathcal{Z}^{\pm}_{\beta\alpha}(p_\beta,p_\alpha^{(o)};W\pm i0)\tau_\alpha^{\pm}({q_\alpha^{(o)}}^2) \tag{8}$$

If we have in addition two-particle bound states with wave functions in the zero range limit $N\exp(-Ky)/y$, we find the further simplification that all four amplitudes (free-free, coalesence, breakup, elastic and rearrangement) are given in terms of the same one- variable function by

$$
\begin{aligned}
3\text{-}3 &: \mathscr{T}^{\pm}_{\beta\alpha} = \tau^{\pm}_{\beta}\ \mathscr{F}^{\pm}_{\beta\alpha}\ \tau^{\pm}_{\alpha} \\
3\text{-}2 &: \tilde{\mathscr{E}}_{\beta\alpha} = N_{\beta}\ \mathscr{F}^{\pm}_{\beta\alpha}\ \tau^{\pm}_{\alpha} \\
2\text{-}3 &: \mathscr{E}_{\beta\alpha} = \tau^{\pm}_{\beta}\mathscr{F}^{\pm}_{\beta\alpha}\ N_{\alpha} \\
2\text{-}2 &: \mathscr{H}_{\beta\alpha} = N_{\beta}\ \mathscr{F}^{\pm}_{\beta\alpha}\ N_{\alpha}
\end{aligned}
\tag{9}
$$

The one-variable functions themselves are determined by the coupled equations

$$
\begin{aligned}
\mathscr{F}_{\beta\alpha}(p_\beta, p_\alpha^{(o)};W) &= \frac{1}{2}\int_{-1}^{1} d\xi\ \bar{\delta}_{\beta\alpha}\ \frac{1}{\tilde{p}_\beta^2 + \tilde{q}_\beta^{(1)2} - W \mp i0} \\
&+ \sum \bar{\delta}_{\beta\gamma}\int_0^\infty p_\gamma'^2 dp_\gamma'\ K^{\pm}_{\beta\gamma}(p_\beta, p_\gamma';W)\,\mathscr{F}^{\pm}_{\gamma\alpha}(p_\gamma', p_\alpha'^{(o)};W) \\
\pm\ \vec{p}^{(1)}_{\beta\beta\pm} &= \frac{m_{\beta\mp}\vec{p}_\beta}{m_{\beta-} + m_{\beta+}} + \vec{p}^{(o)}_{\beta\pm} \\
\alpha = \beta\pm \qquad & \xi = \vec{p}_\beta\cdot\vec{p}_\alpha^{(o)}/p_\beta p_\alpha^{(o)} \\
& \beta-, \beta, \beta+ \text{ cyclic}
\end{aligned}
\tag{10}
$$

where $\bar{\delta}_{\beta\alpha}=1-\delta_{\beta\alpha}$, and the kernels

$$
\begin{aligned}
K^{\pm}_{\beta\gamma}(p_\beta, p_\gamma';W) &= \frac{1}{2}\int_{-1}^{1} d\xi \Bigg\{ \frac{N_\gamma}{\tilde{p}_\gamma'^2 - \kappa_\gamma^2 - W \mp i0}\ \frac{N_\gamma}{q_{\beta\gamma}^{(2)2} + \kappa_\gamma^2} \\
&+ \int_0^\infty dq_\gamma'\ \frac{1}{\tilde{p}'^2 + \tilde{q}_\gamma'^2 - W \mp i0}\left[\frac{\sin\delta'\cos\delta'}{q_\gamma'}\ \delta(q_\gamma' - q_{\beta\gamma}^{(2)}) + \frac{\mathscr{P}}{\pi}\frac{\sin^2\delta'}{q_{\beta\gamma}^{(2)2} - q_\gamma'^2}\right]\Bigg\} \\
\mp\ \vec{q}^{(2)}_{\beta\beta\pm} &= \vec{p}_\beta + \frac{m_\beta \vec{p}'_{\beta\pm}}{m_\beta + m_{\beta\mp}} \\
\gamma = \beta\pm \qquad & \xi = \vec{p}_\beta\cdot\vec{p}_\gamma'/p_\beta p_\gamma'
\end{aligned}
$$

do indeed depend only on the phase shifts at physical energies, binding energies, and reduced widths as promised. It is easy to prove that these equations converge provided only $\sin^2\delta/q^2$ is bounded by const./q^2 as q^2 goes to infinity, again a standard dispersion-theoretic assumption.

If we write the corresponding equation for the function $\mathscr{F}_{\alpha\beta}(p_\alpha^{(o)}, p_\beta;W)$ in which the role of parameter and variable is reversed we find that, thanks to the kinematic identities

$$\vec{q}^{(1)}_{\beta} \xrightarrow[\gamma \leftrightarrow \beta]{} -q^{(2)}_{\beta\gamma} \quad \tilde{p}^2_{\beta} + \tilde{q}^{(1)2}_{\beta\alpha} = \tilde{p}^{(o)2}_{\alpha} + q^{(2)2}_{\alpha\beta} \tag{12}$$

that the driving terms and the kernels in the equations for $\mathcal{T}_{\beta\alpha}$ and $\tilde{\tilde{\mathcal{T}}}_{\alpha\beta}$ are identical. Therefore they can differ at most by a solution of the homogeneous equation. But, since up to the usual Lippmann-Schwinger propagator the kernels are real, the usual arguments [13] suffice to show that these (bound state) solutions are unique and hence that $\mathcal{T}_{\beta\alpha}$ and $\tilde{\tilde{\mathcal{T}}}_{\alpha\beta}$ are identical. Further, $\mathcal{T}(W\pm i0) = \mathcal{T}(W\mp i0)$. Thus we have proved the essential symmetry property

$$\mathcal{T}^{*}_{\beta\alpha}(p_{\beta},p_{\alpha};W\pm i0) = \mathcal{T}_{\alpha\beta}(p_{\alpha},p_{\beta};W\mp i0) \tag{13}$$

By examining the corresponding three-particle wave function in configuration space, it is easy to see that this property, in conjunction with our specific separable representation, suffices to establish time reversal invariance and all other requisite physical symmetry properties, except unitarity.

In order to prove unitarity we cannot, as in the original KZ paper, have recourse to an operator proof. As discussed above, we have gone outside the framework of the conventional Hamiltonian theories by taking the zero range limit in the sense that it is no longer possible for us to construct a fully off-shell t-matrix. However, once the right route is found, it is straightforward, though tedious, to prove three-body on-shell unitarity in the KZ theory using only orthogonality and completeness in addition to the integral equations themselves. One of us (EMZ) has succeeded in constructing such a proof, and will present it elsewhere. [14] Since the requisite kinematic structure of the equations is unaltered in the zero range limit, the same proof would suffice to establish three-body on-shell unitarity in our zero range limit. Unfortunately, the "proof" of zero-range orthogonality presented at the conference was later shown to be incorrect by one of us (EMZ), and the corresponding completeness proof was likewise invalidated by Bengt Karlsson [15].

Therefore, the unitarity of the zero-range amplitudes remains unproven until a satisfactory limiting procedure is found, or alternatively, until we succeed in proving unitarity through a different approach.

It is important to realize that once we can prove unitarity, these equations will be unique, and hence to the extent that we believe we know the two particle on-shell phase shifts (including their extrapolation to infinite energy) and the two-particle binding energies and reduced widths, the three-particle observables predicted by these equations will also be unique. Hence any discrepancies between these predictions and experiment which cannot be realistically attributed to uncertainties in these parameters or to the finite partial wave truncation will provide concrete and unambiguous evidence for hidden degrees of freedom not described by the two-particle on-shell scatterings of the subsystems. Thus one major application of these equations will be to pinpoint where in three-particle systems we have clear evidence for these mesonic effects.

In the three nucleon system we already know that most of the three-particle amplitudes are quite insensitive to off-shell variations. We now have a precise way to prove this. By making a threshold subtraction in our equation for the $^2S_{1/2}$ n-d scattering to fit the sensitive scattering length a_2 (and if necessary a second subtraction to fit the triton binding energy) we can tie down the n-d predictions at threshold in the spirit of an effective range theory. We then anticipate n-d predictions at low energy comparable to the successes already achieved by separable models, once we include enough two-nucleon states. A careful error analysis will then reveal where in the increasingly detailed three-nucleon data there is evidence for genuine mesonic effects, or where we can measure two nucleon phases which are poorly known from two-nucleon elastic scattering experiments (εg. $^1P_1$ and ε_1) in amplitudes where other uncertainties are small. An alternative approach would be to investigate our limit at finite range and include the modifications of the kernel and driving terms arising from the vanishing of the two-particle wave functions inside R. Judging from the success of Brayshaw's boundary condition approach [16], we can hope that this will still provide us with one-variable equations, and can adjust R to fit a_2 and the triton binding energy. If this still does not give good electromagnetic form factors, at the cost of going to two-variable interior equations, we could also postulate nucleonic structure inside R and investigate phenomenologically what is required to fit the electromagnetic properties.

We can hardly expect all three-hadron systems to be as insensitive to hidden degrees of freedom as the three-nucleon system. Therefore we will need a systematic way to introduce phenomenological parameters into the scheme which can be used to analyze the three particle final states relative to an assumed set of phase shifts for the two-particle subsystems. One way to do this would be to go from a 3×3 to a 4×4 component description by including a direct three-particle zero range scattering process. If the new components are then eliminated to get back to a 3×3 description the three-particle phase shifts will appear explicitly in the modified kernels and driving terms. Whether the resulting equations will still retain their one-variable structure is not completely clear, but for some parametrizations this should be possible to achieve. Then we could use the scheme to determine genuine three-particle parameters directly from experiment. This should prove to be particularly useful in the kinematic regions where there are broad overlapping two-particle resonances in the subsystems, since these could be included without approximation.

Another important problem is the extension of the scheme to four-particle systems. Once we have understood in detail the relationship between a direct on-shell three-particle scattering description and the more detailed articulation of the amplitudes in terms of the Faddeev channel decomposition, it might be possible to construct an on-shell four-particle theory using only 3+1 and 2+2 clusters (i.e., a 7×7 channel description). Whether or not this proves to be possible, we can with some effort surely construct a two-variable but on-shell 18×18 component four-particle theory.

Another generalization that is almost immediate is the corresponding covariant three-particle theory. We know this because of Brayshaw's success with the covariant boundary condition approach. [17] In the current scheme we can hope to replace the boundary condition by invoking directly the inelasticity parameter which occurs in the two-particle elastic amplitudes due to the opening up of particle production channels. We also can now use models with left-hand cuts representing particle exchanges in crossed channels, and thus come closer than Brayshaw to conventional elementary particle theories.

Once a four-particle covariant theory exists we can look at the NNπ system below production threshold for the pion, and compare it with NNπ clusters in the NNNπ system similarly restricted. In both cases we can define what we might call a two-nucleon off-shell t-matrix. To the extent that they are the same we could then justify using this similarity to define a "two-nucleon potential"; to the extent that they differ we could begin to obtain realistic estimates of the limitations of the potential concept in nuclear physics.

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