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GAUGE SYMMETRIES IN RANDOM MAGNETIC SYSTEMS*

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Abstract

We study random magnetic systems emphasizing the concept of gauge invariance and gauge invariant disorder (frustration) introduced by Toulouse and Anderson. We formulate our models in a gauge invariant manner and introduce gauge invariant correlation functions to isolate the effects of gauge invariant disorder. Specifically, we study the Ising and XY models in two and three dimensions in a frozen distribution of frustrations. Using duality transformations we obtain expressions for the energetics of frustrations and their effect on correlations. We study simple configurations of frustrations quantitatively. In addition we reformulate the quenching procedure in terms of frustrations.

0. INTRODUCTION

Random magnets are very complicated systems. Since the spin degrees of freedom interact with each neighbor through random bond interactions, naively it could be argued that all possible configurations of bonds are equally important. However, it is well known that there is a class of models, collectively known as Mattis models [1], for which the randomness is trivial since it can be eliminated by a suitable redefinition of the spin variables. This situation led Anderson and Tolouse to the idea of relevant and irrelevant disorder. Anderson [2] advanced the concept of frustration as a measure of relevant disorder and Toulouse [3] realized the existence of a local (gauge) symmetry of random magnetic systems at the microscopical level.

Once the existence of a local symmetry is recognized [4] it becomes apparent that there are certain configurations of bonds which cannot be transformed into that of a pure system by any redefinition of the spin and bond variables (gauge transformation). We say these configurations have frustration.

The idea of frustration is that competing interactions in a random system can lead to configurations where not all the bond interactions can be simultaneously satisfied. In this situation, the ground state energy is always larger than in the "pure" system and the state is highly degenerate.

The purpose of this paper is to present a systematic study of gauge symmetries in random magnetic systems and its consequences. In order to filter out relevant from irrelevant disorder, we make extensive use of the concept of gauge invariance. In fact, many of our ideas have been borrowed from lattice gauge theories studies [5].

In section I, we show that the partition function of a magnet in a frozen configuration of bonds is gauge invariant. Then we conclude that only frustrations can change the nature of the phase transitions allowed for the system. In addition, we construct gauge invariant spin-spin correlation functions. These correlation functions are defined along a path connecting the correlated spins and are path dependent. In fact, gauge invariant correlation functions along two different paths differ by the total amount of frustrations they encircle. Hence, they provide a measure of the effect of the relevant disorder. This result is the analog of the Bohm-Aharonov effect in electrodynamics [6]. This path dependence is also closely related with the "fermionic" character of order and disorder variables, as discussed by Kadanoff and Ceva [7]. At the end of section I, we discuss the problem of quenching the frustrations, i.e. the averaging of the thermodynamic quantities over different configurations of bonds according with some probability weighting factors. It turns out that, when averaging gauge invariant quantities, frustrations behave as if they were an interacting system in thermal equilibrium with each other at an effective temperature. The classical interaction Hamiltonian can be calculated and turns out to be temperature dependent. The rest of the paper is devoted to the analysis of both the frustration network and the nature of their interaction in two and three dimensions. It should be mentioned that we make no attempt to study the possibility of a spin glass phase in any of the systems discussed below.

Section II deals with the random two dimensional Ising model. By performing duality transformations, we derive a relationship between the energy associated with a frozen distribution of N frustrations and the N point correlation function of the spins in the dual lattice. This result is analogous to results of [7]. We then use this relation to calculate explicitly the temperature dependence of the energy associated with having a single frustration and a pair of frustrated plaquettes in a sea of unfrustrated plaquettes. It turns out that single frustrations cannot exist at low temperatures but it is easy to create them in the paramagentic regime. For the case of a frustration pair, we show that at low temperature their energy increases linearly with their separation, with a temperature dependent coefficient that measures the line tension associated with the string that joins them. This confinement picture is lost at the critical temperature, where "melting" of the string leads to a finite energy for simple frustrations. Above T_c, we show that this interaction is effectively screened by the rapid fluctuations of the spins of the lattice. Using again duality transformations, we compute the decrease in the effective magnetization of a trivally disordered Ising system brought about by the presence of frustrations.

In section III, we study the 3d Ising model in a frozen configuration of bonds. Here frustrations are never along but instead they arrange themselves into networks [3]. Using duality transformations, we then calculate the energy associated with a closed tube of frustrations and show that, at low temperatures, it is proportional to the area spanned by the tube, whereas in the paramagnetic phase, it becomes proportional to the length of such a tube. For interacting tubes, at low temperatures the tubes interact with each other through a linear potential at short distances which saturates at large separations.

Sections IV and V deal with the XY model in two and three dimensions, respectively. In the two dimensional case, frustrations turn out to be equivalent to fractional impurities (vortices) in the 2d Coulomb gas. The particular case of half charges has been studied in detail by Villain [8]. As in the Ising case, we study the energetics of frustrations and the gauge invariant correlation function in both dimensionalities.

A collection of appendices provide most of the technical manipulations we have used to derive duality transformations of gauge invariant correlation functions.

I. LOCAL SYMMETRIES AND DISORDER

(A) Disordered Magnets and Gauge Symmetries

Let us consider the problem of describing the behavior of a magnet in an arbitrary configuration of bonds. At first we will discuss **frozen** distributions of them, i.e. the bond distribution is held fixed and not allowed to fluctuate. The problem of quenching (i.e. the averaging of thermodynamic magnitudes over different distribution of bonds) will be discussed at the end of this section.

To be more explicit, consider the case of a disordered Ising magnet in d dimensions [9]. This system consists of interacting Ising spin variables σ_i ($\sigma_i = \pm 1$) residing at the sites {i} of the lattice. The classical Hamiltonian is

$$-\beta H = K \sum_{\langle ij \rangle} \sigma_i A_{ij} \sigma_j \tag{1.1}$$

where K is the coupling ($K = \beta |J|$) and the summation runs over nearest neighbor sites. The variables A_{ij} specify the distribution of bonds and reside at the links {i,j} of the lattice (Fig. 1). In general, the bond variables $\{A_{ij}\}$ may be arbitrary. However, we will only consider the case in which $A_{ij} = \pm 1$. Thus the kind of disorder which may take place will grow from the competition between random ferromagnetic and antiferromagnetic interactions.

We may ask how the thermodynamic quantities differ from one configuration of bonds to another.

Consider the Hamiltonian (1.1) and single out a site i and <u>all</u> the links emerging from this site. Let us perform the <u>local</u> transformation

$$\sigma_i \rightarrow -\sigma_i$$

$$A_{ij} \rightarrow -A_{ij} \tag{1.2}$$

where {(i,j)} is the above mentioned set of links (Fig. 2).

The Hamiltonian (1.1) is invariant under this local transformation. Moreover, it is also invariant under the most general local transformation of this type $G(\{\tau_i\})$ which can be constructed by an arbitrary combination of site transformations like (1.2). $G\{\tau_i\}$ acts on the spins and bond variables as

$$G\{\tau_i\}[\{\sigma_i\};\{A_{ij}\}] = [\{\tau_i\sigma_i\};\{\tau_i A_{ij}\tau_j\}] \tag{1.3}$$

where $\tau_i = \pm 1$ [10]. The local symmetry we have just discussed is called a gauge symmetry, the transformations $G\{\tau_i\}$ are gauge transformation and the A variables are gauge variables. We shall see how this symmetry can be used to get information about order parameters, correlations, etc.

As an example, consider how gauge symmetry works in an annealed spin glass. By an annealed spin glass we mean a system where the σ's and A's are statistical variables to be averaged over at the same time. However, as is well known, this is an uninteresting system. In our "gauge language" this fact can be expressed as follows: the partition function of the annealed spin glass is given by

$$Z_{\text{annealed}} = \sum_{\{\sigma_i\}\{A\}} \exp\{K \sum_{\langle ij\rangle} \sigma_i A_{ij} \sigma_j\} \tag{1.4}$$

Suppose for the moment that we fix all the σ's to be 1. It is easy to see that there is no loss of generality involved in such a choice. In fact, for an arbitrary configuration of σ's and A's $[\{\sigma_i\}\{A_{ij}\}]$, we can always find a gauge transformation which maps this configuration to one another for which all $\sigma_i = 1$. In particular, if we choose for the gauge transformation defined in (1.3) $\tau_i = \sigma_i$, we get

$$G\{\tau_i = \sigma_i\};[\{\sigma_i\};\{A_{ij}\}] = [\{1\};\{\sigma_i A_{ij}\sigma_j\}] = [\{1\};\{A_{ij}'\}] \tag{1.5}$$

Since the Hamiltonian (1.1) is gauge invariant, both configurations have the same energy and therefore give the same contribution to Z. Thus

$$\sum_{\{\sigma_i\}} \sum_{\{A_{ij}\}} \exp\{K \sum_{\langle ij\rangle} \sigma_i A_{ij} \sigma_j\} = 2^N \exp\{K \sum_{\langle ij\rangle} A_{ij}\} \tag{1.6}$$

where N is the number of sites and 2^N is the number of independent gauge transformations. However, (1.6) is just the partition function of a system of independent spins on the links interacting with an external uniform field K, which has a trivial solution.

A more interesting system is the frozen spin glass. In this case we will no longer consider the gauge variables on the same footing with the spin variables but we will first take the thermal average over σ, compute all interesting magnitudes (free energy, correlation functions, magnetization, etc.) in a given field of A's, i.e. in a given distribution of flipped bonds. Later on we will average over distributions of gauge degrees of freedom according to some prescription. At first sight, it appears that the partition function in a given configuration of A's, i.e.

$$Z\{A\} = \sum_{\{\sigma_i\}} \exp \{K \sum_{\langle ij \rangle} \sigma_i A_{ij} \sigma_j\} \tag{1.7}$$

depends on all the details of the configuration of the gauge fields. However, consider two configurations $\{A\}$ and $\{A's\}$, which are related through a gauge transformation,

$$\begin{aligned} Z\{A'_{ij}\} &= \sum_{\{\sigma_i\}} \exp \{K \sum_{\langle ij \rangle} \sigma_i A'_{ij} \sigma_j\} = \\ &\sum_{\{\tau_i\}} \exp\{K \sum_{\langle ij \rangle} \sigma_i \tau_i A_{ij} \tau_i \sigma_j\} = \\ &\sum_{\langle ij \rangle} \exp K_{l}\, \sigma'_i A_{ij} \sigma_j' \equiv Z\{A_{ij}\} \end{aligned} \tag{1.8}$$

where $\sigma'_i = \sigma_i \tau_i$.

Thus, $Z\{A\}$ is invariant under gauge transformations and hence $Z\{A\}$ is not a functional of the configuration $\{A\}$ itself but rather on those features of that configuration which do not change with a gauge transformation.

(B) Frustrations

We now turn to the problem of describing the gauge invariant properties of a configuration of gauge degrees of freedom. For the time being, we shall restrict ourselves to systems with Ising degrees of freedom ($A, \sigma = \pm 1$). Such variables are elements of the Z_2

group. Later on, we shall discuss corresponding generalizations to more complex systems like the XY spin glass (a model with U(1) degrees of freedom).

To begin with, notice that the product of A variables around a closed loop of links on the lattice is invariant under a gauge transformation. In fact, this is the most general gauge invariant quantity that can be constructed from the A's alone. In particular, consider the smallest possible loop, i.e. the loop made of four links surrounding an elementary square ("plaquette") of the lattice. Since the value of the product of the A's around each plaquette of the lattice is a characteristic of the configuration of A's, which is invariant under gauge transformations, it is natural to define a plaquette variable Φ_{ijkl} such that

$$\Phi_{ijkl} = A_{ij}\, A_{jk}\, A_{kl}\, A_{li} \tag{1.9}$$

where i,j,k and l label the corners of the plaquette ijkl. We say that there is a **frustration** located at a plaquette if $\Phi=-1$ at this plaquette. Since $A^2=1$, the product of the A's around an arbitrary loop of the lattice is equal to the product of the Φ's for each plaquette enclosed by the loop. Therefore, the value of all gauge invariant quantities are specified by the values of the plaquette variables Φ.

The plaquette variable Φ is the analog in Z_2 gauge systems of the field strength or gauge curvature of conventional gauge theories. When $\Phi=-1$ at a plaquette, we say, interchangeably, that there is a frustration, curvature or dislocation there. For our purpose, this means that it is impossible to arrange the spins so as to satisfy all bond interactions around this plaquette (Fig. 3).

The partition function (1.7) is gauge invariant. Thus, it is only a function of the value of the Φ variables. This means that the partition function (and the free energy) do not depend on the detailed distribution of flipped bonds but only on the distribution of frustrations. Therefore, the partition function has the property

$$Z\{A\} = Z\{A'\} \equiv Z\{\Phi\} \text{ if } \{A\}\sim\{A'\} \tag{1.10}$$

Since $Z\{\Phi\}$ is the partition function in a fixed distribution of frustrations $\{\Phi_i\}$, we define the free energy in such distribution to be

$$K_1 F\{\Phi_i\} = - \log Z\{\Phi_i\} \tag{1.11}$$

In this language, we can understand the Mattis [1] model ($A_{ij}=\varepsilon_i\varepsilon_j, \varepsilon_i=\pm 1$) as a gauge transformation of the pure Ising model ($A_{ij}=1$). Thus the Mattis model is a random Ising model without frustrations (in fact the most general one) and it has the same (zero external field) free energy as the pure Ising model.

As we have shown frustrations are the only type of disorder that can modify the nature of the phase transitions of the system. Let us give some simple examples of frustrated 2-dimensional Ising models. Consider first the case with only one flipped bond (Fig. 4a). According with definition (1.9), the two plaquettes adjacent to the flipped bond are frustrated. Suppose now that we want to separate the frustrations. One possible way is to put a dual string of flipped bonds between them, as shown in Fig. (4b). However, there are several configurations with the same frustration content. One of them is shown in Fig. 4c. Both configurations differ by a gauge transformation. A closed dual string of flipped bonds (e.g. Fig. 4d) has no frustrations. A gauge transformation performed at all sites enclosed by the string transforms this configuration into all A=1. Notice that the lowest energy configuration for Fig. 4d is just an island of flipped spins whose boundary is the dual string. Analogously an infinite domain wall (Fig. 4e) has no frustrations. In this case the ground state has the spins on each side of the wall pointing in opposite directions.

Constructing a frustration at a single plaquette (Fig. 4f) cannot be accomplished by flipping a finite number of bonds near that plaquette. In fact, it is necessary to make a dual string of flipped bonds running from the frustration to the boundary of the lattice.

In contrast to frustration free configurations (Fig. 4d-4e), where a ground state with all bonds satisifed is possible, configurations with frustrations always have unsatisfied bonds and hence have higher energy. For instance, in Fig. 4b, the lowest energy configuration has all its spins parallel. The difference in energy between that state and the unfrustrated situation is proportional to the length of the string. Thus, a single frustration will have an infinite energy.

The case shown in Fig. 4g has some interesting features. Even with the boundary condition that all spins point up at infinity, there are two degenerate ground states: the central spin up or down. This illustrates the fact that frustrations tend to create additional degeneracies in the ground state since not all bonds can be satisfied simultaneously [11].

(C) Correlation Functions

We have just discussed the meaning of the free energy in a fixed distribution of frustrations. It is thus natural to ask the same kind of questions about the correlation functions. We should point an important difference between both quantities. Suppose we are to compute the correlation function between spins σ at sites i and j. In a fixed distribution of A's, we write

$$\langle\sigma_i\sigma_j\rangle_{\{A\}} = Z\{A\}^{-1}\sum_{\{\sigma_i\}}\sigma_i\sigma_j\exp\{K\sum_{\langle lk\rangle}\sigma_l A_{lk}\sigma_k\} \tag{1.12}$$

This correlation function is not gauge invariant since a local gauge transformation at site i (or j) changes the sign of this function.

We have argued that only gauge invariant disorder (frustrations) can change the nature of the phase transition, and thus we need a gauge invariant correlation function to probe this transition. We may define a gauge invariant analog of $\langle\sigma_i\sigma_j\rangle$ by inserting a "string of A's" between σ_i and σ_j. Then the gauge invariant correlation function is given by

$$\langle\sigma_i(\prod_{\Gamma(i,j)} A)\sigma_j\rangle \equiv Z\{A\}^{-1}\sum_{\{\sigma_i\}}\sigma_i(\prod_{\Gamma(i,j)} A_{lk})\,\sigma_j\exp\{K\sum_{\langle lk\rangle}\sigma_l A_{lk}\sigma_k\} \tag{1.13}$$

where $\Gamma(i,j)$ is a path connecting sites i and j and ΠA means the product of all the A variables along the links of the path (Fig. 5) [12]. Clearly this correlation function is gauge invariant. It depends both on the position of the correlated spins and on the path Γ itself.

Consider two different paths $\Gamma_1(i,j)$ and $\Gamma_2(i,j)$ (Fig. 6) and the corresponding correlation functions $\langle\sigma_i,\sigma_j\rangle_{\Gamma_1}$ and $\langle\sigma_i,\sigma_j\rangle_{\Gamma_2}$. Let us define the closed loop Γ as $\Gamma=\Gamma_1+\Gamma_2$. Then

$$\langle\sigma_i\sigma_j\rangle_{\Gamma_1} = \langle\sigma_i\sigma_j\rangle_{\Gamma_2}\,(\Pi_{\Gamma} A) = \langle\sigma_i\sigma_j\rangle_{\Gamma_2}\,(\prod_S \Phi) \tag{1.14}$$

where S is the region enclosed by Γ.

Thus, the two correlation functions differ by a factor of (-1) raised to the number of frustrations enclosed by the loop. Therefore, the difference of the gauge invariant correlation function along different path with same end point provides a measure of the frustration content within the loop.

Consider now an arbitrary configuration of bonds free of frustrations (pure gauge disorder). Such a configuration is gauge related with the configuration $A_{ij} = 1$ for all links. Then all these configurations of bonds will have the same gauge invariant correlation function. Certainly, gauge non-invariant correlation functions will be different for different configurations. However, those differences are not related with any change in the phase transitions of the system. We know that both the pure ferromagnetic ($A_{ij}=1$) and antiferromagnetic ($A_{ij}=-1$) Ising models are frustration free. In the ferromagnetic case, the gauge invariant correlation function reduces to the ordinary spin-spin correlation function. In the antiferromagnetic case, it reduces to the staggered correlation function.

(D) The Spin Glass as a System of Frustrations

Up to this point, we have only dealt with a frozen distribution of frustrations. Now we wish to make some comments about the spin glass problem, i.e. the averaging of quantitites over different distributions of bonds (quenching).

We will use the usual bond probability weighting factor P(A)

$$P(A) = \begin{cases} p & \text{if } A=1 \\ 1-p & \text{if } A=-1 \end{cases} \tag{1.15}$$

and assume that the total probability distribution $P\{A\}$ for configurations of bonds factorizes, i.e.

$$P\{A\} = \Pi_{\text{links}}\, P(A) \tag{1.16}$$

Consider now the average of a gauge invariant quantity, for instance, the free energy $F\{\Phi_i\}$. As we have already shown, it is gauge invariant and depends only on the distribution of frustrations. The object we want to compute is $\langle F\rangle_{K_l,p}$, where K_L is the inverse spin temperature and p the probability given in (1.14). Clearly,

$$\langle F\rangle_{K_l,p} = \sum_{\{A\}} P\{A\}\, F\{\Phi_{i,K_l}\} \Big/ \sum_{\{A\}} P\{A\} \tag{1.17}$$

Since $F\{\Phi_i\} = F\{A\}$ for all the configurations of bonds (link variables), which have the same distribution of frustration $\{\Phi_i\}$, Eq. (1.17) splits into sums over distributions of frustrations, i.e.

$$\sum_{\{A\}} P\{A\}\, F\{\Phi_{i,k_l}\} = \sum_{\{\Phi_i\}} \Big\{ \sum_{\{A\}}{}' P\{A\} \Big\}\, F\{\Phi_{i,K_l}\} \tag{1.18}$$

where the sum $\sum' P\{A\}$ runs over all the configurations of bonds with the <u>same</u> distribution of frustrations and therefore it weights distributions of frustrations. Let us now define β_f and α to be two parameters such that

$$P(A) = (\alpha/2)\exp(\beta_f A) \tag{1.19}$$

Eqs. (1.15) and (1.19) then give the result

$$\begin{aligned} \alpha/2 &= p(1-p) \\ \beta_f &= \log\,(p/1-p)^{1/2} \end{aligned} \tag{1.20}$$

This change of parameters allows us then to write for the frustration distribution probability weighting factor

$$\sum_{\{A\}}{}' P\{A\} = (\alpha/2)^N \sum_{\{A\}}{}' \exp \{\beta_f \sum_{\langle ij\rangle} A_{ij}\} \tag{1.21}$$

We can now easily recognize the right hand side of (1.21) to be $(\alpha/2)^N$ times the partition function (1.7) written in the gauge $\sigma_i = 1$ for all sites.

From Eqs. (1.11), (1.18) and (1.21), we get

$$\langle F\rangle_{K_l,\beta_f} = \sum_{\{\Phi_i\}} \exp -[\beta_f F\{\Phi_i,\beta_f\}] F\{\Phi_i,k_l\} / \sum_{\{\Phi_i\}} \exp \{-\beta_f F\{\Phi_i,\beta_f\}\} \tag{1.22}$$

This equation is not only valid for the free energy but for all the gauge invariant quantities.

Therefore, when averaging thermodynamic quantities frustrations behave as if they were in thermal equilibrium with each other and interacting through a classical Hamiltonian (configurational energy) given by $F\{\Phi_i,\beta_f\}$. The temperature of the system of frustrations is given by $1/\beta_f$. The Hamiltonian $F\{\Phi_i,\beta_f\}$ can be derived from the correlation functions of the dual system. We will illustrate this procedure in the following sections. We should note, however, that the hamiltonian will not be, in general, a simple sum of pairwise terms. In fact, it is a complicated configurational energy and it will depend on the temperature of the frustrations. Notice that the quantity being averaged in (1.22) is the same Hamiltonian $F\{\Phi_i,\beta_f\}$ evaluated at the spin glass temperature $1/K_L$. Furthermore, the normalization factor can be easily shown to be equal to the partition function of the annealed system.

In the spin glass literature, it is usual to find the phase diagram represented by a plot of K (spin temperature) vs. p (probability) [15]. We can now understand these diagrams in terms of the frustration temperature. At $p = ½$, the frustrations are at infinite temperature (β_f=0). In this situation, the density of frustrations is extremely high. As p increases, the temperature of frustrations decreases and at p = 1 the frustrations are at zero temperature. This state is the pure ferromagnet (p=1) and there are no frustrations here. All the models which are connected through gauge transformations with the pure ferromagnet are also at zero frustration temperature. The situation is symmetric around the point p = ½.

II. THE TWO DIMENSIONAL ISING SPIN GLASS

(A) The Model

We shall first discuss the 2d Ising model in an arbitrary configuration of bonds. The partition function is given by (Eq. 1.7)

$$Z\{A\}=2^{-N}\sum_{\{\sigma_i\}} \exp\{K \sum_{\langle ij\rangle} \sigma_i A_{ij} \sigma_j\} \tag{1.7}$$

In Eq. (1.8) we showed that $Z\{A\}$ is gauge invariant, i.e., if $\{A\}$ and $\{A'\}$ are two bond configurations related through a gauge tranformation, then the partition function is the same for both configurations and so is only dependent on the distribution of frustrations $\{\Phi\}$. Consider now the sum $\sum' Z\{A\}$ restricted to all configurations which have the same distribution of frustrations. From (1.8 - 1.10) we can write

$$\sum_{\{A\}}{}' Z\{A\} = 2^N Z\{\Phi\} \tag{2.1}$$

where 2^N is the total number of gauge transformations (volume of the gauge group).

Therefore, the partition function can be written as [11]

$$Z\{\Phi_i\} = 2^{-N}\sum_{\{A_{ij}\}} [\prod_i \delta(A_{ij}A_{jk}A_{kl}A_{li}\Phi_i-1)] \sum_{\{\sigma_i\}} \exp\{K \sum_{\langle ij\rangle} \sigma_i A_{ij} \sigma_j\} \tag{2.2}$$

where i is the dual site at the center of the plaquette ijkl. The Kronecker's δ replace the constraint in Eq. (2.1). Up to an (infinite) constant Eq. (2.2) takes now the form

$$Z\{\Phi_i\} = 2^{-N}\lim_{K_p\to\infty}\sum_{\{A_{ij}\}}\sum_{\{\sigma_i\}} \exp\{K_L \sum_{\langle ij\rangle}\sigma_i A_{ij}\sigma_j\} \exp\{K_p \sum_i \Phi_i A_{ij}A_{jk}A_{kl}A_{li}\} \tag{2.3}$$

This partition function describes an Ising model with a frozen distribution of frustrations. In order to simplify matters we choose the gauge $\sigma = 1$ (all sites) and, in that gauge, the partition function then reads

$$Z\{\Phi_i\} = \lim_{K_p\to\infty}\sum_{\{A_{ij}\}} \exp\{K_L \sum_{\langle ij\rangle} A_{ij} + K_p \sum_i \Phi_i A_{ij}A_{jk}A_{kl}A_{li}\} \tag{2.4}$$

In what follows we shall always write the partition function in this form.

(B) Duality

The duality properties of models like Eq. (2.4) with $\Phi = 1$ (unfrustrated case) have been extensively discussed by Wegner [14] and Balian, *et al* [15]. In this section, we will show how to extend those methods to the frustrated case, i.e. $\Phi_i = -1$. We will follow Balian quite closely. Let us apply the duality transformation to the model described by the partition function (2.4) in the case $\Phi_i = 1$ (all i). The dual partition function (for K_p finite) is given by

$$Z = [(1/4) \cosh K_p \cosh^2 K_l]^N \sum_{\{s_i\}} \exp [\sum_{\langle ij \rangle} \beta_l^*(s_i s_j - 1) + \Sigma_i H^*(s_i - 1)] \tag{2.5}$$

where N is the total number of lattice sites. The dual coupling β_l^* and dual external magnetic field H^* are related to the original link and plaquette couplings through the relations

$$e^{-2\beta_l^*} = \tanh K_L$$

$$e^{-2H^*} = \tanh K_p \tag{2.6}$$

The dual model is defined on the dual of the square lattice and at each dual site i there is a dual Ising spin s_i. Eq. (2.5) is just the partition function of a 2d Ising model in an external uniform magnetic field.

If we now let $K_p \to \infty$, the external field H^* vanishes. Thus, the system becomes the well known 2d Ising model in zero field. Notice that Eq. (2.6) implies that low temperatures and high temperatures are exchanged through a duality transformation.

In the previous discussion the system was uniform (i.e., all the bonds were the same). However, the duality transformation holds even in the case that the couplings K_l, K_p, vary throughout the lattice. In this case Eq. (2.6) becomes a local relationship between dual couplings. Since in two dimensions links are dual to links and plaquettes are dual to sites, the coupling at each link transforms into the coupling on its dual link and the plaquette coupling transforms into a local external field.

We now turn our attention to the case $\Phi = -1$ at some plaquette which means to flip the sign of the coupling K_p at that plaquette. Thus a system with some $\Phi_i = -1$ is just a system with some K_p negative.

From Eq. (2.6) we get the equivalency

$$K_p \to -K_p \rightleftarrows H^* \to H^* + i\pi/2 \tag{2.7}$$

So that, in general, the following identity is true

$$e^{-(2H^* + i[\pi/2][1 - \Phi_i])} = \tanh(K_p \Phi_i) \tag{2.8}$$

To flip the sign of a coupling is equivalent to shift the dual coupling by $i\pi/2$. This trick has been exploited by Kadanoff and Ceva in their discussion of disorder variables in the 2d Ising Model. The identity

$$\exp(i\pi(1-s)/2) = s \qquad\qquad s = \pm 1 \tag{2.9}$$

combined with Eq. (2.8) leads us to the conclusion that when we flip a plaquette coupling in the original model we are bringing down a dual spin variable (at the site dual to that plaquette) in the dual system. Thus, for arbitrary $\Phi_{\underset{\sim}{i}}$, the normalized function (2.4) (for finite K_p) after a duality transformation (2.6) - (2.8) becomes

$$\frac{Z\{\Phi_{\underset{\sim}{i}}\}}{Z\{\Phi_{\underset{\sim}{i}}=1\}} = \frac{\sum_{\{s_{\underset{\sim}{i}}\}} \left[\prod_{\underset{\sim}{i}} s_{\underset{\sim}{i}}^{(1-\Phi_{\underset{\sim}{i}})/2}\right] \exp\left\{\sum_{\langle ij\rangle} \beta_I^* s_{\underset{\sim}{i}} s_{\underset{\sim}{j}} + \sum_{\underset{\sim}{i}} H^* s_{\underset{\sim}{i}}\right\}}{\sum_{\{s_{\underset{\sim}{i}}\}} \exp\left\{\sum_{\langle ij\rangle} \beta_I^* s_{\underset{\sim}{i}} s_{\underset{\sim}{j}} + \sum_{\underset{\sim}{i}} H^* s_{\underset{\sim}{i}}\right\}} = \left\langle \prod_{\underset{\sim}{i}} s_{\underset{\sim}{i}}^{(1-\Phi_{\underset{\sim}{i}})/2} \right\rangle_{\beta_I^*, H^*} \tag{2.10}$$

where the average is taken in the dual system. In order to fix a distribution of frustrations we now let $K_p \to \infty$. Then from Eq. (2.10) the normalized partition function (2.4) in a specified distribution of frustrations turns out to be equal to the N point correlation function of the dual zero field Ising model at the temperature given by β_I^*. Notice that the limit $K_p \to \infty$ is essential not only to specify the distribution of frustrations but also to avoid the destruction of the phase transition of the 2d Ising model. In summary

$$(Z_{K_1}\{\Phi_i\})/(Z_{K_1}\{\Phi_i = 1\}) = \langle \Pi s_i^{(1-\Phi_i)/2} \rangle_{\beta_1^*} \qquad (2.11)$$

Since $Z_{K_1}\{\Phi_i\} \equiv e^{-K_L F\{\Phi_i\}}$ ($F \equiv$ free energy), then (2.11) gives the change in the free energy due to the effect of the frustrations as

$$\exp[-K_1 \Delta F\{\Phi_i\}] = \langle \Pi_i s_i^{(1-\Phi_i)/2} \rangle_{\beta_1^*} \qquad (2.12)$$

(C) The Energetics of Frustrations

Let us now discuss some specific examples. Unfortunately little is known about the behavior of this general N point correlation function. Nevertheless, some of the known general features are important for us. In the unmagnetized phase of the dual Ising model (i.e., high temperatures in the dual Ising $\equiv$ low temperatures in the spin glass), the N point correlation function vanishes identically if N is odd. Thus, frustrations come in even numbers (neutral configurations) in the low spin-glass temperature phase.

We now take full advantage of all the available information about the magnetization and the two point correlation function of the 2d Ising Model in zero external field in order to study the energetics of frustration systems [16].

The change in the free energy of the system due to the presence a single frustration is given by

$$\Delta F_{single} = -(1/K_1) \log \langle s \rangle_{\beta_1^*} = -(1/K_L) \log M(\beta_1^*) \qquad (2.13)$$

where $M(\beta_1^*)$ is the magnetization. Since the latter is exactly known we obtain

$$\Delta F_{single} = \begin{cases} -(1/8K_L \log (1-\sinh^4 2K_1) & K_1 < K_c \text{ (high spin glass temp.)} \\ \infty & K_1 > K_c \text{ (low spin glass temp.)} \end{cases} \qquad (2.14)$$

where $\sinh 2K_c = 1$ is the critical point of the 2d Ising model.

In fact, at low temperatures (spin glass), a single frustration is strictly forbidden since the excess free energy is infinite. At high temperatures of the spin glass system a single frustration costs a finite amount of free energy (it has a finite "mass"). The fluctuating Ising spins screen the frustration at high temperatures.

Let us now study the interaction energy for a pair of frustrations in an unfrustrated sea of spins. The change in the free energy due to two frustrations can be obtained from the two point correlation function of the dual Ising model. There are two regimes. 1) At low spin glass temperatures ($K_l > K_c$) the dual system is in its disordered phase (high temperature implies $\beta_l^* < K^{-1}{}_c$). The correlation function decays exponentially at large distances with a correlation length, ξ, given by

$$\xi(\beta_l^*) = (2\sqrt{2}|\log \sinh 2\beta_l^*|)^{-1} = (2\sqrt{2}|\log \sinh 2K_L|)^{-1} \tag{2.15}$$

Thus the excess free energy associated with a frustration pair separated by a distance R is given by

$$\Delta F(R) = -(1/K_l) \log \langle s_o s_R \rangle \sim (R/K_l\xi) + 0(\log R) \tag{2.16}$$

As Eq. (2.16) shows, the excess free energy grows linearly with R and therefore the energy necessary to separate two frustrations by an infinite distance is divergent. Thus, in the low (spin glass) temperature phase frustrations are "confined". One can picture the two frustrations as held together by a "string" whose tension τ is given by the coefficient of the linear term in (2.16), i.e.

$$\tau = 1/k_L\xi = (\sqrt{2}/K_L)|\log \sinh 2K_l| \tag{2.17}$$

At the critical point the correlation length diverges and the string tension goes to zero like $|K_l - K_c|$. In other words, "melting" of the string holding the frustrations together leads to a change in the force law.

In the high (spin glass) temperature phase "confinement" is lost. Here the dual system is in its ordered phase and hence the dual spin correlation function approaches a constant value at infinite distance

$$\langle s_o s_R \rangle \sim M^2(\beta_l^*)\{1+(V_o/R^2) \exp\{-R/\xi\} + \ldots\} \tag{2.18}$$

where $M(\beta_l^*)$ is the magnetization which is given by

$$M(\beta_l^*) = (1-\sinh^{-4} 2\beta_l^*)^{1/8} = (1-\sinh^{4} 2K_l)^{1/8} \tag{2.19}$$

ξ is the correlation length (2.15) and V_o is the constant

$$V_o = (\sinh {}^4 2K_1)/4\pi(1-\sinh {}^4 2k_1)^2 \tag{2.20}$$

Therefore, the excess free energy at high (spin glass) temperatures is given by

$$\triangle F(R) = 2\triangle F_{single} - (V_o/R^2) \exp(-R/\xi) \tag{2.21}$$

This means that at high temperatures frustrations are "free" and they interact through an attractive short ranged screened potential. The range of the potential is just the correlation length ξ. Notice that this range is strongly temperature dependent.

(D) Correlation Functions of the Ising model with Frustrations

We now wish to study the effect of frustrations on the two point correlation function of spins on the original spin glass. Since we are not interested in the effect of non-serious disorder (i.e., the disorder which is not associated with frustrations), we have to study the behavior of the gauge invariant correlation function in the presence of frustrations.

The gauge invariant correlation function is given by

$$C_{\Gamma(i,j)}\{\underset{\sim}{\Phi_i}\} = \langle \sigma_i (\prod_{\Gamma(i,j)} A_{lk}) \sigma_j \rangle_{\{\underset{\sim}{\Phi_i}\}} \tag{2.20}$$

The average is taken as explained in Eq. (1.12). $C_{\Gamma(i,j)}\{\Phi_i\}$ is a gauge invariant quantity. Thus all the arguments made for the partition function (1.7) which lead to the form given in (2.5) are valid in this case.

$C_{\Gamma(i,j)}\{\underset{\sim}{\Phi_i}\}$ as given by (2.20) can be rewritten as

$$C_{\Gamma(i,j)}(\underset{\sim}{\Phi_i}) = \lim_{K_p\to\infty} \frac{\sum_{\{A_{ij}\}}\sum_{\{\sigma_i\}} \sigma_i\left(\prod_{\Gamma(i,j)} A_{lk}\right)\sigma_j \exp\left\{K_t\sum_{\langle i'j'\rangle}\sigma_{i'}A_{i'j'}\sigma_{j'} + K_p\sum_{i'} AAAA\,\underset{\sim}{\Phi_{i'}}\right\}}{\sum_{\{A_{ij}\}}\sum_{\{\sigma_i\}} \exp\left\{K_t\sum_{\langle i'j'\rangle}\sigma_{i'}A_{i'j'}\sigma_{j'} + K_p\sum_{i'} AAAA\,\underset{\sim}{\Phi_{i'}}\right\}} \tag{2.21}$$

where AAAA means the product of all the link variables around the plaquette i'.

These gauge invariant two point correlation function obey a duality transformation. Through this transformation the two point (gauge invariant) correlation function in the presence of a distribution of N frustrations maps into the N point gauge invariant correlation function of the dual system in the presence of two frustrations, where the positions of frustrations and correlated spins are interchanged. Notice that the gauge invariant dual N point correlation function has strings of dual link variables a joining the dual spins pair wise.

The relation is given by

$$\frac{\langle \sigma_i \left(\prod_{\Gamma(i,j)} A \right) \sigma_j \rangle_{K_\ell} \{ \bar{\Phi}_{\underset{\sim}{i}} \}}{\langle \sigma_i \left(\prod_{\Gamma(i,j)} A \right) \sigma_j \rangle_{K_\ell} \{ \bar{\Phi}_{\underset{\sim}{i}} = 1 \}} = (-1)^n \frac{\langle \prod_{\alpha=1}^{N/2} \left[s_{\underset{\sim}{i}_\alpha} \left(\prod_{\tilde{\Gamma}_\alpha(\underset{\sim}{i}_\alpha, \underset{\sim}{j}_\alpha)} a \right) s_{\underset{\sim}{j}_\alpha} \right] \rangle_{\beta^*} \{ \tilde{\bar{\Phi}}_{i'} \}}{\langle \prod_{\alpha=1}^{N/2} \left[s_{\underset{\sim}{i}_\alpha} \left(\prod_{\tilde{\Gamma}_\alpha(\underset{\sim}{i}_\alpha, \underset{\sim}{j}_\alpha)} a \right) s_{\underset{\sim}{j}_\alpha} \right] \rangle_{\beta^*} \{ \tilde{\bar{\Phi}}_{i'} = 1 \}} \tag{2.22}$$

with a pictorial description given by Fig. 7 and

(a) $\Phi_{\underset{\sim}{i}}$ is the frustration field of the dual system, and $\Phi_{i'} = \begin{cases} -1 & \text{if } \underset{\sim}{i}' = i, j \\ 1 & \text{otherwise} \end{cases}$

(b) $\Gamma_\alpha\ (\underset{\sim}{i}_\alpha, \underset{\sim}{j}_\alpha)$ is a path of dual links which goes from site $\underset{\sim}{i}_\alpha$ to $\underset{\sim}{j}_\alpha$. The parameter α labels the different paths. In the case when N is odd, one of these paths runs to the boundary;

(c) Again remember that $e^{-2\beta^*} = \tanh K_\ell$;

(d) In the factor $(-1)^n$, n is the total number of intersections between the path $\Gamma_{(i,j)}$ and all the dual paths $\Gamma\ (\underset{\sim}{i}_\alpha, \underset{\sim}{j}_\alpha)$. The derivation of Eq. (2.22) is given in Appendix A.

Let us now discuss the influence of frustrations on the asymptotic behavior of the gauge invariant correlation function. We shall restrict ourselves to the case of two frustrations.

i) High Temperature Behavior

The behavior at high (spin glass) temperatures can be most easily studied directly by means of the high temperature expansion.

In. Eq. (2.14) we showed that one free frustration can exist at temperatures higher than the transition temperature. Let us study the effect of one single frustration on the behavior of the gauge invariant two point correlation function. Consider the simple case of a frustration in between the correlated spins. The string of link variables is a straight line of links joining the spins with the frustration adjacent to the string. To make an explicit high temperature calculation we choose the special flipped bonds representation of the frustration shown in Fig. 8. To the first non-trival power in $x \equiv \tanh K_l$, we get

$$\langle \sigma_i (\prod_{\Gamma} A) \sigma_j \rangle_{K_L} \{ \underset{\sim}{\Phi_l} \} \; / \; \langle \sigma_i \sigma_j \rangle_{K_L} = 1 - 2d\,(R-d+1)x^2 \qquad (2.23)$$

where $|i-j| = R$ and $|l-i| = d$ measures the distance of the frustration to one of the spins. Of course, this formula is only valid when $Rx<<1$. As expected, the correlation function decreases in the presence of the frustration.

An analogous computation for two nearest neighbor frustrations lying between the two spins (Fig. 9) gives the result

$$\langle \sigma_i (\prod_{\Gamma} A) \sigma_j \rangle_{K_l} \{ \underset{\sim}{\Phi_l} \} \; / \; \langle \sigma_i \sigma_j \rangle_{K_l} = 1 - 4d\,(R-d+1)x^2 \qquad (2.24)$$

$x \equiv \tanh K_l$

Notice that the effect of a frustration pair on the correlation function is bigger than in that of a simple isolated frustration.

(ii) Low Temperatures

At temperatures lower than T_c, there is long range order and the random system is "magnetized". It is interesting to see how the magnetization is affected by the frustrations.

Consider two frustrations a distance R apart and let us compute the (gauge invariant) magnetization at a point between them at a distance d from one of the frustrations (Fig.

10). This situation is the dual of that shown in Fig. A for which we obtained the high temperature result above.

The duality relation (2.22), together with (2.23), allows us to write

$$\langle\sigma_i(\Pi_{\Gamma(i,\infty)}A\rangle_{K1}\{\Phi\} / \langle\sigma_i\rangle_{K1} = (-1)\,[1 - 2d\,(R-d+1)(x^*)^2] \tag{2.25}$$

$x^* = \tanh \beta_1^* = e^{-2K_1}$

Once again Eq. (2.25) gives the answer to first non-trivial order. The magnetization is locally decreased and the effect is non-uniform. In fact, this decrease is largest midway between the two frustration. The minus sign of Eq. (2.25) arises from the fact that the string of A's (Fig. 10) we have chosen crosses the string of μ's in the dual case (Fig. 8).

(C) Flipped Bonds, Frustrations and Disorder Parameters

The results we have obtained in Section 2.c can also be understood in terms of an Ising model with flipped bonds.

One possible alternative realization of an Ising model with 2 frustrations is an Ising model with a dual string of flipped bonds connecting the two frustrated plaquettes along some path Γ (Fig. 4d). Any path is equally good; models with different paths differ only by a gauge transformation. Such a configuration of flipped bonds is actually an interface or domain wall, as discussed by Fisher and Ferdinand [17]. Our string tension is nothing more than the interfacial tension of the domain wall that appears in their work.

Kadanoff and Ceva have shown that there is a duality transformation connecting the normalized partition function in the presence of a dual string of flipped bonds and the correlation function in the dual system. This quantity, which for us is the partition function in the presence of frustrations, in their language is the correlation function of the disorder variables. Thus, frustrations have a close connection with disorder variables [7,18]. In fact, the "fermionic" character of the order and disorder variables (i.e., the fact that one picks up a factor of (-1) by moving disorder variables strings through spin variables) is just the path dependence of the gauge invariant correlation function, as discussed in Section I.

III. THE 3 DIMENSIONAL ISING SPIN GLASS

(A) Frustrations in 3 Dimensions

We want to discuss now frustrations in the 3 dimensional Ising model. Frustrations will also be introduced here in the same way as in the 2d case. There is an A variable at each link and a frustration variable at each plaquette defined by

$$\Phi_{ijkl} = A_{ij} A_{jk} A_{kl} A_{li} \tag{3.1}$$

where i,j,k,l are sites in the 3 dimensional cubic lattice which define the plaquette. Unlike the situation in 2 dimensions where the plaquettes are associated to dual sites, in 3 dimensions plaquettes are associated with dual links (see Fig. 10). So the frustration in 3d has a vector character. From its definition, it is clear that the Φ variables obey the constraint:

$$\Pi'_{\text{faces}}\Phi = 1 \tag{3.2}$$

where the product is taken over all faces composing a closed surface on the lattice. [Consider, for instance, an elementary cube of the lattice (Fig. 10). If we consider the product (3.2) on that surface, it is clear the A variable at each link of that cube occurs twice in the product. Since $A^2 = 1$, Eq. (3.2) is an identity.]

From the view point of the dual lattice (not the dual model), this constraint says that there should be an even number of dual frustrated links associated entering each dual site. Thus, the only allowed configurations of frustrations correspond to closed loops of dual links on the dual lattice; a result already pointed out by Toulouse [3].

(B) Duality

We start with a 3d Ising model with a fixed distribution of frustrations. In analogy with Eq. (2.4), we write for the partition function [15]

$$\frac{Z\{\Phi_{ijkl}\}}{Z\{\Phi=1\}} = \lim_{K_p\to\infty} \frac{\sum_{\{A_{ij}\}} \exp\{\sum_{\langle ij\rangle} K_l A_{ij} + \sum_{\langle ijkl\rangle} K_p A_{ij} A_{jk} A_{kl} A_{li} \Phi_{ijkl}\}}{\sum_{\{A_{ij}\}} \exp\{\sum_{\langle ij\rangle} K_l A_{ij} + \sum_{\langle ijkl\rangle} K_p A_{ij} A_{jk} A_{kl} A_{li}\}} \tag{3.3}$$

We define the excess free energy of the frustrated system in analogy with the 2d case. It is known that the partition function (3.3), for K_p finite, and $\Phi_{ijkl} = 1$ at all plaquettes, is self dual. Link interactions transform into plaquette interactions, and vice versa, through the duality relationship

$$\begin{aligned} e^{-2K_p^*} &= \tanh K_l \\ e^{-2K_l^*} &= \tanh K_p \end{aligned} \tag{3.4}$$

Let A_{ij} to be the gauge variable associated with the dual link i j (i and j are two neighboring sites in the dual lattice). The normalized partition function (3.3) (finite K_p) takes the form

$$\frac{Z\{\Phi_{ij}\}}{Z\{\Phi=1\}} = \frac{\sum_{\{a_{ij}\}} \left(\prod_{\langle ij\rangle} a_{ij}^{(1-\Phi_{ij})/2}\right) \exp\{K_l^* \sum_{\langle ij\rangle} a_{ij} + K_p^* \sum_{\langle ijkl\rangle} a_{ij} a_{jk} a_{kl} a_{li}\}}{\sum_{\{a_{ij}\}} \exp\{K_l^* \sum_{\langle ij\rangle} a_{ij} + K_p^* \sum_{\langle ijkl\rangle} a_{ij} a_{jk} a_{kl} a_{li}\}} \tag{3.5}$$

which is equal to the correlation function

$$Z\{\Phi_{ij}\}/Z\{\Phi_{ij}=1\} = \left\langle \prod_{\langle ij\rangle} a_{ij}^{(1-\Phi_{ij})/2} \right\rangle_{K_l^*, K_p^*} \tag{3.6}$$

The proof of (3.5) is entirely analogous to the proof of the 2d case. Again we are interested in the constrained situation $K_p \to \infty$ and (3.4) implies that $K_l^* \to 0$. Thus the averages (3.6) are taken in the pure gauge system described by the partition function

$$Z_{gauge} = \sum_{\{a_{ij}\}} \exp\{K_p^* \sum_{\langle ijkl\rangle} a_{ij} a_{jk} a_{kl} a_{li}\} \tag{3.7}$$

The Hamiltonian of (3.7) is gauge invariant [i.e. it is invariant under the transformations (1.2)-(1.3)]. Elitzur [19] has shown that such local symmetries are never broken, i.e. the expectation value of any gauge non-invariant quantity θ is identically zero for all values of the coupling constant i.e.

$$\langle\theta\rangle \equiv 0 \qquad \text{all } K_p^* \tag{3.8}$$

Therefore, one may ask in which case is the quantity in (3.6) gauge invariant. As we have already discussed in Section I.B only the product of a variables around any closed loop of links is gauge invariant. This is the way in which the constraint discussed in (3.2) is realized in the dual system. Notice that in contrast to the 2d situation the partition function for one frustrated plaquette is zero for all temperatures.

(C) Energetics of Frustrations

As discussed above, the simplest configuration of frustrations is a **closed tube of frustrated plaquettes** (Fig. 11).

The normalized partition function of this tube is equal to the expectation value of the product of the a variables along the loop Γ of dual links threading the frustration tube. This expectation value is a familar object in gauge theories: the Wilson loop integral [5].

The excess free energy for this tube is given by

$$\Delta F(K_I) = -(1/K_I) \log \langle \prod_{\Gamma} a \rangle_{K_p^*} \tag{3.9}$$

From high and low temperature expansions (in the dual model), it is known that the loop integral has the asymptotic behavior [5,14,18]

$$\langle \prod_{\Gamma} a \rangle_{K_p^*} \sim \begin{cases} \exp\{-\alpha A\} & K_p^* < K_c^* \\ \exp\{-\beta L\} & K_p^* > K_c^* \end{cases} \tag{3.10}$$

where A is the minimal area spanned by the loop Γ and L is the perimeter of that loop. K_c^* is the critical coupling of the dual model and is the dual of the critical coupling K_c of the 3d Ising model. The coefficients α and β are temperature dependent. In the original Ising model, this implies that the excess free energy of a closed tube of frustrations behaves like

$$\Delta F(k_l) = \begin{cases} (\alpha/K_l)\ A & K_L > K_c \text{ (low spin glass temperatures)} \\ (\beta/K_l)\ L & K_l < K_c \text{ (high spin glass temperatures)} \end{cases} \tag{3.11}$$

Let us now look at the interaction between two tubes of frustrations in various relative orientations.

Consider first two face-to-face tubes (Fig. 12). In order to compute the excess free energy of that configuration of frustrations at low temperatures, it is useful to go to the dual system and consider there the expectation value of the two dual loops of a variables at high temperature. The leading diagram in the high temperature expansion of the dual system is that one which covers the minimal area surface spanned by the loops. This is the just dual analog of the statement made by Toulouse [3] and Kirpatrick [11] that the ground state configurations correspond to covering surfaces of minimum area. For one loop, we then obtain

$$(\tanh K_p^*)^A = e^{-2K_l A} \tag{3.12}$$

which is the area law quoted above.

For two loops, the character of the minimal surface changes with the distance R between them. The two situations are shown in Fig. 13(b,c). If d is the linear dimension of the loop, we get, to leading order,

$$\Delta F_{K_L}(R,d) = 8dR \quad R \ll d \tag{3.13a}$$

$$\Delta K_{k_L}(R,d) = 4d^2 \quad R \gg d \tag{3.13b}$$

at low spin-glass temperatures ($K_l \gg K_c$). At high spin temperatures, the excess free energy can be evaluated directly through the high temperature expansion in the Ising spin glass model. The result is to (leading order)

$$\Delta F_{K_l}(R_1 d) = 8d, \ K_l \ll K_c \tag{3.14}$$

Eq. (3.13a) shows that at low temperatures, R<<d, there is a linear potential between the

loops whose strength is proportional to the perimeter d of the loops, a result suggested by the 2d results. In contrast with the 2d case, though, this potential saturates at a distance R~d and, for R>>d, has only a weak R dependence. Thus, loops of frustrations tend to bind but they are not confined. There is also an orientation effect in the interaction between tubes. For two loops oriented as in Fig. 13(d), the minimal surface does not change character so there is no strong distance dependence. In analogy with 2d case, at high (spin-glass) temperatures, the R dependence is weak for all distances.

In the 3d case, it is also possible to compute gauge invariant correlation functions using duality transformations, as we did in the 2-D case. The proof and results are given in Appendix A.

IV. THE 2D XY SPIN GLASS

(A) Gauge Symmetries in Random XY Models

Up to now, we have discussed random Ising spin systems. We can extend our treatment to XY systems for which the degrees of freedom are fixed-length 2 dimensional planar rotors $S = (\cos\theta, \sin\theta)$ sitting at the sites of the lattice.

The standard nearest neighbor ferromagnetic coupling is usually written as

$$K_l(\vec{S}_i, \vec{S}_j) \equiv K_l \vec{S}_i \cdot \vec{S}_j = K_l \cos(\theta_i - \theta_j) \tag{4.1}$$

where i,j are nearest neighbor lattice sites and K_l is the coupling constant for this link. This interaction favors configurations with neighboring spin parallel to each other.

We can introduce disorder in the system by adjusting the interaction to favor configurations with neighboring spins tilted by an angle ψ_{ij} at each link (i,j). The form of the interaction is now

$$K_l \cos(\theta_i - \theta_j - \psi_{ij}) \tag{4.2}$$

In particular $\psi_{ij} = \pi$ corresponds to flipping the sign of the interaction.

Define a link gauge degree of freedom U_{ij} such that [5]

$$U_{ij} = \exp\{i\psi_{ij}\} \tag{4.3}$$

Then Eq. (4.2) can be rewritten as [20]

$$(K_l/2)\,[S_i U_{ij}^* S_j^* + \text{h. c.}] \tag{4.4}$$

with $S_i \equiv e^{i\theta_i}$.

The partition function in a fixed configuration of gauge degrees of freedom $\{U_{ij}\}$ is given by

$$Z\{U_{ij}\} = \int_{\{Si\}} \exp\{(K_1/2) \sum_{\langle ij\rangle} [S_i U_{ij}^* S_j^* + h.c.]\} \tag{4.5}$$

where $\int_{\{S_i\}}$ means a normalized integration over all the angles between $-\pi$ to π.

Define a local gauge transformation $G\{Vi\}$, with $V_i = \exp\{i\chi_i\}$, such that the spin and link degrees of freedom transform under $G\{V_i\}$ like

$$S_i \rightarrow V_i S_i \tag{4.6}$$

$$U_{ij} \rightarrow V_i U_{ij} V_j^*$$

In compact notation

$$G(V)\{S_i,U_{ij}\} = \{V_iS_i, V_i U_{ij} V_j^*\} \tag{4.7}$$

Hence, each spin is rotated by χ_i and each link-angle by the difference $\chi_i-\chi_j$. Again the key point is that the interaction (4.3) is invariant under the gauge transformation (4.6).

With the above definitions all the remarks already made in Section I apply to XY systems with almost trivial modifications. In particular, the partition function (4.3) is gauge invariant and we only need the gauge invariant features of the gauge configuration.

Let us define a frustration angle $2\pi\Phi_{ijkl}$ at plaquette ijkl such that

$$e^{i2\pi\Phi_{ijkl}} = U_{ij} U_{jk} U_{kl} U_{li} \tag{4.8}$$

around that plaquette.

Thus, from Eq. (4.3), the frustration angle may be written as

$$2\pi\Phi_{ijkl} = \psi_{ij} + \psi_{jk} + \psi_{kl} + \psi_{li} \pmod{2\pi} \tag{4.9}$$

From the periodicity of the interaction (4.2) we arrive to the conclusion that only fractional values of Φ are meaningful. For instance, if we want to reverse the sign at the link ij, (i.e. $\psi_{ij}=\pi$, $\psi_{lk}=0$ otherwise) is equivalent to set Φ_{ijkl} = ½+integer for all the plaquettes which contain the reversed link as shown by Villain [8].

Thus, the random XY model is a frozen configuration of frustrations Φ_{ijkl}, derived from the angles ψ_{ij}, has a Hamiltonian given by

$$H = K_1 \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j - \psi_{ij}) \tag{4.10}$$

Instead of writing Eq. (4.10) as a constrained Hamiltonian, as we did in the Ising case, it will prove to be more convenient to deal with a Hamiltonian depending explicitly on the angles ψ_{ij}.

(B) The 2d XY Spin Glass

It is convenient to change our notation. A link whose ends are the sites i and j can be equivalently described by one of the sites (i) and a direction (μ). Thus we write [22]

$$\psi_{ij} \equiv \psi_\mu(i) \tag{4.11}$$

A plaquette is defined by a corner (i) and two directions μ and ν, i.e. (i,$\mu\nu$). In particular, the frustration field can be written as

$$2\pi\Phi_{\mu\nu}(i) = \triangle_\mu \psi_\nu(i) - \triangle_\nu \psi_\mu(i) \tag{4.12}$$

where $\triangle_\mu$ is the finite difference operator

$$\triangle_\mu \chi(i) = \chi(i) - \chi(i - \hat{e}_\mu) \tag{4.13}$$

where $\hat{e}_\mu$ is the unit vector pointing in the μ direction. In 2 dimensions a plaquette (i,$\mu\nu$) is uniquely associated to the dual site i at its center and we often consider the scalar frustration $\Phi(i)$ residing there

$$\Phi(i) = \tfrac{1}{2}\, \varepsilon_{\mu\nu}\, \Phi_{\mu\nu}(i) \tag{4.14}$$

where $\varepsilon_{\mu\nu}$ is the 2d Levi-Civita tensor.

In this notation, the resemblance between the frustration field $\Phi_{\mu\nu}(i)$ and the electromagnetic field tensor is evident (Eq. 4.12).

Consider the partition function of the 2 dimensional random XY model in an arbitrary field of frustrations (Eq. 4.10)

$$Z\{\Phi(i)\} = \int D\theta \exp\{\sum_{(i,\mu)} K_l \cos(\Delta_\mu \theta(i) - \psi_\mu(i))\} \tag{4.15}$$

Following references 2 & 3, we consider the Villain approximation of (4.15), i.e.

$$e^{\beta\cos\theta} \simeq e^{\beta} \sum_{l} \exp\{-\beta/2\, (\theta-2\pi l)^2\} \tag{4.16}$$

Performing a Fourier expansion at each link, we obtain a system of integer valued variables $l_\mu(i)$ residing on links. The partition function now reads

$$Z\{\Phi(i)\} = \sum_{\{l_\mu(i)\}} \exp\{-K_l/2\} \sum_{\{i,\mu\}} l_\mu^2(i)\} \exp\{-i \sum_{\{i,\mu\}} \psi_\mu(i)\, l_\mu(i)\} \prod_i \delta(\Delta_\mu l_\mu(i)) \tag{4.17}$$

By solving the constraint

$$l_\mu(i) = \varepsilon_{\mu\nu}\, \Delta_\nu\, n(i) \tag{4.18}$$

we can map the (normalized) partition function (4.17) into a correlation function of the surface roughening model [24] whose partition function is given by

$$Z_{SR} = \sum_{\{n_\mu(i)\}} \exp\{-K_l/2\} \sum_{(i,\mu)} (\Delta_\mu n(i))^2\} \tag{4.19}$$

Thus, we obtain the result

$$\frac{Z\{\Phi(i)\}}{Z\{\Phi(i)=i\}} = \left\langle \prod_i \exp\{-2\pi i\, n(i)\,\Phi(i)\} \right\rangle_{S.R.} \tag{4.20}$$

If we perform a global shift by m of all the n(i) variables the partition function (4.19) is left unchanged, but the expectation value (4.20) picks up a phase $e^{2\pi i m \Sigma \Phi(i)}$ Hence, for arbitrary boundary conditions, the expectation value (4.20) vanishes identically unless the frustration system is "neutral" i.e.

$$\sum_i \Phi(i) = 0 \quad \text{(mod. integer)} \tag{4.21}$$

This result is analogous to the vanishing of $\langle\sigma\rangle$ in the Ising model due to the global symmetry $\sigma \to -\sigma$. In both models fixing boundary conditions at infinity allows symmetry breaking quantities such as Eq. (4.20) to develop non-vanishing expectation values.

By using the Poisson summation formula [23], we can write in terms of the coulomb gas picture

$$Z\{\Phi(\underset{\sim}{i})\} = \sum_{\{m(\underset{\sim}{i})\}} \int_{-\infty}^{+\infty} \mathcal{D}\varphi \; \exp\left\{2\pi i \sum_{\underset{\sim}{i}} \left(\Phi(\underset{\sim}{i}) + m(\underset{\sim}{i})\right)\varphi(\underset{\sim}{i})\right\} \exp\left\{-\frac{1}{2K_2} \sum_{\underset{\sim}{i},\mu} \left(\Delta_\mu \varphi(\underset{\sim}{i})\right)^2\right\} \tag{4.22}$$

The evaluation of the Gaussian path integral then gives the result

$$Z\{\Phi(\underset{\sim}{i})\} = Z_{SW} \sum_{\{m(\underset{\sim}{i})\}}{}' \exp\left\{\pi K_2 \sum_{\underset{\sim}{i}\underset{\sim}{j}} \left(m(\underset{\sim}{i}) + \Phi(\underset{\sim}{i})\right) D(\underset{\sim}{i}-\underset{\sim}{j}) \left(m(\underset{\sim}{j}) + \Phi(\underset{\sim}{j})\right)\right\} \tag{4.23.a}$$

where Z_{SW} is a spin wave partition function and the summation is restricted to strictly neutral configurations, i.e.

$$\sum_{i} (m(i) + \Phi(i)) = 0 \tag{4.23.b}$$

The propagator $D(\underset{\sim}{i}-\underset{\sim}{j})$ is the lattice coulomb Green's function and in 2d has the asymptotic behavior

$$D(\underset{\sim}{i}-\underset{\sim}{j}) \sim \log |\underset{\sim}{i} - \underset{\sim}{j}| + \pi/2 \tag{4.24}$$

where $\pi^2/2$ provides an effective chemical potential for the vortices $m(\underset{\sim}{i})$.

Therefore frustrations in the XY model map into fractionally charged impurities in the coulomb gas.

(C) Energetics of Frustrations

Consider two frustrations (charges) located at dual sites i and j with strength $\Phi_{\underset{\sim}{i}} = q$ and $\Phi_{\underset{\sim}{j}} = -q$.

At low temperatures, the 2d Coulomb gas is a dielectric [25] (a dilute gas of dipoles). Thus, the two frustrations interact with their original logarithmic Coulomb interaction renormalized by dielectric constant given by [26]

$$\varepsilon = 1 + 8\pi/K_\ell \exp\{-2K_\ell\mu\},\ K_\ell >> 1 \tag{4.25}$$

For the single frustration, the excess of free energy is logarithmically divergent and excludes its existence at low temperatures.

On the other hand, at high temperatures, the 2d Coulomb gas is a plasma and we expect Debye screening to take place. Therefore, the impurities interact via a short range screened Yukawa potential and the excess free energy of an isolated frustration is now finite. The excess free energy of a single frustration at high temperatures is easily computable in the low temperature expansion of the dual (surface roughening) model.

(D) Gauge Invariant Correlation Functions

As in the Ising model, the spin-spin correlation function is distorted by the presence of frustrations. We define the gauge invariant correlation function for this model $\langle S_i \cdot S_j \rangle_{\Gamma_{ij}}$ as

$$\langle S_i \cdot S_j \rangle_{\Gamma_{ij}} = \langle S_i (\Pi\, U_{lk}^*)\, S_j^* + \text{h.c.} \rangle = \langle \cos(\theta_i - \theta_j - \sum_{\Gamma(i,j)} \psi_{lk}) \rangle \tag{4.26}$$

where $\sum_{\Gamma(i,j)}$ means the summation of the ψ variables along all links on the path $\Gamma(ij)$ between sites i and j.

Just as in the Ising model, this correlation function is path dependent. Consider two different paths $\Gamma_1(ij)$ and $\Gamma_2(ij)$ such that inside the area enclosed between them there are frustrations of total strength $2\pi Q$. Then

$$\sum_{\Gamma_1(i,j)} \psi_{lk} = \sum_{\Gamma_2(i,j)} \psi_{lk} + 2\pi Q \tag{4.27}$$

Thus, the phase of the cosine in (4.26) is shifted by $2\pi Q$. When $Q = \frac{1}{2}$, this result gives the usual (-1) factor that we obtained in the 2d Ising model.

For the pure XY system, at low temperatures, there is strong evidence that the two point correlation function falls off with a power law [23]. In order to evaluate the effect of frustrations on the correlation function, we go to the Coulomb gas picture which gives (see Appendix B).

$$\frac{\left\langle \cos\left(\theta_i - \theta_j - \sum_{\Gamma(i,j)} \varphi_{\ell k}\right)\right\rangle_{K_\ell}\{\Phi(\underset{\sim}{j})\}}{\left\langle \cos(\theta_i - \theta_j)\right\rangle_{K_\ell}} = \frac{\left\langle \cos\left(\sum_{\underset{\sim}{j}} m(\underset{\sim}{j})\left[\theta(\underset{\sim}{j}-i) - \theta(\underset{\sim}{j}-j)\right]_{\Gamma(i,j)}\right)\right\rangle_{c.g.}\{\Phi(\underset{\sim}{j})\}}{\left\langle \cos\left(\sum_{\underset{\sim}{j}} m(\underset{\sim}{j})\left[\theta(\underset{\sim}{j}-i) - \theta(\underset{\sim}{j}-j)\right]\right)\right\rangle_{c.g.}} \tag{4.28}$$

where the left hand side averages are taken in the XY model and the right hand side in the Coulomb gas.

The angle $\theta(\underset{\sim}{j}-j)$ is the polar angle of the vector $\underset{\sim}{j}-j$, where j and $\underset{\sim}{j}$ are the positions of the correlated spin and the frustration and $[\theta(\underset{\sim}{j}-i) - \theta(\underset{\sim}{j}-j)]_\Gamma$ is the angular paralax of the frustration (or vortex) as seen from the ends of the path Γ [23,27]. The rules for computing these paralaxes are given in Appendix B. The fact that the gauge invariant correlation function is path dependent resides entirely in the way the paralaxes are computed. For instance, if a frustration Q lies to the right of a path and to the left of one another, the argument of Eq. (4.28) differs by $2\pi Q$ between both paths. The reason is that the paralax is spanned counterclockwise in the first case and clockwise in the second (see Fig. 14).

For instance, let us compute the spin correlation function in the presence of two frustrations q and -q (Fig. 14). At very low temperatures, the leading term (all m=0) gives the result

$$\frac{\left\langle \cos\left(\theta_R - \theta_0 - \sum_{\Gamma(R,0)} \varphi_{\ell k}\right)\right\rangle_{K_\ell}\{\Phi(\tfrac{L}{2})=q,\ \Phi(-\tfrac{L}{2})=-q\}}{\left\langle \cos(\theta_R - \theta_0)\right\rangle_{K_\ell}} = \cos(4q\omega) \tag{4.29}$$

Thus, the correlation function has the asymptotic behavior (R large)

$$\frac{\langle \cos(\theta_R - \theta_0 - \sum_{R,O} \text{[illegible]}) \rangle}{\langle \cos(\theta_R - \theta_0) \rangle_{K_\ell}} = \frac{\text{const}}{R^{\left(\frac{1}{2\pi K_\ell}\right)}} \cos(4q\omega) \tag{4.30}$$

For R>>d, we can approximate $\omega \simeq \pi/2 - d/R$. If we confine ourselves to the case $q = ½$ (flipped bonds) then Eq. (4.29) becomes

$$\langle \cos(\theta_R - \theta_O - \Gamma_{R,O} \psi) \rangle_{K_\ell} \{q,-q\} = (-1)\ \text{const.}\ [1 - ½(4q\, d/R)^2] / R^{(2\pi K_\ell)^{-1}} \tag{4.31}$$

Again, as in the Ising case, we pick up a (-1) in front of the gauge invariant correlation function signaling the existence of frustrations in the system.

Finally, it is interesting to see what the XY model with all plaquettes frustrated looks like. In the Coulomb gas representation, this means studying system with charged impurities at every site. One system with well behaved energetics is a "salt-crystal" with an impurity charge q on one sublattice and -q one the other (Fig. 15). The ground state (zero temperature) configuration has no vortices present (all m=0). When q = 1/2, though, there is another state degenerate with this one: m = 1 at all dual sites for which q = - 1/2 and m = -1 when q = 1/2. This has the effect of shifting one sublattice into the other. Villain has studied this model with q = 1/2 (the "odd model") and has also found this double degeneracy.

V. THE 3 DIMENSIONAL XY SPIN GLASS

(A) Frustrations in the 3d XY model

The definition of the 3 dimensional XY spin glass follows naturally from its definition in 2 dimensions. The different dimensionality, however, changes the structure of the frustrations as well as the properties of the dual models.

To begin with, let us consider the frustration network for this model. As in the 3d Ising model, frustrations arrange themselves into spatial networks. The reason is that the frustration field in 3 dimension is a pseudovector, as follows from the definition (4.12)

$$2\pi \Phi_{\mu\nu}(i) = \Delta_\mu \psi_\nu(i) - \Delta_\nu \psi_\mu(i) \tag{4.12}$$

This relationship also shows that the frustration field obeys a constraint, which is analogous to the one we discussed in section III. Eq. (4.12) makes $\Phi_{\mu\nu}(i)$ the circulation of ψ_μ around the plaquette. In 3d we can describe this circulation as a pseudovector, i.e., the flux of the field strength of the gauge variable ψ_μ through the plaquette. Therefore, we can define a pseudovector ψ_α, which lives on the dual link piercing the plaquette, and describes the direction of the frustration flux across the plaquette, as

$$\varphi_\alpha(\underset{\sim}{i}) = \tfrac{1}{2}\varepsilon_{\alpha\mu\nu}\Phi_{\mu\nu}(i) \tag{5.1}$$

where $(\underset{\sim}{i},\alpha)$ is the link dual to the plaquette $(i,\mu\nu)$ [27]. From Eq. (4.12) we see that $\varphi_\alpha(\underset{\sim}{i})$ is just the curl of $\psi_\mu(i)$

$$2\pi\varphi_\alpha(\underset{\sim}{i}) = \varepsilon_{\alpha\mu\nu}\, \Delta_\mu\, \psi_\nu(i) \tag{5.2}$$

This expression has the same form as the magnetic field of electrodynamics. Notice that since $\varphi_\alpha(\underset{\sim}{i})$ is a curl it is divergence free

$$\Delta_\alpha \varphi_\alpha(\underset{\sim}{i}) = 0 \quad (\text{mod. integer}) \tag{5.3}$$

at each dual site $\underset{\sim}{i}$. This is the analog of the constraint we found in the 3d Ising model. If we now interpret the 3d XY model as a lattice version of the Ginzburg-Landau theory for a superconductor, we can regard these structures as tubes of frozen fractional magnetic flux.

Because of the continuous nature of the degrees of freedom of the XY model, many other types of configurations are also possible. In particular, since the frustration flux Φ_μ is a continuous variable, the flux can spread out with the result that any configuration of magnetostatic fields is possible for the frustrations themselves. For instance, we can construct configurations for which $\Phi_\mu \sim 1/r^2$, which is analogous to a magnetic monopole and its associated strings.

Let us now perform the duality transformations to partition function of the random 3d XY model, which is given by

$$Z\{\Psi_\mu(i)\} = \int_0^{2\pi} \mathcal{D}\theta \; \exp\left\{ k_1 \sum_{(i,\mu)} \cos(\Delta_\mu \theta(i) - \Psi_\mu(i) \right\} \tag{5.4}$$

The procedure is essentially analogous to that we have already employed in the 2d XY case.

The first step is again a Fourier expansion of (5.4) per link. After integrating out the angular degrees of freedom $\theta(i)$ we are left with the constrained system

$$Z\{\Phi_{\mu\nu}(i)\} = \sum_{\{l_\mu(i)\}} \exp\left\{-\frac{1}{2k_1}\sum_{(i,\mu)} l_\mu^2(i)\right\} \exp\left\{-i\sum_{(i,\mu)} l_\mu(i)\Psi_\mu(i)\right\} \prod_i \delta(\Delta_\mu l_\mu(i)) \tag{5.5}$$

which is the same as (4.17). The differences, however, become apparent as soon as one solves the constraint condition. In this case, we obtain

$$l_\mu(i) = \epsilon_{\mu\nu\lambda} \, \Delta_\lambda \, n_\lambda(\tilde{i}) \tag{5.6}$$

where the integer valued variables $n_\lambda(\tilde{i})$ reside on the links of the dual lattice. Therefore, after solving the constraint of Eq. (5.6), the normalized partition function (5.5) can be written as an expectation value of the dual system, which is a gauge theory with integer valued degrees of freedom $n_\lambda(\tilde{i})$. Its partition function is given by

$$Z_{gauge} = \sum_{\{\eta_\mu(i)\}} \exp\{-(1/2K_l) \sum_{(i,\mu\nu)} (\Delta_\mu \eta_\nu(i) - \Delta_\nu \eta_\mu(i))^2\} \tag{5.7}$$

Then [28]

$$\frac{Z\{\varphi_\mu(i)\}}{Z\{\varphi_\mu(i)=0\}} = \left\langle \exp\{2\pi i \sum_{(i,\mu)} \eta_\mu(i)\, \varphi_\mu(i)\} \right\rangle_{gauge} \tag{5.8}$$

where the relationship between $\varphi_\mu(i)$ is given by (5.1). We should note that the partition function (5.7) is invariant under the local gauge transformation

$$\eta_\mu(i) \to \eta_\mu(i) + \Delta S(i) \tag{5.9}$$

Thus, Eq. (5.9) picks up a phase factor $e^{-2\pi i \sum_i S(i) \Delta_\mu \varphi_\mu(i)}$ under the transformation (5.9), and therefore it is not a gauge invariant quantity

Hence, we can write

$$\left\langle \exp\{2\pi i \sum_{(i,\mu)} \eta_\mu(i)\, \varphi_\mu(i)\} \right\rangle_{gauge} = 0 \tag{5.10}$$

unless the frustration field obeys the constaint (5.3). We should also note there is a fundamental difference between the constaint (5.3) and the "neutrality" condition that we discussed in the section dealing with the 2d XY model. While the global symmetry involved in (4.20) can be broken by specifying suitable boundary conditions, the local symmetry (5.9) can never be broken. Thus Eq. (5.10) is an identity which is valid for all values of the coupling K_l regardless of boundary conditions.

The partition function (5.8) can also be written in terms of the topological excitations of the 3d XY model (quantized vortex strings) interacting via Coulomb interactions. Applying the Poisson summation formula to (5.7) we obtain

$$Z\{\varphi_\mu(i)\} = \sum_{\{n_\mu(i)\}} \int_{-\infty}^{+\infty} \mathcal{D}\theta_\mu \; \exp\{-\frac{1}{2K_l} \sum_{(i,\mu\nu)} (\Delta_\mu \theta_\nu(i) - \Delta_\nu \theta_\mu(i))^2\} \; \exp\{\sum_{(i,\mu)} 2\pi i\, \theta_\mu(i) (n_\mu(i) + \varphi_\mu(i))\} \tag{5.11}$$

where gauge invariance once again demands that (5.11) vanish unless the following constraint is satisified

$$\Delta_\mu\,(\varphi_\mu(\underset{\sim}{i}) + m_\mu(\underset{\sim}{i})) = 0 \tag{5.12}$$

Performing the integrals in Eq. (5.11), we obtain [29]

$$Z[\varphi_\mu(\underset{\sim}{i})] = Z_{SW} \sum_{\{m_\mu(\underset{\sim}{i})\}}' \exp\Big\{-\frac{\pi\beta}{2}\sum_{\underset{\sim}{i},\underset{\sim}{j}} (\varphi_\mu(\underset{\sim}{i}) + m_\mu(\underset{\sim}{i}))\, D(\underset{\sim}{i}-\underset{\sim}{j})\,(\varphi_\mu(\underset{\sim}{j}) + m_\mu(\underset{\sim}{j}))\Big\} \tag{5.13}$$

where Z_{SW} is a spin wave partition function and the summation is restricted to those configurations which satisify (5.12), and in 3 dimensions the Coulomb lattice propagator $D(\underset{\sim}{i}-\underset{\sim}{j})$ has the asymptotic form

$$D(\underset{\sim}{i}-\underset{\sim}{j}) \sim -\big(1/|\underset{\sim}{i}-\underset{\sim}{j}|\big) + \text{const.} \tag{5.14}$$

In the absence of frustrations, the constraint (5.12) requires the topological excitations of the 3d XY model to form closed loops. When frustrations are present and $\Delta_\mu\varphi_\mu(\underset{\sim}{i})$ = integer, at a point one can have a monopole at that point and a vortex string that can begin (or terminate) there. This situation has already been discussed by Einhorn and Savit [21].

(B) Energetics of Frustrations

(i) Low Temperatures

Consider first the excess free energy associated with a closed tube of frustration flux, analogous to the configuration we have already discussed in the 3d Ising model.

At low temperatures, the representation (5.11) is most convenient. The leading contribution to the free energy (all $m_\lambda(\underset{\sim}{i})=0$) is given by Eq. 5-13 and 5.14 as

$$\Delta F\{\varphi_\lambda(\underset{\sim}{i})\} = -\frac{\pi}{2}\sum_{\underset{\sim}{i},\underset{\sim}{j},\lambda} \varphi_\lambda(\underset{\sim}{i})\,\frac{1}{|\underset{\sim}{i}-\underset{\sim}{j}|}\,\varphi_\lambda(\underset{\sim}{j}) \tag{5.15}$$

For a tube of linear dimension R, a simple electrostatic-like calculation that gives

$$\Delta F = \text{const.}\, Q^2 R \log R + O(R) \tag{5.16}$$

where Q is the frustration flux in the tube.

The result (5.16) has a weaker dependence in R than the area law we found in th 3d Ising model. The RlogR behavior arises from the fact that XY rotators can always relax continuously around a frustration tube.

(ii) **High Temperatures**

A convenient representation to calculate the high temperature properties of the XY model is the constrained system described by the partition function

$$\frac{Z\{\psi_\mu(i)\}}{Z\{\psi_\mu(i)=0\}} = \frac{\sum_{\{\ell_\mu(i)\}} \exp\{-i\sum_{(i,\mu)} \psi_\mu(i)\ell_\mu(i)\} \exp\{-\frac{1}{2K_\ell}\sum_{(i,\mu)} \ell_\mu^2(i)\} \prod_i \delta(\Delta_\mu \ell_\mu(i))}{\sum_{\{\ell_\mu(i)\}} \exp\{-\frac{1}{2K_\ell}\sum_{(i,\mu)} \ell_\mu^2(i)\} \prod_i \delta(\Delta_\mu \ell_\mu(i))} \tag{5.17}$$

The integer valued variable $l_\mu(i)$ lives on the (i,μ) link of the original lattice and $\psi_\mu(i)$ is the gauge field angle. Therefore, to study the high temperature (K_ℓ small) behavior of the XY-Villain model in the presence of frustrations is equivalent to studying the constrained model (5.17) at low temperature ($1/K_\ell$ large).

The lowest energy excitations of this model are elementary plaquettes with l_μ=1 or -1 around the plaquette. The leading term in the low temperature expansion of (5.15) gives

$$\frac{Z\{\psi_\mu\}}{Z\{\psi_\mu=0\}} = 1 - 2\exp\left(-\frac{2}{K_\ell}\right) \sum_{(i,\mu\nu)} \left[1-\cos\left(2\pi\bar{\Phi}_{\mu\nu}(i)\right)\right] \tag{5.18}$$

where $\Phi_{\mu\nu}(i)$ is defined in Eq. (4.12). Let us look at the configurations of frustrations we examined at low temperatures. Consider a closed tube of frustration flux of strength Q and perimeter length L. For this case (5.18) gives an excess free energy

$$\Delta F = \frac{2}{K_\ell} (1 - \cos 2\pi Q) L \quad \exp\left[-\frac{2}{K_\ell}\right] \tag{5.19}$$

if $L\ e^{-2/K_l} << 1$.

As in the 3d Ising model, we get a perimeter law.

Concluding Remarks

Let us summarize the results of the above sections.

In two dimensions, at low temperatures, Ising frustrations have a linear interaction energy; XY frustrations, logarithmic. At high temperatures large spin fluctuations "screen" the frustrations and we have exponentially damped interactions. In both models frustrations decrease the magnitude of the spin-spin correlations, as expected.

In three dimensions the constraints require the frustrations to form divergenceless configurations: closed tubes in the Ising model and more general spread flux configurations in the XY model. At low temperatures we have an area law (L^2) for an Ising tube, L logL for the XY tube. At high temperatures fluctuations give a perimeter law (L) for both cases.

We would like to stress once again the importance of studying gauge invariant quantities. In particular, the gauge invariant correlation function emerges naturally as the correlation function to be studied when relevant disorder is present in the system.

Finally we observe that frustrations can be regarded as fractional topological excitations or merons of each model. In two dimensions a single frustration is a disorder variable in the sense that it breaks the symmetry of the dual model.

In three dimensions the situation is somewhat different due to the existence of constraints on the possible configurations of frustrations. In any case, it is always possible to construct a frustration network which behaves as a disorder variable in the sense that it has a non-vanishing expectation value in the disordered phase.

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APPENDIX A: DUALITY RELATIONS FOR THE GAUGE INVARIANT SPIN-SPIN CORRELATION FUNCTIONS IN THE PRESENCE OF FRUSTRATIONS: ISING MODEL (d = 2,3)

The gauge invariant spin-spin correlation function $\langle \sigma_i (\prod_{\Gamma(i,j)} A) \sigma_j \rangle$, in the $\sigma = 1$ gauge, can be written as

$$\langle \sigma_i (\prod_{\Gamma(i,J)} A) \sigma_J \rangle = \lim_{k_p \to \infty} \frac{\sum_{\{A\}} (\prod_{\Gamma(i,J)} A) \exp\left\{ k_1 \sum_{\text{links}} A + k_p \sum_{\text{plaquettes}} AAAA\, \Phi_i \right\}}{\sum_{\{A\}} \exp\left\{ k_1 \sum_{\text{links}} A + \sum_{\text{plaquettes}} AAAA\, \Phi_i \right\}} .$$

(A.1)

We want to derive a duality relation for (2.21) - (3.9). The first step is to bring all the A variables into the exponentials. The following identify is useful for that purpose.

$$A\, e^{\beta A} = i\, e^{(\beta - i\frac{\pi}{2})A}$$

Let us define a *shifted link coupling* $K_\ell(\Gamma)$ along the path Γ.

$$k_\ell(\Gamma_{iJ}) \equiv \begin{cases} k_\ell & ; \quad \ell \in \Gamma_{iJ} \\ k_\ell - i\pi/2 & ; \quad \ell \notin \Gamma_{iJ} \end{cases} \tag{A.2}$$

$$k_p(\Phi_i) \equiv k_p\, \Phi_i \tag{A.3}$$

Define R to be the distance (= number of links) between sites i and j: $R = |i - j|$. With the definitions given above, the gauge invariant correlation function reads (for finite K_p):

$$\langle \sigma_i (\prod_{\Gamma(i,J)} A) \sigma_J \rangle = \frac{\sum_{\{A\}} i^R \exp\left\{ \sum_{\text{links}} k_\ell(\Gamma_{iJ}) A + \sum_{\text{plaquettes}} k_p(\Phi_i)\, AAAA \right\}}{\sum_{\{A\}} \exp\left\{ k_\ell \sum_{\text{links}} A + \sum_{\text{plaquettes}} k_p(\Phi_i)\, AAAA \right\}}$$

(A.4)

which turns out to be equal to

$$\langle \sigma_i (\prod_\Gamma A) \sigma_j \rangle = i^R \frac{\sum_{\{A\}} \left[\prod_{\text{links}} \cosh K_\ell(\Gamma_j) \left(1 + \tanh K_\ell(\Gamma) A\right)\right] \cdot \left[\prod_{\text{plaquettes}} \cosh K_p(\Phi) \left(1 + AAAA \tanh K_p(\Phi)\right)\right]}{\sum_{\{A\}} \left[\prod_\ell \cosh K_\ell \left(1 + \tanh K_\ell A\right)\right] \cdot \left[\prod_p \cosh K_p(\Phi) \left(1 + AAAA \tanh K_p(\Phi)\right)\right]} \tag{A.5}$$

A.1 Two Dimensions

In 2-D the duality relation takes the form (see Eqs. (2.5) - (2.10))

$$\langle \sigma_i (\prod_\Gamma A) \sigma_J \rangle = \frac{i^R \left(\prod_\ell \cosh k_\ell(\Gamma)\right)\left(\prod_p \cosh k_p(\phi_{\underset{\sim}{i}})\right) \sum_{\{s_{\underset{\sim}{i}}\}} \exp\left\{\sum_{\langle ij \rangle} K_\ell^*(\Gamma)(s_{\underset{\sim}{i}} s_{\underset{\sim}{j}} - 1) + \sum_{\underset{\sim}{i}} H^*(\Phi_{\underset{\sim}{i}})(s_{\underset{\sim}{i}} - 1)\right\}}{\left(\prod_\ell \cosh k_\ell\right)\left(\prod_p \cosh k_p(\Phi_{\underset{\sim}{i}})\right) \sum_{\{s_i\}} \exp\left\{\sum_{\langle ij \rangle} k_\ell^*(s_{\underset{\sim}{i}} s_{\underset{\sim}{j}} - 1) + \sum_i H^*(\phi_{\underset{\sim}{i}})(s_{\underset{\sim}{i}} - 1)\right\}} \tag{A.6}$$

where $\{S_{\underset{\sim}{i}}\}$ are Ising spins residing at the sites $\underset{\sim}{i}$ of the dual square lattice. $K_\ell^*(\Gamma)$ and $H^*(\Phi_{\underset{\sim}{i}})$ are defined as

$$\begin{aligned} k_\ell^*(\Gamma) &= -\tfrac{1}{2} \ln \tanh k_\ell(\Gamma) \\ H^*(\Phi) &= -\tfrac{1}{2} \ln \tanh k_p(\Phi) \end{aligned} \tag{A.7}$$

Remember that K_ℓ^* and K_ℓ follow the same duality relationship. Since

$$i \cosh\left(\beta - \frac{i\pi}{2}\right) e^{\frac{1}{2} \ln \tanh\left(\beta - \frac{i\pi}{2}\right)} = \cosh \beta \, e^{\frac{1}{2} \ln \tanh \beta} \tag{A.8}$$

is an exact identity, (A.6) can be written as

$$\langle \sigma_i (\prod_\Gamma A) \sigma_j \rangle \equiv \frac{\sum_{\{s_i\}} \left[\prod_i s_i^{(1-\xi_i)/2} \right] \exp\left\{ \sum_{\langle i,j \rangle} K_\ell^*(\Gamma)\, s_i s_j + \sum_i H^* s_i \right\}}{\sum_{\{s_i\}} \left[\prod_i s_i^{(1-\xi_i)/2} \right] \exp\left\{ \sum_{\langle i,j \rangle} K_\ell^*\, s_i s_j + \sum_i H^* s_i \right\}} \tag{A.9}$$

The duality relations (A.7) together with the fact that

$$\tanh\left(\beta - \frac{i\pi}{2}\right) = (\tanh\beta)^{-1} \tag{A.10}$$

lead us to the conclusion that the link dual couplings $K_\ell^*(\Gamma)$ satisfy

$$K_\ell^*(\Gamma) = \begin{cases} -K_\ell^* & \text{if} \quad \ell \in \Gamma \\ K_\ell^* & \text{if} \quad \ell \notin \Gamma \end{cases} \tag{A.11}$$

We conclude that the string of gauge variables ΠA transforms into a *string of flipped dual bonds* (Fig. 8). The flipped dual bonds are those which are pierced by the string.

We now return to the constrained situation $K_p \to \infty$. Thus we set $H^* = 0$.

The correlation function, in the dual system (with a dual string of flipped bonds), is written as

$$\langle \sigma_i (\prod_\Gamma A) \sigma_j \rangle = \frac{\sum_{\{s_i\}} \left[\prod_i s_i^{(1-\xi_i)/2} \right] \exp\left\{ \sum_{\langle i,j \rangle} K_\ell^*(\Gamma)\, s_i s_j \right\}}{\sum_{\{s_i\}} \left[\prod_i s_i^{(1-\xi_i)/2} \right] \exp\left\{ \sum_{\langle i,j \rangle} K_\ell^*\, s_i s_j \right\}} \tag{A.12}$$

We now proceed to write (A.12) in a gauge invariant manner.

Define $A_{\underset{\sim}{i}\underset{\sim}{j}}$ to be the dual gauge variable of dual link $\underset{\sim}{i}\underset{\sim}{j}$. Then (A.12) holds for the configuration

$$a_{\underset{\sim}{i}\underset{\sim}{j}} = \frac{K_\ell^*(\Gamma)}{K_\ell^*}$$

Introducing strings of A's between the s variables pairwise, Eq. (A.12) can be written (for this configuration of A's)

$$\langle \sigma_i (\prod_\Gamma A) \sigma_j \rangle = (-1)^n \frac{\sum_{\{s_{\underset{\sim}{i}}\}} \left\{ \prod_{\alpha=1}^{N/2} \left[s_{\underset{\sim}{i}_\alpha} \left(\prod_{\tilde{\Gamma}_\alpha(\underset{\sim}{i}_\alpha, \underset{\sim}{j}_\alpha)} a \right) s_{\underset{\sim}{j}_\alpha} \right] \right\} \exp\left\{ \sum_{\langle \underset{\sim}{i}, \underset{\sim}{j} \rangle} K_\ell^*(\Gamma) s_{\underset{\sim}{i}} s_{\underset{\sim}{j}} \right\}}{\sum_{\{s_{\underset{\sim}{i}}\}} \left\{ \prod_{\alpha=1}^{N/2} \left[s_{\underset{\sim}{i}_\alpha} \left(\prod_{\tilde{\Gamma}_\alpha(\underset{\sim}{i}_\alpha, \underset{\sim}{j}_\alpha)} a \right) s_{\underset{\sim}{j}_\alpha} \right] \right\} \exp\left\{ \sum_{\langle \underset{\sim}{i}, \underset{\sim}{j} \rangle} K_\ell^* s_{\underset{\sim}{i}} s_{\underset{\sim}{j}} \right\}} \tag{A.13}$$

with the same conventions used for Eq. (2.22). Again n is the number of times the path Γ connecting the correlated spins i and j crosses the paths $\tilde{\Gamma}$ joining the frustrations. Eq. (A.13) is manifestly gauge invariant. Thus, from Eq. (2.1), we can write

$$\frac{\langle \sigma_i (\prod_\Gamma A) \sigma_j \rangle_{K_\ell} (\tilde{\Phi}_{\underset{\sim}{i}})}{\langle \sigma_i (\prod_\Gamma A) \sigma_j \rangle_{K_\ell} \{\tilde{\Phi}_{\underset{\sim}{i}} = 1\}} = (-1)^n \frac{\left\langle \prod_{\alpha=1}^{N/2} \left\{ s_{\underset{\sim}{i}_\alpha} \left(\prod_{\tilde{\Gamma}_\alpha(\underset{\sim}{i}_\alpha, \underset{\sim}{j}_\alpha)} a \right) s_{\underset{\sim}{j}_\alpha} \right\} \right\rangle_{K_\ell^*} \{ \tilde{\tilde{\Phi}}(i') \}}{\left\langle \prod_{\alpha=1}^{N/2} \left\{ s_{\underset{\sim}{i}_\alpha} \left(\prod_{\tilde{\Gamma}_\alpha(\underset{\sim}{i}_\alpha, \underset{\sim}{j}_\alpha)} a \right) s_{\underset{\sim}{j}_\alpha} \right\} \right\rangle_{K_\ell^*} \{ \tilde{\tilde{\Phi}}(i') = 1 \}} \tag{A.14}$$

where

$$\tilde{\tilde{\Phi}}(i') = \begin{cases} -1 & i' = i, j \end{cases}$$

A.2 Three Dimensions

We have already pointed out in Section III that the 3-D Ising model is dual to the 3-D Ising gauge theory (Eq. (3.7)). Therefore, all the manipulations we have performed from Eq. (A.5) up to Eq. (A.12) can be paralleled here too. Thus, in analogy with (A.12) we can write

$$\frac{\langle \sigma_i (\prod_\Gamma A) \sigma_j \rangle_{K_\ell} \{\Phi_{ij}\}}{\langle \sigma_i (\prod_\Gamma A) \sigma_j \rangle_{K_\ell} \{\Phi_{ij}=1\}} = \frac{\langle \prod_{\langle i,j \rangle} a_{ij}^{(1-\Phi_{ij})/2} \rangle_{K_p^*(\Gamma)}}{\langle \prod_{\langle i,j \rangle} a_{ij}^{(1-\Phi_{ij})/2} \rangle_{K_p^*}} \tag{A.15}$$

where

$$K_p^*(\Gamma) = \begin{cases} -K_p^* & \text{if the plaquette } p \text{ is pierced by } \Gamma \\ K_p^* & \text{otherwise} \end{cases}$$

Since frustrations come in tubes (see Section III) from Eq. (A.15) it follows that the gauge invariant spin-spin correlation function in the presence of frustrations dualizes into the loop integrals in the presence of a tube of overturned plaquette couplings. This tube begins and ends at the correlated spins and follows the path Γ of A variables.

APPENDIX B: DUALITY RELATIONS FOR THE GAUGE INVARIANT SPIN-SPIN CORRELATION FUNCTION IN THE PRESENCE OF FRUSTRATIONS: XY MODEL (d = 2,3)

The gauge invariant function for an XY system is given by

$$C_\Gamma = \left\langle \cos\left[\sum_{(i,\mu)} (\Delta_\mu \theta(i) - \psi_\mu(i))\, s_\mu(i)\right] \right\rangle_{\{\psi_\mu(i)\}} \qquad \text{(B.1)}$$

where $\psi_\mu(i)$ are the gauge variables, $\phi_\mu(\underset{\sim}{i})$ is the frustration field $(2\pi\phi_\mu(\underset{\sim}{i}) = \varepsilon_{\mu\nu\lambda}\Delta_\nu\psi_\lambda(i)$ and $s_\mu(i)$ is an integer variable which specifies the path Γ connecting the correlated spins, i.e.,

$$s_\mu(i) = \begin{cases} 1 & \text{if } (i,\mu) \in \Gamma \\ 0 & \text{otherwise} \end{cases} \qquad \text{(B.2)}$$

The thermal average is taken in a fixed distribution of frustrations. So

$$C_\Gamma = \frac{\mathcal{R}e \int \mathcal{D}\theta \; \exp\left[i \sum_{(i,\mu)} s_\mu(i)(\Delta_\mu \theta(i) - \psi_\mu(i))\right] \exp\left[K_\ell \sum_{(i,\mu)} \cos(\Delta_\mu \theta(i) - \psi_\mu(i))\right]}{\int \mathcal{D}\theta \; \exp\left[K_\ell \sum_{(i,\mu)} \cos(\Delta_\mu \theta(i) - \psi_\mu(i))\right]}$$

(B.3)

The first step in the duality transformation for an XY model is to perform a Fourier expansion at each link. Further integration over the angular XY variables θ at each site leads to the existence of constraints in the transformed model. Within the Villain approximation the correlation function (B.3) takes the form

$$C_\Gamma = \frac{1}{Z\{S_\mu\}} \sum_{\{\ell_\mu(i)\}} \exp\left\{-i \sum_{(i,\mu)} \left(\ell_\mu(i) + S_\mu(i)\right)\varphi_\mu(i)\right\} \exp\left\{-\frac{1}{2K_2}\sum_{(i,\mu)} \ell_\mu^2(i)\right\} \prod_i \delta(\Delta_\mu[\ell_\mu(i) + S_\mu(i)]) \tag{B.4}$$

where $Z\{S_\mu\}$ is the partition function

$$Z\{S_\mu\} = \sum_{\{\ell_\mu(i)\}} \exp\left\{-\frac{1}{2K_2}\sum_{(i,\mu)} \ell_\mu^2(i)\right\} \prod_i \delta\left(\Delta_\mu[\ell_\mu(i) + S_\mu(i)]\right) \tag{B.5}$$

These expressions are valid regardless of the dimensionality d of space. In fact, Jose, et al.,[23] have obtained Equation (B.5) in their discussion of the correlation functions of the pure 2-D XY model. We follow closely their approach. Space dimensionality becomes important in solving the constraints.

B.1 Two Dimensions

In two dimensions the constraint that the integer valued link variables $\ell_\mu(i)$ must satisfy

$$\Delta_\mu[\ell_\mu(i) + S_\mu(i)] = 0 \tag{B.6}$$

can be satisfied if we write $\ell_\mu + S_\mu$ as a curl, i.e.,

$$\ell_\mu(i) = \varepsilon_{\mu\nu}\, \Delta_\nu n(\underset{\sim}{i}) - S_\mu(i) \tag{B.7}$$

where the dual variables $n(\underset{\sim}{i})$ are the (integer valued) degrees of freedom of the (dual) surface roughening model in two dimensions.

Hence, Eq. (B.3), written in terms of the surface roughening variables $n(\underset{\sim}{i})$, reads

$$C_\Gamma = \frac{1}{Z_{S.R.}\{S_\mu\}} \, \mathcal{R}e \sum_{\{n(\underset{\sim}{i})\}} \exp\{-2\pi i \sum_{\underset{\sim}{i}} n(\underset{\sim}{i}) \Psi(\underset{\sim}{i})\} \exp\{-\frac{1}{2K_\varepsilon} \sum_{(\underset{\sim}{i},\mu)} (\Delta_\mu n(\underset{\sim}{i}) + \varepsilon_{\mu\nu} S_\nu(i))^2\}$$

$$2\pi \Psi(\underset{\sim}{i}) = \varepsilon_{\mu\nu} \Delta_\mu \Psi_\nu(\underset{\sim}{i})$$

(B.8)

The partition function $Z_{SR}\{S_\mu\}$ represents a surface roughening model with all its integer variables shifted by one unit on those dual links perpendicular to the original path Γ. Defining the dual of the string variable $S_\mu(i)$ as

$$t_\mu(\underset{\sim}{i}) = \varepsilon_{\mu\nu} S_\nu(i) \tag{B.9}$$

the partition function $Z_{SR}\{t_\mu(\underset{\sim}{i})\}$ takes the form

$$Z_{S.R.}\{t_\mu(\underset{\sim}{i})\} = \sum_{\{n(\underset{\sim}{i})\}} \exp\{-\frac{1}{2K_\varepsilon} \sum_{(\underset{\sim}{i},\mu)} (\Delta_\mu n(\underset{\sim}{i}) + t_\mu(\underset{\sim}{i}))^2\} \tag{B.10}$$

Thus,

$$C_\Gamma = \langle \cos[\sum_{\underset{\sim}{i}} 2\pi n(\underset{\sim}{i}) \Psi(\underset{\sim}{i})] \rangle_{S.R.\{t_\mu(\underset{\sim}{i})\}} \tag{B.11}$$

For arbitrary boundary conditions, the frustration field $\phi(\underset{\sim}{i})$ must satisfy the "neutrality" condition $\sum_{\underset{\sim}{i}} \phi(\underset{\sim}{i}) = 0$ (mod. integer) (Eq. (4.21)).

From (B.11) we see that, through a duality transformation, shifted bonds are mapped into correlated dual variables and vice versa. We have already found this result within the Ising model duality transformations.

Finally, we can perform a further duality transformation: the mapping to the Coulomb gas. The Poisson summation formula transforms (B.11) into the expression

$$C_\Gamma = \frac{1}{Z\{\varphi(\underset{\sim}{i})\}} \mathcal{R}e \sum_{\{m(\underset{\sim}{i})\}} \int_{-\infty}^{+\infty} \mathcal{D}\chi \; \exp\Big\{-2\pi i \sum_{\underset{\sim}{i}} (\varphi(\underset{\sim}{i}) + m(\underset{\sim}{i}))\, \chi(\underset{\sim}{i})\Big\} \exp\Big\{-\frac{1}{2K_\ell} \sum_{(\underset{\sim}{i},\mu)} (\Delta_\mu \chi(\underset{\sim}{i}) + t_\mu(\underset{\sim}{i}))^2\Big\}$$

(B.12)

and

$$Z\{\varphi\} = \sum_{\{m(\underset{\sim}{i})\}} \int_{-\infty}^{+\infty} \mathcal{D}\chi \; \exp\Big\{-2\pi i \sum_{\underset{\sim}{i}} m(\underset{\sim}{i})\, \chi(\underset{\sim}{i})\Big\} \exp\Big\{-\frac{1}{2K_\ell} \sum_{(\underset{\sim}{i},\mu)} (\Delta_\mu \chi(\underset{\sim}{i}) + t_\mu(\underset{\sim}{i}))^2\Big\}$$

(B.13)

In order to integrate out the χ variables, it is useful to expand the square and to rewrite (B.12) - (B.13) in the compact form

$$C_\Gamma = \frac{1}{Z\{\varphi\}} \mathcal{R}e \Big[\sum_{\{m(\underset{\sim}{i})\}} \exp\Big\{-\frac{1}{2K_\ell} \sum_{(\underset{\sim}{i},\mu)} t_\mu^2(\underset{\sim}{i})\Big\} \int_{-\infty}^{+\infty} \mathcal{D}\chi \; \exp\Big\{-\frac{1}{2K_\ell} \sum_{(\underset{\sim}{i},\mu)} (\Delta_\mu \chi(\underset{\sim}{i}))^2 - \sum_{\underset{\sim}{i}} J\{\varphi\}\chi(\underset{\sim}{i})\Big\}\Big]$$

(B.14)

where the source $J\{\phi\}$ is given by

$$J\{\varphi\} = 2\pi i\,(\varphi(\underset{\sim}{i}) + m(\underset{\sim}{i})) - \frac{1}{K_\ell} \Delta_\mu t_\mu(\underset{\sim}{i}) \tag{B.15}$$

After performing the path integral, the expression between brackets in (B.14) becomes

$$C_{\Gamma} = \frac{C_{SW}(\Gamma)}{Z\{\varphi\}} \, \mathrm{Re} \sum_{\{m(\underset{\sim}{j})\}}' \exp\Big\{-i \sum_{\underset{\sim}{i},\underset{\sim}{j}} (\varphi(\underset{\sim}{i}) + m(\underset{\sim}{i})) \frac{1}{2\pi} \Delta_{\mu}^{j} G'(\underset{\sim}{i}-\underset{\sim}{j}) \, t_{\mu}(\underset{\sim}{j})\Big\} \;\cdot$$

$$\cdot \exp\Big\{2\pi K_{\ell} \sum_{\underset{\sim}{i},\underset{\sim}{j}} (\varphi(\underset{\sim}{i}) + m(\underset{\sim}{i})) \, G'(\underset{\sim}{i}-\underset{\sim}{j}) (\varphi(\underset{\sim}{j}) + m(\underset{\sim}{j}))\Big\} \qquad \text{(B.16)}$$

where $C_{SW}(\Gamma)$ is the spinwave result for the correlation function of the pure system

$$C_{SW}(\Gamma) = \exp\Big\{-\pi K_{\ell} \sum_{(\underset{\sim}{i},\underset{\sim}{j})} \Delta_{\mu} S_{\mu}(i) \; G'(i-j) \; \Delta_{\nu} S_{\nu}(j)\Big\} \qquad \text{(B.17)}$$

$Z\{\phi\}$ is the partition function of the Coulomb gas with impurities, $G(\underset{\sim}{i} - \underset{\sim}{j})$ is the lattice Coulomb propagator and

$$G'(\underset{\sim}{i}-\underset{\sim}{j}) = G(\underset{\sim}{i}-\underset{\sim}{j}) - G(0) \qquad \text{(B.18)}$$

The reduced lattice Coulomb propagator $G'(\overline{r})$ is asymptotically equal to log $|\overline{r}|$ $(d = 2)$. Following JKKN[23] we make use of the Cauchy Riemann equations for $G'(z)$.

$$\varepsilon_{\mu\nu} \Delta_{\nu}^{j} \, \mathrm{Re}\, G'(\underset{\sim}{i}-j) = -\Delta_{\mu}^{j} \, \mathrm{Im}\, G'(\underset{\sim}{i}-j) \qquad \text{(B.19)}$$

We also define the angle $\theta(\vec{r})$

$$\theta(\vec{r}) = \mathrm{Im}\, G'(\vec{r}) \qquad \text{(B.20)}$$

which represents the angular position of the vortex or frustration respect to the integration site j. The branch of the logarithm is chosen in such a way that θ is measured accordingly with the usual convention from the positive X axis and ranges between 0 and 2π. Thus,

$$
\begin{aligned}
\sum_{j} \Delta_{\mu}^{j} G'(i-j)\, t_{\mu}(j) &= \sum_{j} \varepsilon_{\mu\nu} \Delta_{\mu}^{j} G'(i-j)\, s_{\nu}(j) = \\
&= -\sum_{j} \Delta_{\nu}^{j} \, \mathrm{Im}\, G'(i-j)\, S_{\nu}(j) = \\
&= -\left[\Theta(i-i) - \Theta(i-j)\right]_{\Gamma}
\end{aligned}
\tag{B.21}
$$

where the expression between brackets is the parallax angle of the frustration (or vortex) as seen from both ends of the path Γ. However, it has to be specified whether this angle is being scanned clockwisely or counterclockwisely. Such a specification depends on the position of the path (i.e., is path dependent). Consider the case in which the path is a straight line Γ_o from site i to site j. Then for all frustrations lying to the left (right) of the path, the parallax angle has to be computed clockwisely (counterclockwisely). For an arbitrary path Γ the rule goes as follows: Compute first the parallax for the straight path Γ_o. Then compute the *closed* line integral (B.20) along the path $\Gamma + \Gamma_o^-$ (where Γ_o^- is the negatively oriented path Γ_o). If the frustration is left inside the closed path and the orientation of that path is positive (negative) then the line integral (B.20) along an arbitrary path Γ is shifted by 2π (-2π). For a pure vortex, all these considerations are unimportant since they imply shifting the argument of (B.15) by $2\pi M$. Since frustrations are fractional vortices, these shifts are detectable. In fact, they are in analog of the $(-1)^n$ factor already found in the 2-D Ising model.

Finally, the gauge invariant correlation function $C_\Gamma\{\phi\}$ in the Coulomb gas representation is

$$\frac{C_\Gamma\{\varphi(\underset{\sim}{i})\}}{C_\Gamma\{\varphi(\underset{\sim}{i})=0\}} = \frac{\left\langle \cos\left\{\sum_{\underset{\sim}{i}'} (\varphi(\underset{\sim}{i}')+m(\underset{\sim}{i}'))\left[\Theta(\underset{\sim}{i}'-\underset{\sim}{i})-\Theta(\underset{\sim}{i}'-\underset{\sim}{j})\right]_\Gamma\right\}\right\rangle_{c.g.\{\varphi(\underset{\sim}{i})\}}}{\left\langle \cos\left\{\sum_{\underset{\sim}{i}'} m(\underset{\sim}{i}')\left[\Theta(\underset{\sim}{i}'-\underset{\sim}{i})-\Theta(\underset{\sim}{i}'-\underset{\sim}{j})\right]\right\}\right\rangle_{c.g.\{\varphi(\underset{\sim}{i})=0\}}} \tag{B.22}$$

where the numerator is averaged in a Coulomb gas with a fixed distribution of impurities (frustrations) $\phi(\underset{\sim}{i})$ and the denominator is the pure Coulomb gas. The sites i and j are the endpoints of the path Γ. Notice that since the denominator is evaluated in the pure Coulomb gas, it is path independent.

B.2 Three Dimensions

In three dimensions the constraint

$$\Delta_\mu(\ell_\mu(\underset{\sim}{i})+S_\mu(\underset{\sim}{i}))=0 \tag{B.23}$$

can be solved by requiring $\ell_\mu + S_\mu$ to be a curl, i.e.,

$$\ell_\mu(\underset{\sim}{i})=\varepsilon_{\mu\nu\lambda}\Delta_\nu n_\lambda(\underset{\sim}{i})-S_\mu(\underset{\sim}{i}) \tag{B.24}$$

where the dual variables $n_\mu(\underset{\sim}{i})$ are the integer valued degrees of freedom of the (dual) gauge theory in three dimensions.

Hence, Eq. (B.3), written in terms of the (dual) gauge variables $n_\mu(\underset{\sim}{i})$, reads

$$C_\Gamma=\frac{1}{Z_{G.T.}\{t_{\mu\nu}(\underset{\sim}{i})\}}\,\mathrm{Re}\sum_{\{n_\mu(\underset{\sim}{i})\}}\exp\Big\{-2\pi i\sum_{(\underset{\sim}{i},\mu)} n_\mu(\underset{\sim}{i})\varphi_\mu(\underset{\sim}{i})\Big\}\exp\Big\{-\frac{1}{2K_2}\sum_{(\underset{\sim}{i},\mu\nu)}(\Delta_\mu n_\nu(\underset{\sim}{i})-\Delta_\nu n_\mu(\underset{\sim}{i})-t_{\mu\nu}(\underset{\sim}{i}))^2 \tag{B.25}$$

where

$$t_{\mu\nu}(\underset{\sim}{i}) = \varepsilon_{\mu\nu\lambda} S_\lambda(\underset{\sim}{i}) \tag{B.26}$$

and $Z_{GT}\{t_{\mu\nu}(\underset{\sim}{i})\}$ represents the partition function of the (dual) gauge theory *with a tube of shifted plaquette interactions.*

$$Z_{G.T.}\{t_{\mu\nu}(\underset{\sim}{i})\} = \sum_{\{n_\mu(\underset{\sim}{i})\}} \exp\{-\frac{1}{2K_\ell}\sum_{(\underset{\sim}{i},\mu\nu)} (\Delta_\mu n_\nu(\underset{\sim}{i}) - \Delta_\nu n_\mu(\underset{\sim}{i}) - t_{\mu\nu}(\underset{\sim}{i}))^2\} \tag{B.27}$$

Thus, the gauge invariant correlation function is

$$C_\Gamma\{\varphi_\mu(\underset{\sim}{i})\} = \left\langle \cos\{2\pi \sum_{(\underset{\sim}{i},\mu)} n_\mu(\underset{\sim}{i})\,\varphi_\mu(\underset{\sim}{i})\}\right\rangle_{G.T.}\{t_{\mu\nu}(\underset{\sim}{i})\} \tag{B.28}$$

In the gauge theory picture, the line $S_\mu(i)$, defining the path of the correlation function in the 3-D XY model, is just a line (tube) of external magnetic flux "injected" in the system by sources residing at the endpoints of the path Γ. Here again the frustration field is constrained by the condition $\Delta_\mu \phi_\mu(\underset{\sim}{i}) = 0$ (mod. integer).

To write Eq. (B.28) in terms of the topological excitations of the 3-D XY model, we use once again the Poisson summation formula

$$Z_{G.T.}\{t_{\mu\nu}(\underset{\sim}{i})\}\; C_\Gamma\{\varphi_\mu(\underset{\sim}{i})\} =$$

$$= \mathrm{Re} \sum_{\{m_\mu(\underset{\sim}{i})\}} \int_{-\infty}^{+\infty} \mathcal{D}\chi_\mu \exp\{2\pi i \sum_{(\underset{\sim}{i},\mu)} (\varphi_\mu(\underset{\sim}{i}) + m_\mu(\underset{\sim}{i}))\,\chi_\mu(\underset{\sim}{i})\}$$

$$\times \exp\{-\frac{1}{2K_\ell}\sum_{(\underset{\sim}{i},\mu\nu)} (\Delta_\mu \chi_\nu(\underset{\sim}{i}) - \Delta_\nu \chi_\mu(\underset{\sim}{i}) - t_{\mu\nu}(\underset{\sim}{i}))^2\} \tag{B.29}$$

Expanding the square in the exponent we obtain

$$Z_{G.T.}\{t_{\mu\nu}(\underset{\sim}{i})\}\; C_p\{\varphi_\mu(\underset{\sim}{i})\} =$$

$$= \mathcal{R}e \sum_{\{m_\mu(\underset{\sim}{i})\}} \exp\Big\{-\frac{1}{2K_\ell}\sum_{\underset{\sim}{i},\mu\nu} t^2_{\mu\nu}(\underset{\sim}{i})\Big\} \int_{-\infty}^{+\infty} d\chi_\mu \; \exp\Big\{\sum_{\underset{\sim}{i},\mu} J_\mu(\underset{\sim}{i})\, \chi_\mu(\underset{\sim}{i})\Big\} \; \exp\Big\{-\frac{1}{2K_\ell}\sum_{\underset{\sim}{i},\mu\nu}\big(\Delta_\nu \chi_\mu(\underset{\sim}{i}) - \Delta_\mu \chi_\nu(\underset{\sim}{i})\big)^2\Big\}$$

(B.30)

where the current $J_\mu(\underset{\sim}{i})$ is given by

$$J_\mu(\underset{\sim}{i}) = 2\pi i\,(\varphi_\mu(\underset{\sim}{i}) + m_\mu(\underset{\sim}{i})) + \frac{1}{K_\ell}\;\Delta_\nu t_{\mu\nu}(\underset{\sim}{i}) \tag{B.31}$$

Since $t_{\mu\nu}(\underset{\sim}{i})$ is an antisymmetric tensor, the last term in the current does not affect the constraint equation (5.11)

$$\Delta_\mu J_\mu(\underset{\sim}{i}) = 0 \tag{5.11}$$

Hence, we obtain

$$Z_{G.T.}\{t_{\mu\nu}(\underset{\sim}{i})\}\; C_p\{\varphi_\mu(\underset{\sim}{i})\} =$$

$$= \mathcal{R}e \sideset{}{'}\sum_{\{m_\mu(\underset{\sim}{i})\}} \exp\Big\{-\frac{1}{2K_\ell}\sum_{(\underset{\sim}{i},\mu\nu)} t^2_{\mu\nu}(\underset{\sim}{i})\Big\} \;\; \exp\Big\{-\frac{K_\ell}{8\pi}\sum_{(\underset{\sim}{i}\mu)(\underset{\sim}{j}\mu)} J_\mu(\underset{\sim}{i})\; G'(\underset{\sim}{i}-\underset{\sim}{j})\; J_\mu(\underset{\sim}{j})\Big\}$$

(B.32)

where $G'(\underset{\sim}{i} - \underset{\sim}{j})$ is the reduced Coulomb lattice propagator in three dimensions. and the summation is restricted to those configurations which obey Eq. (5.11).

After some algebra, Eq. (B.30) takes the form

$$\frac{C_p\{\varphi_\mu(i)\}}{C_p\{\varphi_\mu(i)=0\}} = \frac{\left\langle \cos\left\{\frac{1}{2}\sum_{i,j}(\varphi_\mu(i)+m_\mu(i))\, G'(i-j)\, \varepsilon_{\mu\nu\lambda}\, \Delta_\nu^j\, s_\lambda(j)\right\}\right\rangle_{t.e.}\{\varphi_\mu(i)\}}{\left\langle \cos\left\{\frac{1}{2}\sum_{i,j} m_\mu(i)\, G'(i-j)\, \varepsilon_{\mu\nu\lambda}\, \Delta_\nu^j\, s_\lambda(j)\right\}\right\rangle_{t.e.}\{\varphi_\mu(i)=0\}}$$

(B.33)

where the averages are taken in the gas of topological excitations with impurities (numerator) and without impurities (denominator).

The spin-spin correlation function of the pure system is given by

$$C_p\{\varphi_\mu(i)=0\} = C_{SW}\left\langle \cos\left\{\frac{1}{2}\sum_{i,j} m_\mu(i)\, G'(i-j)\, \varepsilon_{\mu\nu\lambda}\, \Delta_\nu^j\, s_\lambda(j)\right\}\right\rangle_{t.e.}\{\varphi_\mu(i)=0\}$$

(B.34)

The factor C_{SW} is the spinwave approximation to the correlation function

$$C_{SW} = \exp\left\{-\frac{1}{2K}\sum_{i,j} \Delta_\alpha s_\alpha(i)\; G'(i-j)\, \Delta_\beta s_\beta(j)\right\} \qquad (B.35)$$

At large distances, C_{SW} is given by

(B.36)

$$C_{SW} \simeq \exp\left\{\frac{1}{4\pi K_2 R}\right\}$$

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27. A vector φ_α related with a tensor $\phi_{\mu\nu}$ through Eq. (5.1) is called the dual of the tensor. This geometrical duality is related to but is distinct from the algebraic duality we have been discussing.

28. Strictly speaking, it is necessary to choose a gauge if we want to identify the partition function of the XY model with that of the gauge theory. The difference with the gauge non-fixed partition function in the (infinite) number V^N, where V is the volume of the gauge group and N the total number of sites. All our arguments invoking gauge invariance can be rephrased as statements concerning independence under gauge fixing. Despite these considerations statements like Eq. (5.8) remain valid.

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FIGURE CAPTIONS

Fig. 1 Location of the degrees of freedom of the spin glass system. Here the dark dots represent the spin (site) variables σ and the crosses represent the link (gauge) variables A.

Fig. 2 A gauge transformation. Dark circles are spins pointing up and white circles are spins down. The signs on the links denote the values of the link variables A.

Fig. 3 A frustrated plaquette.

Fig. 4 (a) One flipped bond creates two frustrations.

(b,c) Two separated frustrations are created by a dual string of flipped bonds between them. The broken line is the dual string. The strings shown in (b) and (c) are equivalent.

(d) A closed dual string of flipped bonds is a closed domain wall. It does not create frustrations.

(e) An infinite domain wall is a dual string running from one side of the boundary to one another.

(f) A single frustration has a infinite string of flipped bonds.

(g) Four frustrations created by 2 flipped bonds. The orientation of the central spin (dark dot) is the degeneracy of the ground state.

Fig. 5 A gauge invariant correlation function is defined for two lattice sites (i and j) and the path Γ_{ij} of links joining both sites. There is a σ variable at each end and an A variable at each link of the path Γ_{ij}.

Fig. 6 Two different paths between sites i and j. The correlation function for both paths differ in a factor $(-1)^N$ where N is the number of frustrations within the dashed area.

Fig. 7 (a) The gauge invariant spin-spin correlation function. The correlated spins reside at sites i and j, and $\Gamma_{(ij)}$ is the path of the string of gauge variables A. The frustrations, here denoted by crosses, are linked together by strings with paths Γ.

(b) The dual transformed of the situation described in Fig. (7.a). Correlated spins and frustrations exchange their roles. The string Γ intersects the paths Γ three times: the correlation function picks up a minus sign (eq. 2.22).

Fig. 8 A frustration lying between two correlated spins. The dark links represent the flipped bonds we choose as a representation of the frustration (cross). The broken line is the path Γ of link variables.

Fig. 9 Two nearest neighbor frustrations pierced by the string of link variables.

Fig. 10 Magnetization of a region between two frustrations. The situation is the dual of that depicted in Fig. 9.

Fig. 11 A cube and the six links dual to its faces.

Fig. 12 A closed tube of frustrated plaquettes. The broken line represents the loop of dual links involved in the loop integral (3.10).

Fig. 13 (a) Two face to face tubes frustrated plaquettes here represented by the closed loop of dual link variables threading the plaquettes together.

(b) When $R \ll d$ the minimal surface spanned by the loops is the lateral surface (shaded in the figure).

(c) When $R >> d$ the minimal surface is the surface spanned by each loop independently.

(d) The minimal surface spanned by two orthogonal loops.

Fig. 14 The gauge invariant spin-spin correlation function in the presence of two frustrations (impurities) of strength q and -q. The paralax is 2ω. The dark line represents the string of gauge variables ψ.

Fig. 15 The "salt crystal". Each plaquette is frustrated. The strength is q. This is Villain's "odd model".

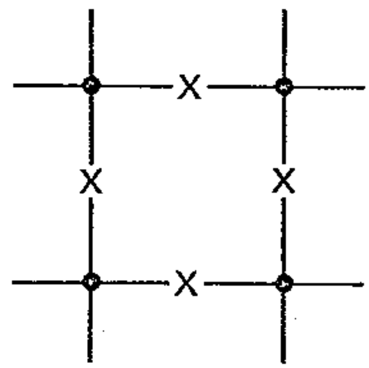


Fig. 1

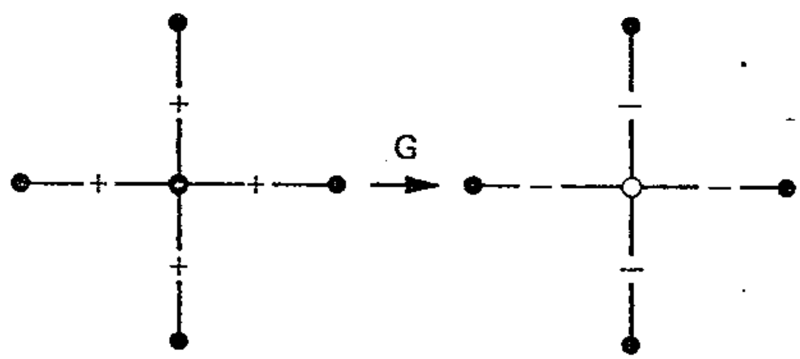


Fig. 2

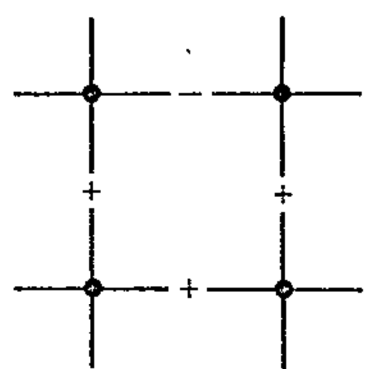


Fig. 3

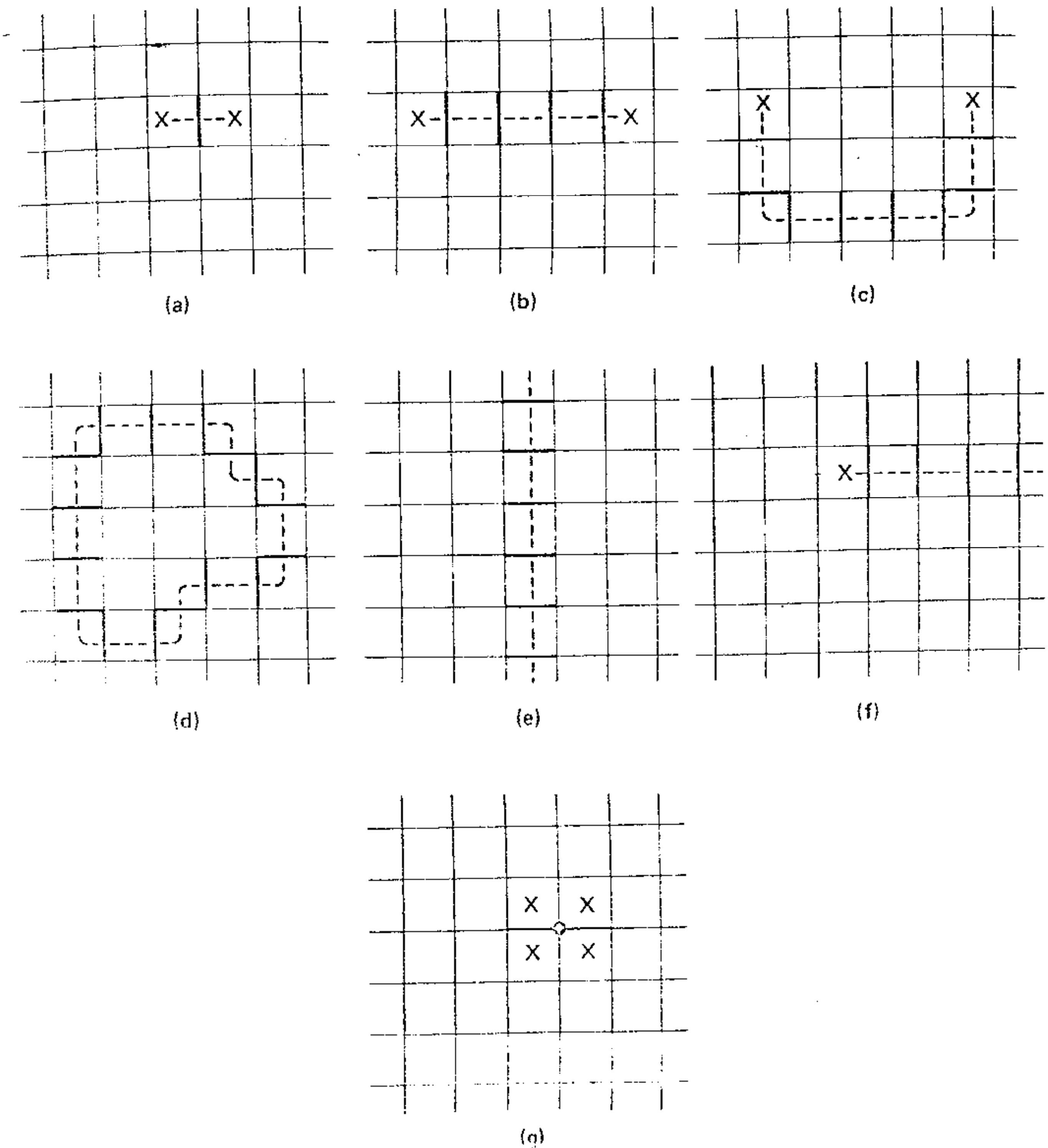


Fig. 4

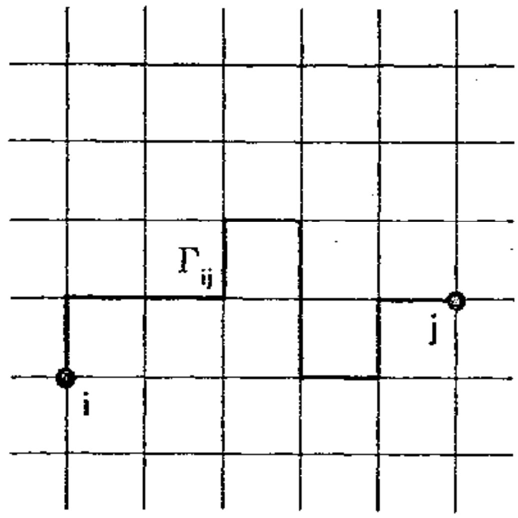


Fig. 5

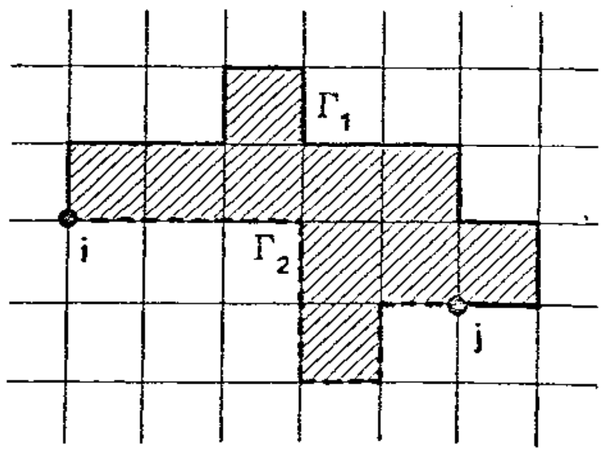
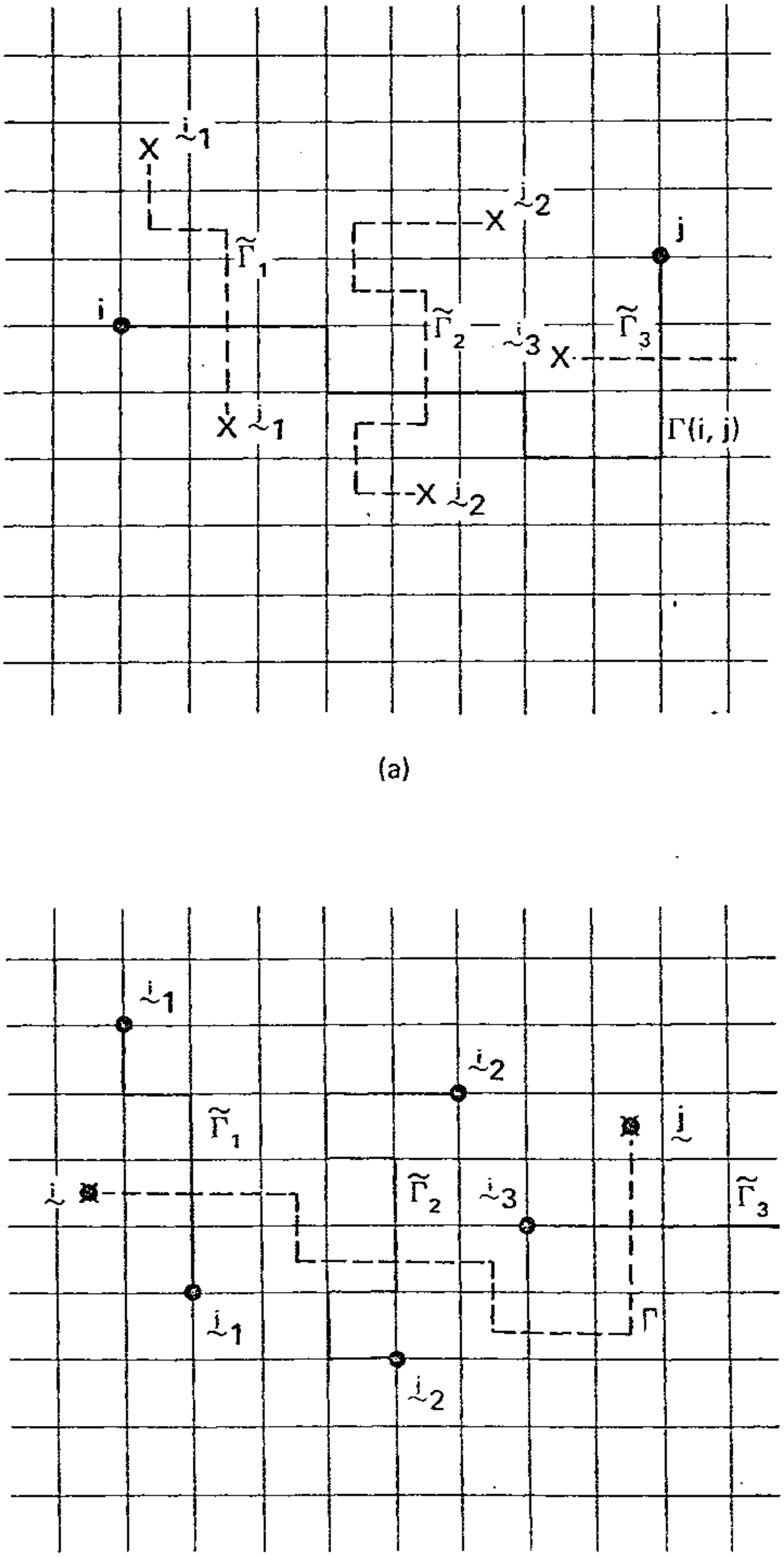


Fig. 6



i
j
$\underset{\sim}{i}_1$
$\underset{\sim}{j}_1$
$\underset{\sim}{i}_2$
$\underset{\sim}{j}_2$
$\underset{\sim}{i}_3$
$\tilde{\Gamma}_1$
$\tilde{\Gamma}_2$
$\tilde{\Gamma}_3$
$\Gamma(i, j)$

(a)

(b)

Fig. 7

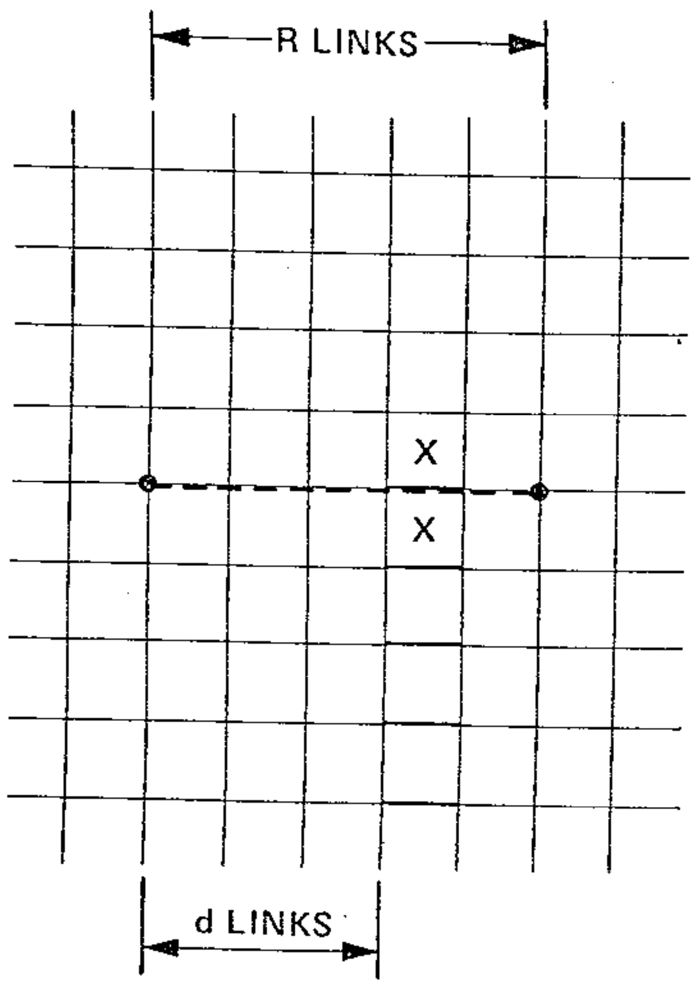


Fig. 8

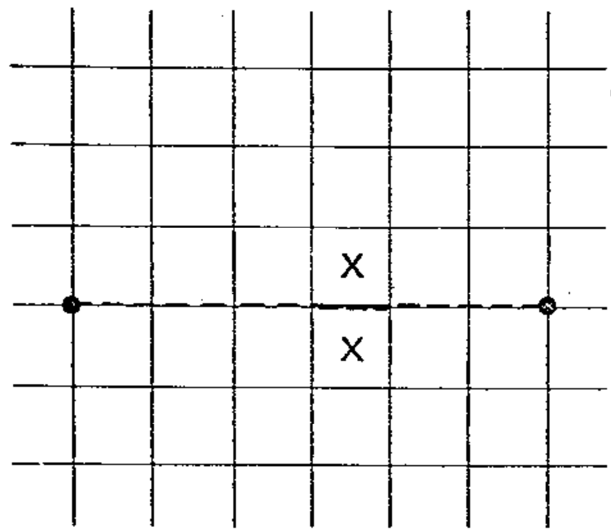


Fig. 9

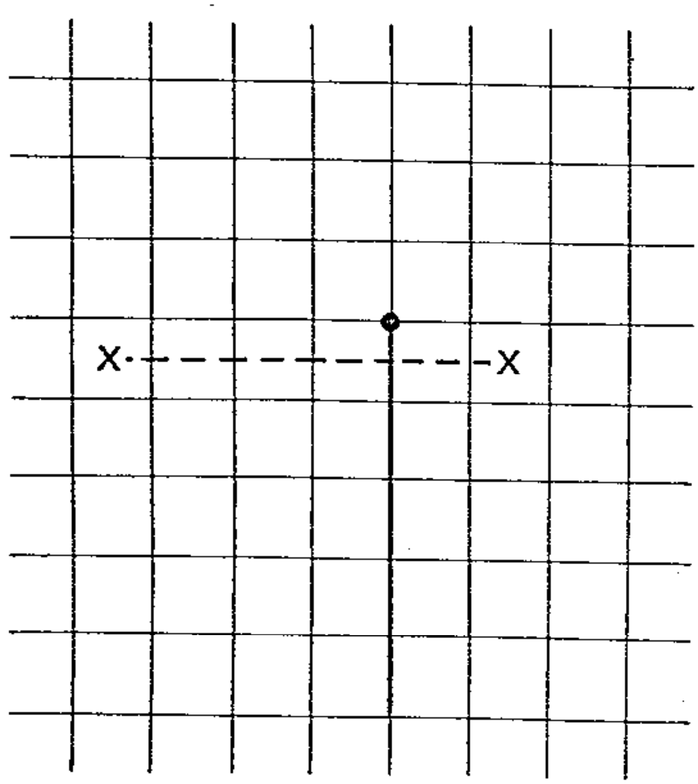


Fig. 10

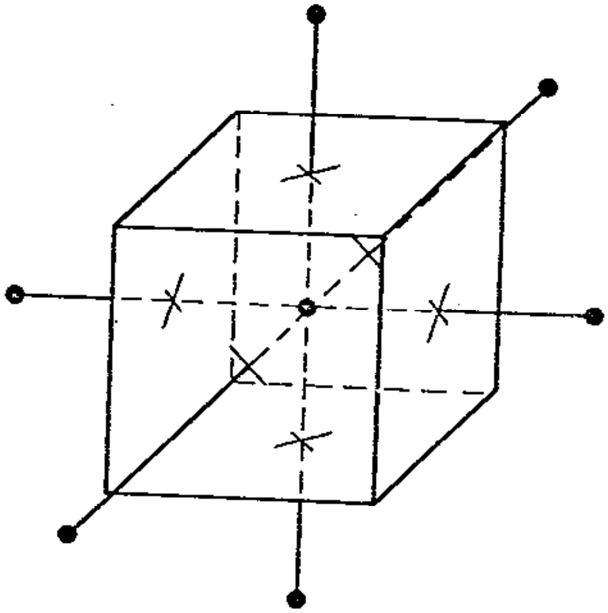


Fig. 11

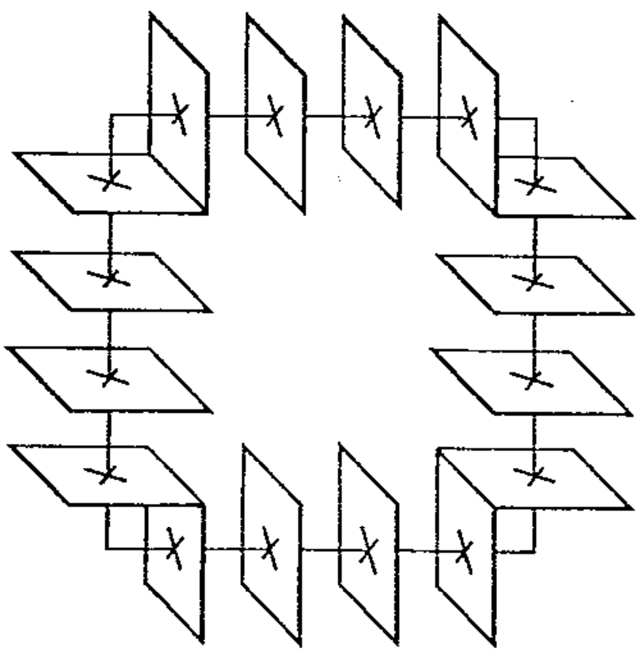


Fig. 12

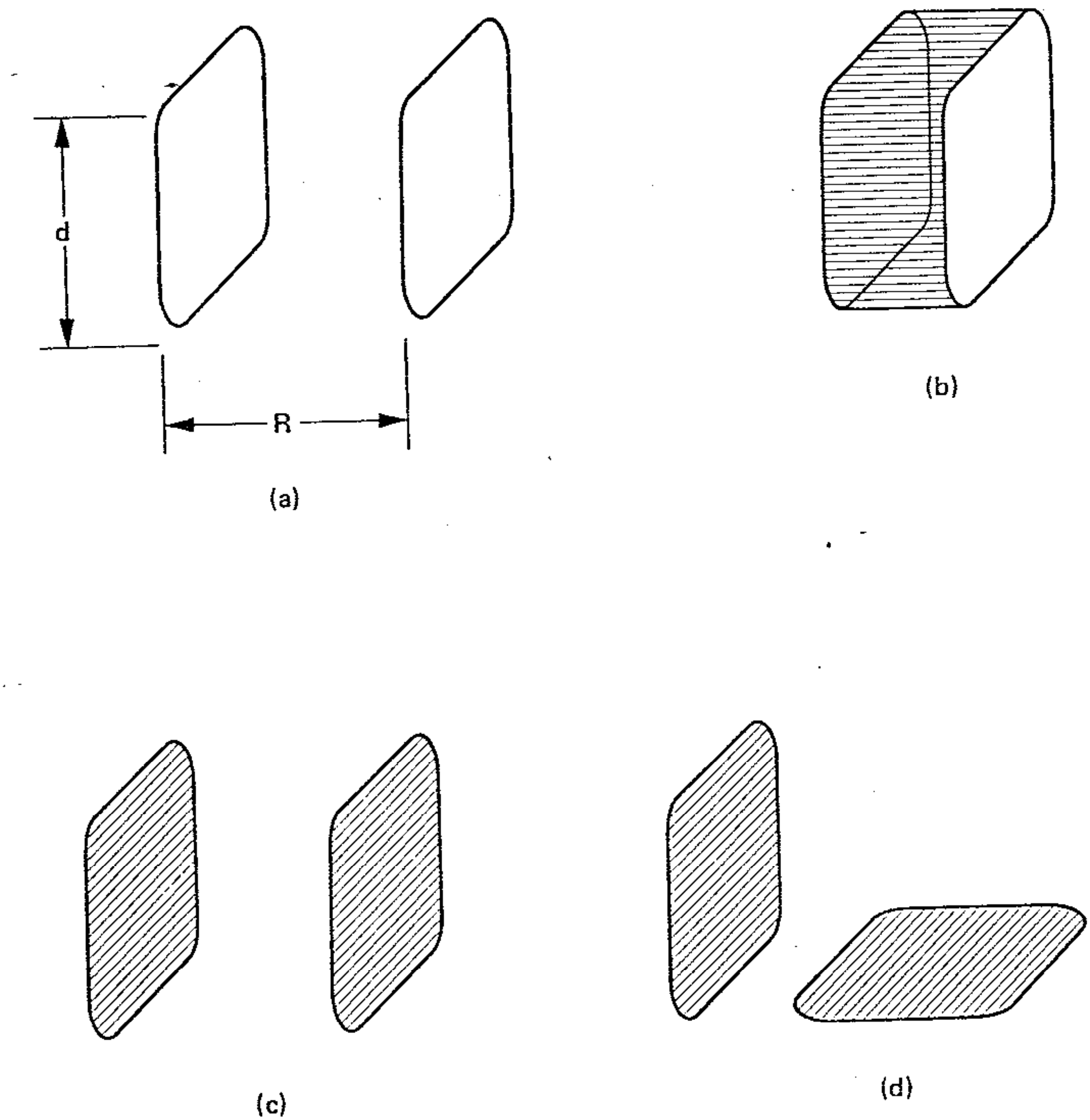


Fig. 13

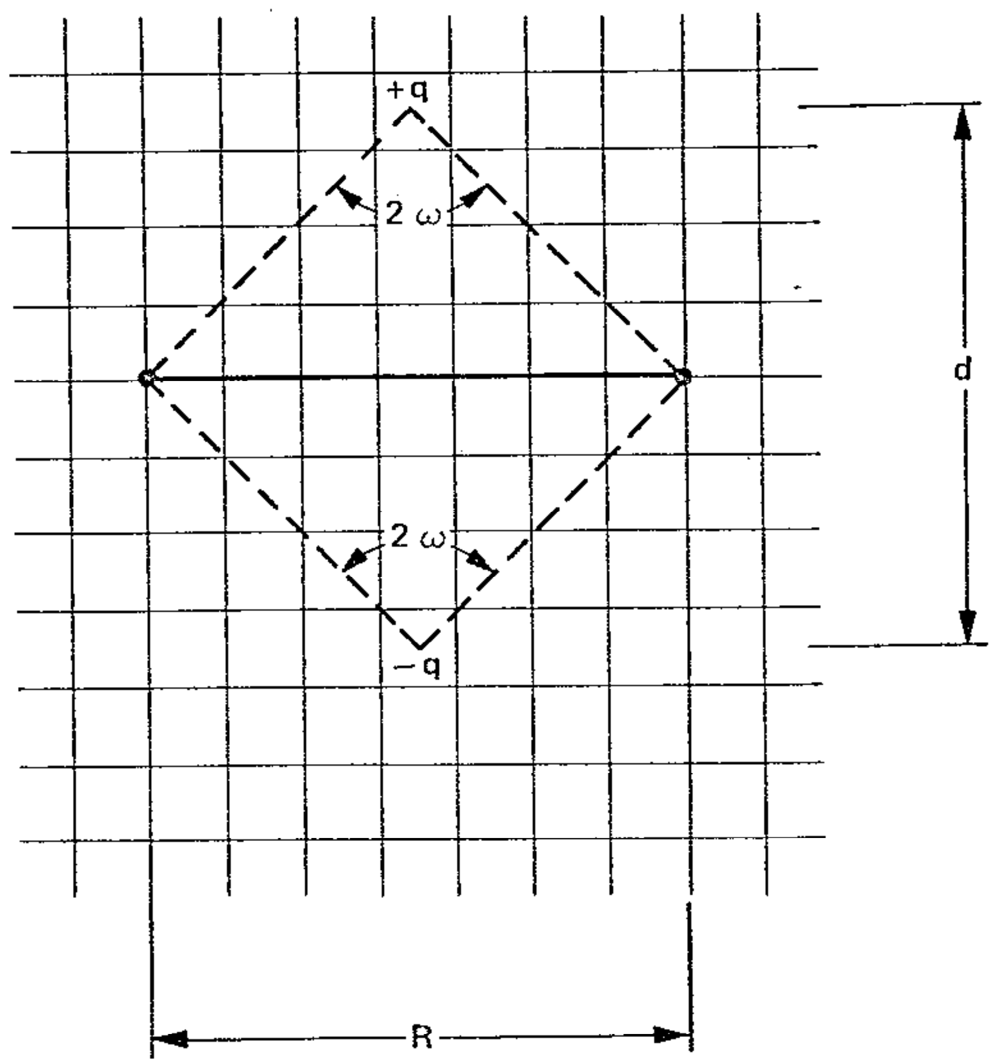


Fig. 14

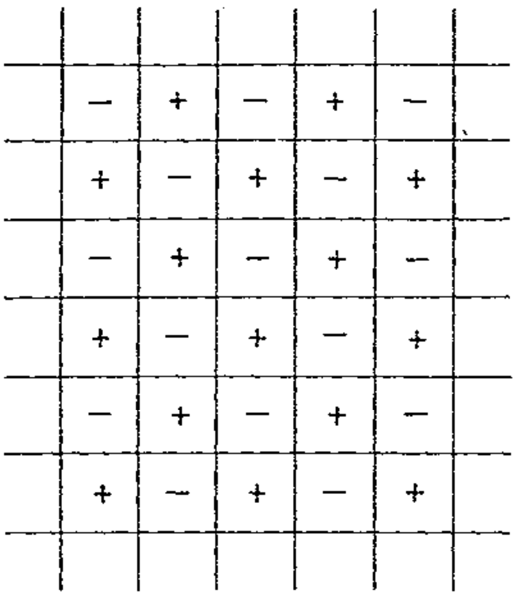


Fig. 15