SLAC-PUB-2101
March 1978
(T)

SCREENING SOLUTIONS TO CLASSICAL YANG-MILLS THEORY *

P. Sikivie and N. Weiss †
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305

ABSTRACT

We present two new solutions to the classical Yang-Mills field equations in the presence of a localized external source. These solutions totally screen the charge of the source. They have lower energy than the corresponding Coulomb solution.

(Submitted to Physical Review Letters.)

---

* Work supported in part by the Department of Energy.

† Work supported in part by the National Research Council of Canada.

Non-Abelian gauge theories offer the greatest promise to describe the elementary forces in nature. We here investigate the solutions to the classical Yang-Mills equations in the presence of a static external source in Minkowski space:

$$(D_\mu F^{\mu\nu})^a = j^{a\nu}(x) = \delta^{\nu o} q^a(x) \tag{1.a}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g c^{abc} A^b_\mu A^c_\nu \tag{1.b}$$

where $q^a(x)q^a(x)$ is time-independent. By a local gauge transformation, one can always line up the source into commuting directions of color space, e.g. $q^a(x) \rightarrow \delta^{a3}\sqrt{q^b(x)\,q^b(x)} = \delta^{a3} q(\vec{x})$ for SU(2) which for simplicity we will study first. The ansatz[1] $A^a_\mu = \delta^{a3} A_\mu$ then reduces Eqs. (1) to the Maxwell equations of electrodynamics. We call the corresponding solution the Coulomb solution for the source $q^a(x)$.

However, various results in the literature have already shown that classical unbroken Yang-Mills theories in Minkowski space are qualitatively different from electrodynamics, e.g. the Wu-Yang monopole[2] and Coleman's non-Abelian plane wave[3] which are both non-trivial solutions to Eqs. (1) with $q^a(x) = 0$. Moreover, Mandula[4] has shown that the Coulomb solution corresponding to a static source distributed over a thin spherical shell is unstable if $gQ > 3/2$, where $Q = \int d^3x \sqrt{q^a(x)\,q^a(x)}$. Mandula also showed that the instability modes produce an inward flow of charge that tends to screen the external source. Since the energy is positive definite, Eqs. (1) must admit static solutions of lower energy than the Coulomb one. Below we exhibit two new types of solutions to Eqs. (1) with localized and integrable static sources. The first type has the long-range behaviour of a magnetic dipole field, and has lower energy than the Coulomb solution once gQ is large enough. The second type has no long-range field strengths at all, and its energy can be made arbitrarily small.

1. The magnetic dipole solution:

The ansatz:

$$A_0^1 = A_0^2 = A_i^2 = A_i^3 = 0$$

$$A_0^3 = \phi(\rho, x_3)\,, \quad A_i^1 = \epsilon_{i3j} \frac{x_j}{\rho} A(\rho, x_3) \tag{2}$$

where $\rho = \sqrt{x_1^2 + x_2^2}$ assures that all the Eqs. (1) are automatically satisfied provided:

$$-\nabla^2 \phi + g^2 A^2 \phi = q \tag{3.a}$$

$$+\nabla^2 A - \frac{1}{\rho^2} A + g^2 \phi^2 A = 0\,. \tag{3.b}$$

The Coulomb solution corresponds to setting $A = 0$. Outside of this ansatz, the full non-linearity of the equations comes into play and there are no analytical methods available. It is nevertheless possible to show that there exists a whole class (a continuous infinity) of charge distributions $q(\vec{x})$ which are localized and integrable (i. e. $Q < \infty$) and which admit besides the Coulomb potential a new type of solution with $A \neq 0$ and $\phi \neq 0$ and finite total energy. To this end, let us consider any field $A(\rho, x_3)$ which satisfies the following two conditions:

1. $A(\rho, x_3)$ goes to zero as $r = \sqrt{x_1^2 + x_2^2 + x_3^2} \to 0$.
2. Away from the origin, $A(\rho, x_3)$ approaches exponentially fast the solution $\mathscr{A} = \rho/r^3$ of $\nabla^2 \mathscr{A} - \frac{1}{\rho^2}\mathscr{A} = 0$.

For that given $A(\rho, x_3)$, let us successively solve Eq. (3.b) for $\phi(\vec{x})$ and calculate $q(\vec{x})$ from $\phi(\vec{x})$, $A(\vec{x})$ and Eq. (3.a). For the charge distribution $q(\vec{x})$ thus found, $\phi(\vec{x})$ and $A(\vec{x})$ will be an exact solution of the field equations. The second condition on $A(\vec{x})$ assures that both $\phi(\vec{x})$ and $q(\vec{x})$ vanish exponentially fast away from the origin. The first and second condition together assure finiteness of the energy. Let us give a particular example:

$$A(\rho, x_3) = c\, a\, \frac{\rho}{r^3}\, \mathrm{th}\left(\frac{r}{a}\right)^3$$

$$\phi(\rho, x_3) = \frac{\sqrt{18}}{a^3 g} \frac{r^2}{\mathrm{ch}^2\left(\frac{r}{a}\right)^3} \tag{4}$$

are solutions of Eqs. (3) for a rather complicated but non-singular charge distribution $q(\vec{x})$ spread over a region of width a, and of total charge:

$$Q = \int d^3x\, g^2 A^2 \phi = c^2 g I_1 \tag{5}$$

where $I_1 = \frac{8\pi}{3}\sqrt{18} \int_0^\infty dx \frac{\mathrm{th}^2 x^3}{\mathrm{ch}\, x^3}$ for our particular example. The particular charge distribution we obtain depends of course on the particular choice we made for the way $A(\rho, x_3)$ approaches ρ/r^3 in the transition region between $r << a$ $r >> a$. The point is that to the continuous infinity of ways in which $A(\rho, x_3)$ can approach ρ/r^3 corresponds a continuous infinity of localized charge distributions which admit solutions of the new type. Presumably the thin spherical shell studied by Mandula is among these charge distributions.

The new solution has the long-range behaviour of a magnetic dipole field. Indeed, using a vector notation for the spatial components, we have for $r >> a$:

$$\vec{A}^1 \cong c\, a(\hat{3} \times \vec{x}) \frac{1}{r^3} = -\vec{m} \times \vec{\nabla} \frac{1}{r}$$

$$\vec{B}^1 = \vec{\nabla} \times \vec{A}^1 \cong \frac{3(\vec{m} \times \vec{x}) - \vec{m}\, r^2}{r^5} \tag{6}$$

where $\vec{m} = c\,a\,\hat{3}$. In Eqs. (2) and (6) the orientation of the magnetic dipole has been arbitrarily chosen to be along the $\hat{3}$ direction of space and the $\hat{1}$ direction of isospin space (it could have been any linear combination of the $\hat{1}$ and $\hat{2}$ isospin directions). The other field strengths are either zero or short range. The physical situation is as follows. The Yang-Mills fields $\vec{A}^1$ and ϕ create a charge distribution $-g^2(\vec{A}^1)^2\phi$ whose total charge exactly cancels Q. The electric field strengths thus become short range. On the other hand, the Yang-Mills fields create a current loop distribution:

$$\vec{j}^{\,1} = g^2\phi^2\vec{A}^1 = g^2\phi^2 A\,\frac{1}{\rho}\,(\hat{3}\times\vec{x}) \tag{7}$$

whose total magnetic moment is precisely $\vec{m} = c\,a\,\hat{3}$.

The energy of the magnetic dipole solution has the following form:

$$\begin{aligned} H^{m.d.} &= \int d^3x\,\tfrac{1}{2}\left[|\vec{\nabla}\phi|^2 + g^2\phi^2(\vec{A}^1)^2 + \left(\vec{\nabla}\times\vec{A}^1\right)^2\right] \\ &= \int d^3x\left[\tfrac{1}{2}|\vec{\nabla}\phi|^2 + g^2\phi^2(\vec{A}^1)^2\right] \\ &= \frac{1}{a}\left(\frac{1}{g^2}I_2 + c^2 I_3\right) = \frac{1}{a}\left(\frac{1}{g^2}I_2 + \frac{Q}{g}\frac{I_3}{I_1}\right) \end{aligned} \tag{8}$$

where I_2 and I_3 (like I_1) are calculable numbers which depend on the shape of the charge distribution but not on its norm (Q) nor its spatial extension (a). Since the energy of the Coulomb solution has the general form:

$$H^c = \frac{1}{a}\,Q^2 I_4 \tag{9}$$

we find that the magnetic dipole solution has lower energy than the Coulomb solution when

$$Qg > \frac{1}{2I_4}\left(\frac{I_3}{I_1} + \sqrt{\left(\frac{I_3}{I_1}\right)^2 + 4I_2I_4}\right). \qquad (10)$$

2. The total screening solution:

The magnetic dipole solution thus appears to be precisely the new type of solution whose existence had been implied by Mandula's work. However, we will now show, by merely exploiting our knowledge of the Coulomb solution and gauge invariance, that any extended charge distribution admits solutions of energy as low as one wishes. Indeed, while in Abelian gauge theories the sign of a charge is unambiguously defined, this is not so in non-Abelian gauge theories where the direction in isospin space of a charge distribution can be locally reversed by a gauge transformation. The only gauge invariant quantity that characterizes a source is $q^2(x) = q^a(x)\,q^a(x)$ for SU(2).[5] Thus for any given extended source $q^2(x)$, we can choose a gauge where half of the source is lined up in the positive $\hat{3}$ direction of isospin space and the other half in the negative $\hat{3}$ direction. We can then make the ansatz $A_\mu^a = \delta^{a3} A_\mu$ which will yield a Coulomb solution corresponding to an electric dipole. By rotating back into the gauge where q^a is completely lined up in the positive $\hat{3}$ direction, we find a solution whose energy is that of a dipole field although q^a is in the monopole configuration. It is clear that from dipole we can go to quadrupole and so on, lowering the energy indefinitely in the process. Let us illustrate this by giving a particular example, in which all fields will be free of discontinuities. Equations (1) are solved by:

$$A_0^a = 0 \qquad\qquad A_i^a = E_i^a t$$

$$E_i^a = \frac{Q}{4\pi} \frac{x_i}{r^3} \frac{1}{2\pi n}\left[\delta^{a2}(\cos 2\pi nh(r)-1) - \delta^{a3}\sin 2\pi nh(r)\right]$$

$$q^a = \frac{Q}{4\pi}\left(\frac{-1}{r^2}\frac{dh}{dr}\right)\left[\delta^{a2}\sin 2\pi nh(r) + \delta^{a3}\cos 2\pi nh(r)\right] \tag{11}$$

where t = time, n is an integer and h(r) is an arbitrary function that goes to one as $r \to 0$ and goes to zero as $r \to \infty$, say $h(r) = e^{-\frac{1}{2}(r/a)^2}$. Rotated back into the gauge where q^a is completely lined up in the positive $\hat{3}$ direction of isospin space, the solution has the form:

$$A'^a_0 = 0 \qquad\qquad A'^a_i = E'^a_i t - \delta^{a1}\frac{1}{g}\,\partial_i(2\pi nh(r))$$

$$E'^a_i = \frac{Q}{4\pi}\frac{x_i}{r^3}\frac{1}{2\pi n}\left[\delta^{a2}(1-\cos 2\pi nh(r)) - \delta^{a3}\sin 2\pi nh(r)\right]$$

$$q'^a = \delta^{a3}\frac{Q}{4\pi}\left(\frac{-1}{r^2}\frac{dh}{dr}\right). \tag{12}$$

The electric field is completely screened because the charge distribution $-g\,\epsilon^{abc}A'^b_i E'^c_i$ carried by the Yang-Mills fields exactly cancels the external source. There is no magnetic field. The energy of this total screening solution:

$$H^{t.s.} = \frac{Q^2}{2\pi}\left(\frac{1}{2\pi n}\right)^2 \frac{1}{a}\int_0^\infty \frac{dx}{x^2}\sin^2 \pi n h(xa) \tag{13}$$

is finite provided $1 - h(r) \sim r^{\frac{1}{2}+E}$, with $E > 0$, as $r \to 0$ in which case $H^{t.s.}$ goes to zero an $n \to \infty$.[6]

In conclusion, we have shown by exploiting our knowledge of the Coulomb solution and gauge invariance that the Yang-Mills field equations in the presence of a static extended external source admit solutions which completely screen the external source and which have energy as low as one wishes. But we have also shown that there is yet more structure to the Yang-Mills equations in the presence of external sources: they also admit solutions of the magnetic dipole type whose energy becomes lower than that of the Coulomb solution when gQ is larger than some critical value. These solutions cannot be transformed to a Coulomb solution by any gauge transformation since $B_i^a \neq 0$ and $\epsilon^{abc} E_i^b E_j^c \neq 0$. The generalization of the above results to larger gauge groups is trivial only if the source lies completely within a SU(2) subgroup. This and other questions related to this work will be expanded upon in a later publication.

Acknowledgements

We would like to thank our colleagues at SLAC for many useful and illuminating discussions. In particular we are grateful to L. Abbott, S. Brodsky, S. Drell, T. Eguchi, Y. Nambu, H. Quinn, J. Richardson, L. Susskind, and M. Weinstein. Work supported in part by the Department of Energy and by the National Research Council of Canada.

References and Footnotes

1. See, for example, H. G. Loos, Nucl. Phys. 72, 677 (1965) and references therein.
2. T. T. Wu and C. N. Yang, in "Properties of Matter Under Unusual Conditions," ed. by K. Mack and S. Fernbach (Interscience, New York, 1969).
3. S. Coleman, Phys. Lett. 70B, 59 (1977).
4. J. E. Mandula, Phys. Lett. 67B, 175 (1977). See also M. Magg, Aachen Preprint 78-0082 (January 1978).
5. In general, the number of invariants that characterize a source equals the rank of the group; e. g. for SU(3), the invariants are $q^a(x)\, q^a(x)$ and $d_{abc}\, q^a(x)\, q^b(x)\, q^c(x)$.