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RENORMALIZATION OF THE ONE-DIMENSIONAL QUANTUM ISING MODEL*

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ABSTRACT

Using the framework of the SLAC lattice approach it is shown that a symmetric block spinning between link and site terms in the Hamiltonian of the one-dimensional quantum Ising Model provides exact known results for both critical coupling constant and thermal exponent. The ground state energy density calculated in this way exhibits in the critical point a divergence in its second derivative.

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It is well known that the Ising Model (I.M.) in a transverse magnetic field at zero temperature and d dimensions is equivalent to the classical I.M. in d+1 dimensions, the transverse field in the former playing the role of the disordering field, the temperature in the latter. This was shown for d=1 by Pfeuty[1] and Suzuki[2]. The generalization to an arbitrary d was conjectured by many authors[3,4] and it is justified using the transfer matrix technique.[5]

For d=1 the Hamiltonian of the system is

$$H = \sum_i \left[-\frac{\varepsilon}{2}\sigma_z(i) - \Delta\sigma_x(i)\sigma_x(i+1) \right]$$

$$= \sum_i \left[-\frac{\varepsilon}{2} \right](i) + \varepsilon\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}(i) - \Delta\sigma_x(i)\sigma_x(i+1) \Big] \qquad (1)$$

$$K \equiv \frac{\varepsilon}{\Delta}$$

where σ_x and σ_z are the usual Pauli matrices. From the exact solution[1] is known that a second order phase transition occurs for K=2. The behavior of the order parameter or magnetization in this model is given by $\langle\sigma_x\rangle = \left[1-(\frac{K}{2})^2\right]^{1/8}$ for K<2 (ferromagnetic phase) and $\langle\sigma_x\rangle=0$ for K>2 (paramagnetic phase). The explicit dependence of the magnetization function shows the value of the β exponent (.125), identical to the case of the two-dimensional I.M. All the other critical exponents are also the same.

The two-dimensional I.M. and equivalent one-dimensional quantum problem have the remarkable property of self-duality.[6] For a one-dimensional lattice the sites of its dual correspond to the links of the original, and the first system can be redescribed by a new system with degrees of freedom attached to the dual lattice. In our case, the Hamiltonian (1)

is self-dual because it is invariant on passing from the original variables to their duals.[7] This is an important result because it sets a mapping between high and low K energy eigenstates and indicates the critical point K=2 as the fixed point of that correspondence.

Here we will describe a way of doing blockspinning in this model by a progressive thinning of degrees of freedom so that the site-link symmetry of (1) is maintained at every stage. This will render some exact known results, which is quite remarkable bearing in mind that we are using an intrinsically approximate method. Specifically we propose a modification of the SLAC blockspinning[8] in its application to the one dimension quantum I.M.[9]. This is a variational renormalization group (R.G.) method[10,11] in which the lattice is dissected into small blocks, each containing a few spins, which are coupled to one another by the link terms in the Hamiltonian. The Hamiltonian for the resulting few-degree of freedom problem within each block is diagonalized and the degrees of freedom thinned by keeping an appropriate set of low lying states. A new effective Hamiltonian is then constructed by computing the matrix elements of the original Hamiltonian in the space of states spanned by eigenvectors having the lowest energy eigenvalues in each block. The process is then repeated for the new effective Hamiltonian, whose coupling parameters change at each step. The procedure is iterated until reaching a regime that can be solved trivially or by perturbation theory. The important thing to emphasize here is that in this method the block Hamiltonian is formed in general by n sites and n-1 links. Our modification consists simply in omitting one of the sites in order to have a perfect balance between

both kinds of energy terms. That is, our idea is to bring to the blocks the link-site symmetry of the model. For n=2 both methods are pictorially shown in Figs. 1a and 1b.

To illustrate the new way let us find the R.G. equations using the minimum block size. So let us proceed to diagonalize one of the block Hamiltonians that appear in Fig. 1b.

$$B = \varepsilon \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} (2) - \Delta \sigma_x(1)\sigma_x(2) \tag{2}$$

Its representation in the space subtended by $|\downarrow\downarrow>$, $|\uparrow\uparrow>$, $|\downarrow\uparrow>$ and $|\uparrow\downarrow>$ is decoupled in two 2X2 matrices because B is invariant under a rotation about the z axis of π, hence the state $|\downarrow\downarrow>$ only mixes with $|\uparrow\uparrow>$ and $|\uparrow\downarrow|$ only with $|\downarrow\uparrow>$. Furthermore the two matrices we get are identical

$$\begin{bmatrix} <\downarrow\downarrow|B|\downarrow\downarrow> & <\downarrow\downarrow|B|\uparrow\uparrow> \\ <\uparrow\uparrow|B|\downarrow\downarrow> & <\uparrow\uparrow|B|\uparrow\uparrow> \end{bmatrix} = \begin{bmatrix} <\uparrow\downarrow|B|\uparrow\downarrow> & <\uparrow\downarrow|B|\downarrow\uparrow> \\ <\downarrow\uparrow|B|\uparrow\downarrow> & <\downarrow\uparrow|B|\downarrow\uparrow> \end{bmatrix} = \begin{bmatrix} \varepsilon & -\Delta \\ -\Delta & 0 \end{bmatrix}$$

The eigenvalues[12] of this matrix are

$$\lambda\pm = \frac{\varepsilon}{2} \pm \sqrt{\frac{\varepsilon^2}{4} + \Delta^2} \tag{4}$$

corresponding to the eigenvectors.

$$|\lambda+> = \frac{1}{\sqrt{1+b^2}} \begin{pmatrix} 1 \\ -b \end{pmatrix} \qquad |\lambda-> = \frac{1}{\sqrt{1+b^2}} \begin{pmatrix} b \\ 1 \end{pmatrix}$$

$$b \equiv \frac{-\frac{\varepsilon}{2} + \sqrt{\frac{\varepsilon^2}{4} + \Delta^2}}{\Delta} \tag{5}$$

The truncation is done on retaining the two lowest degenerate eigenstates and dropping the other two. In this way we obtain a new effective Hamiltonian by computing all matrix elements of H in the truncated basis generated by the block states.

$$|E1\rangle = \frac{1}{\sqrt{1+b^2}}\left(b|\downarrow\downarrow\rangle+|\uparrow\uparrow\rangle\right)$$

$$|E'_1\rangle = \frac{1}{\sqrt{1+b^2}}\left(b|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \tag{6}$$

We therefore obtain the coefficient of $\begin{pmatrix}0&0\\0&1\end{pmatrix}(j)^{13}$ as the difference

$$\varepsilon\to\varepsilon' = \langle E'_1|\varepsilon\begin{pmatrix}0&0\\0&1\end{pmatrix}(1)|E'_1\rangle - \langle E_1|\varepsilon\begin{pmatrix}0&0\\0&1\end{pmatrix}(1)|E_1\rangle = \frac{\varepsilon(1-b^2)}{1+b^2}$$

While the coefficient of $\sigma_x(j)\sigma_x(j+1)$ comes from calculating the representation of say $\Delta\sigma_x(2,j)\sigma_x(1,j+1)$ in the truncated basis, i.e.

$$\Delta\to\Delta' = \Delta z_1 z_2 = \Delta\,\frac{2b}{1+b^2} \tag{8}$$

where

$$z_1 \equiv \langle E'_1|\sigma_x(1)\ E_1\rangle = 1$$

$$z_2 \equiv \langle E'_1|\sigma_x(2)|E_2\rangle = \frac{2b}{1+b^2} \tag{9}$$

Summarizing the procedure, we have

$$H = \sum_i \left[-\frac{\varepsilon}{2}\right](i) + \varepsilon\begin{pmatrix}0&0\\0&1\end{pmatrix}(i) - \Delta\sigma_x(i)\sigma_x(i+1)\Big] \longrightarrow$$

$$\longrightarrow H' = \sum_i \left[\left\{-\varepsilon + \frac{\varepsilon}{2} - \sqrt{\frac{\varepsilon^2}{4}+\Delta^2} + \frac{\varepsilon b^2}{1+b^2}\right\}\right](j) + \tag{10}$$

$$+\left\{\varepsilon\ \frac{1-b^2}{1+b^2}\right\}\begin{pmatrix}0&0\\0&1\end{pmatrix}(j) - \left\{\Delta\ \frac{2b}{1+b^2}\right\}\sigma_x(j)\sigma_x(j+1)\Big]$$

So that the R.G. equation for K will be

$$K\to K' = K\,\frac{1-b^2}{2b} = \frac{K^2}{2} \tag{11}$$

which has as fixed points the two trivial K=0 and K=∞ and the nontrivial K*=2 coincident with the exact known result for the critical constant of the model.

The properties of the R.G. transformation near the critical point are determined by the derivative λ_T

$$\lambda_T \equiv \left.\frac{dK'}{dK}\right|_{K=K^*} \tag{12}$$

which is known as "thermal eigenvalue." For (11) $\lambda_T=2$. The "thermal exponent" y_T is related to the previous eigenvalue but it is independent of the block size we choose:

$$y_T \equiv \frac{\ell n(\lambda_T)}{\ell n(n)} \quad , \tag{13}$$

n being the number of spins per block used in the method (in this case n=2). From this we obtain $y_T=1$ which is also the exact known result of the model.

The "magnetic eigenvalue" λ_H is defined as the response to a perturbation of the form

$$h\sum_i \sigma_x(i) \tag{14}$$

due to the existence of a longitudinal magnetic field[14], of the fixed point Hamiltonian H(K*). We determine this number in our scheme by representing this perturbation in the space of the truncated basis and dividing by $\left(\frac{\Delta'}{\Delta}\right)$ in order to have the coefficient of $\left[\sum_j \sigma_x(j)\sigma_x(j+1)\right]$ always normalized to -1:

$$h \to h' = h(z_1+z_2) \tag{15}$$

$$\lambda_H = \frac{z_1+z_2}{z_1 z_2} \tag{16}$$

Analogously with the thermal case, the "magnetic exponent" is defined as

$$y_H \equiv \frac{\ell n(\lambda_H)}{\ell n(n)} \tag{17}$$

As it is well known from the R.G. theory the "physical exponents" ($\alpha,\beta,\gamma,\delta$,etc) can be calculated using Y_T and Y_H.[15] However we are going to calculate here the magnetization exponent β directly and also the ground state energy density using the iterative way described in detail in 9

The magnetization function is found after a not large (10 renormalization iterations is typically enough to reach a stable value) number q of steps in

$$M = \prod_q \left[\frac{1}{2}(z_1+z_2)\right]_q \tag{18}$$

β is found by fitting the numerical points obtained in the iteration of (18) with the function $\left[1-\left(\frac{K}{K^*}\right)^2\right]^\beta$ at points K very near to K*. In Fig. 2 we show comparatively the magnetization calculated using the symmetric and the nonsymmetric blockspinning, and the exact result.

To get the ground state energy density it is necessary to calculate the function

$$\mathscr{E}_0 = -\frac{\varepsilon}{2} + \sum_q \frac{1}{2^q}\left[\frac{\varepsilon}{2} - \sqrt{\frac{\varepsilon^2}{4}+\Delta^2} + \frac{\varepsilon b^2}{1+b^2}\right]_q \tag{19}$$

for a q large enough to obtain stationary results (q=15 is usually good). We see that this formula collects the successive coefficients of the] matrix divided by the volume of the block in each step. The final results of the energy density in the critical area are presented in Table 1 and compared with the results of the nonsymmetric blockspinning. Both methods are compared graphically with the exact result in Fig. 3. It is not

surprising that the lower result is obtained using the nonsymmetric block-spinning because in that method the quantity of energy stored in each step is

$$\varepsilon - \sqrt{\varepsilon^2+\Delta^2} \tag{20}$$

which is lower than ours

$$\frac{\varepsilon}{2} - \sqrt{\frac{\varepsilon^2}{4} + \Delta^2} + \frac{\varepsilon b^2}{1+b^2} \tag{21}$$

However the nonsymmetric blockspinning has a serious disadvantage on missing completely the divergence of the second derivative of the energy density, which is a property of the model, and which does appear with our method[16]. See Fig. 4.

The repetition of the previous analysis for a bigger block (n=3,4...) must render better results. In fact, this is the case as can be seen in Table 2. Our method maintains the exact values for K* and y_T improving progressively y_H and β. The energy density is also improved. It is an interesting fact that if the site is excluded in an inner position of the block the results are clearly better for y_H and β.

For completeness we shall put the original nonsymmetric and the new symmetric blockspinning in a general scheme. To do that let us define a variable two spins block Hamiltonian T as

$$T \equiv K\left[x \begin{pmatrix} \circ & \circ \\ \circ & 1 \end{pmatrix}(1) + y \begin{pmatrix} \circ & \circ \\ \circ & 1 \end{pmatrix}(2)\right] - \sigma_x(1)\,\sigma_x(2) \tag{22}$$

depending on two variables x and y. As usual in this method, to do a R.G. transformation, the two lowest eigenstates of T are retained and a new Hamiltonian H' is obtained on representing H in the truncated block basis.

Proceeding along this way, the fixed point Hamiltonian H* characterized by a K such that K'=K will vary for every (x,y) choice in (22), being symmetric under the $x \leftrightarrow y$ exchange.

In this scheme the original SLAC blockspinning corresponds to work in the (1,1) point and our method to use (1,0) or (0,1) as variational renormalization point.

In Fig. 5 what we call "isocritical trajectories" are displayed for a few representative cases. All the points of the XY plane that are in the same trajectory provide the same critical coupling constant in the renormalization process.

Amusingly the trajectory corresponding to K*=2 has a borderlike behavior because it is asymptotic with y=x. It is mathematical shape is specifically $y= \sqrt{x^2-1}$ and $y= \sqrt{x^2+1}$. It separates the one branch trajectories (those with K*>2) from the two branch trajectories (those with K*<2). In this case $x=y \to \infty$ in (22) leads to K*=2 simply because 2 is the coordination number of the lattice we are describing.

In general for quantum I.M. in any dimension in any sort of lattice, if we do the blockspinning 'a la SLAC' using a block where all the spins are symmetrical (all of them in an outer position) and we choose as variational point the corresponding $x=y=z=\ldots \to \infty$, the result for K* is always the coordination number of that lattice, as can be trivially checked.

The contemplation of Fig. 5 illustrates what would be a "supervariational" method somehow parallel to the one we comment in 16. In it the renormalization point would be free to run along the exact critical trajectory, $y= \sqrt{x^2-1}$ for instance, so that the final energy density be minimum at the end of the iterative path. This procedure would render

the lowest possible energy density (for a two spins block) without spoiling the correct critical constant.

We conclude that for a self-dual model like the quantum 1-dimension I.M. our method has some clear advantages over the previous nonsymmetric one. In a subsequent paper we will study the two-dimension quantum I.M. and will compare both approaches. That will be very interesting because working for a not self-dual model, both methods will compete in a more fair ground.

It seems also very promising to study another more ambitious self-dual model like the Z2 gauge in 3+1 dimensions,[16] where a self-dual block-spinning could in principle render exact results as the ones presented here.

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Finally, it is a pleasure to thank L. Kadanoff for a very positive and encouraging conversation and E. Saiz for initial computer help.

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9. See the third paper of Ref. 8.

10. Z. Friedman, Phys. Rev. Lett. 36, 1326 (1976) and K. Subbarao, Phys. Rev. Lett. 37, 1712 (1976) have developed a renormalization method a la Niemeijer and Van-Leeuwen to treat quantum Ising Models that provide fairly good results.

11. A renormalization method very close to the SLAC approach has been applied to some solid state problems. See R. Julien, J. N. Fields and S. Doniach, Phys. Rev. B16, 4889 (1977).

12. In the SLAC method described in Ref. 9 the block spectrum is not degenerate.

13. The j index is used to label block spins and the i index for original spins.

14. Notice that the addition to (1) of this kind of interaction spoils the self-duality of the model. So it is reasonable to expect for the "magnetic exponent" only approximate results.

15. See for example T. Niemeijer and J. M. Van Leeuwen in Phase Transitions and Critical Phenomena, Ed. Domb and Green, Academic Press (1976).

16. In Ref. 9 is also explained a powerful algorithm where the choice of retained states in the block is made variationally among the most general set of states such that the final energy density at the end of the iterative path be minimum. This method besides giving extremely good energy density results provides the divergence of its second derivative but in a displaced position, K*=2.75.

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TABLE 1

Ground State Energy Density

K		
.5	-1.0031	-1.0005
.6	-1.0059	-1.0009
.7	-1.0101	-1.0017
.8	-1.0156	-1.0028
.9	-1.0227	-1.0043
1.0	-1.0315	-1.0064
1.1	-1.0420	-1.0092
1.2	-1.0544	-1.0126
1.3	-1.0686	-1.0170
1.4	-1.0848	-1.0224
1.5	-1.1029	-1.0290
1.6	-1.1230	-1.0370
1.7	-1.1451	-1.0468
1.8	-1.1693	-1.0588
1.9	-1.1956	-1.0737
2.0	-1.2240	-1.0938
2.1	-1.2545	-1.1282
2.2	-1.2872	-1.1673
2.3	-1.3220	-1.2089
2.4	-1.3590	-1.2520
2.5	-1.3982	-1.2963
2.6	-1.4395	-1.3415
2.7	-1.4822	-1.3873
2.8	-1.5258	-1.4338
2.9	-1.5700	-1.4807
3.0	-1.6148	-1.5280
3.1	-1.6601	-1.5756
3.2	-1.7058	-1.6235
3.3	-1.7519	-1.6717
3.4	-1.7982	-1.7200
3.5	-1.8448	-1.7685

TABLE 2

Summary of Results for the Critical Constant and Exponents

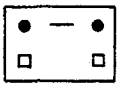
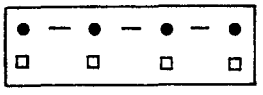
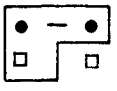
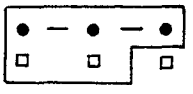
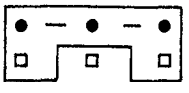
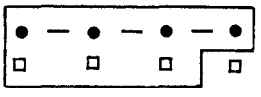
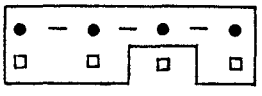
Exact Result	K*	y_T	y_H	β
	2	1	1.8750	.125
	2.55	.67	1.27	.39
	2.31	.76	1.37	.34
	2.21	.80	1.43	.30
	2	1	1.27	.23
	2	1	1.28	.21
	2	1	1.43	.20
	2	1	1.29	.20
	2	1	1.45	.19

FIGURE CAPTIONS

1. (a) Nonsymmetric blockspinning

   (b) Symmetric blockspinning

   • stands for the spin degree of freedom of the lattice

   □ for the energy site term in the (1) Hamiltonian

   —— for the energy link term in (1).

2. Magnetization

   ∘∘∘∘ Nonsymmetric block spinning

   •••• Symmetric blockspinning

   —— Exact result

3. Ground state energy density

   ∘∘∘∘ Nonsymmetric blockspinning

   •••• Symmetric blockspinning

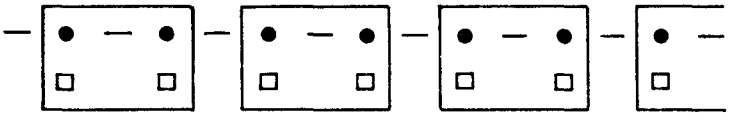
   —— Exact result

4. Second derivative of the ground state energy density

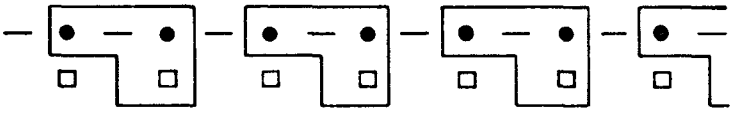
   ∘∘∘∘ Nonsymmetric blockspinning

   •••• Symmetric block spinning

5. "Isocritical map" in the space of variational parameters (see explanation in text).



(a)



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(b)

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Fig 1

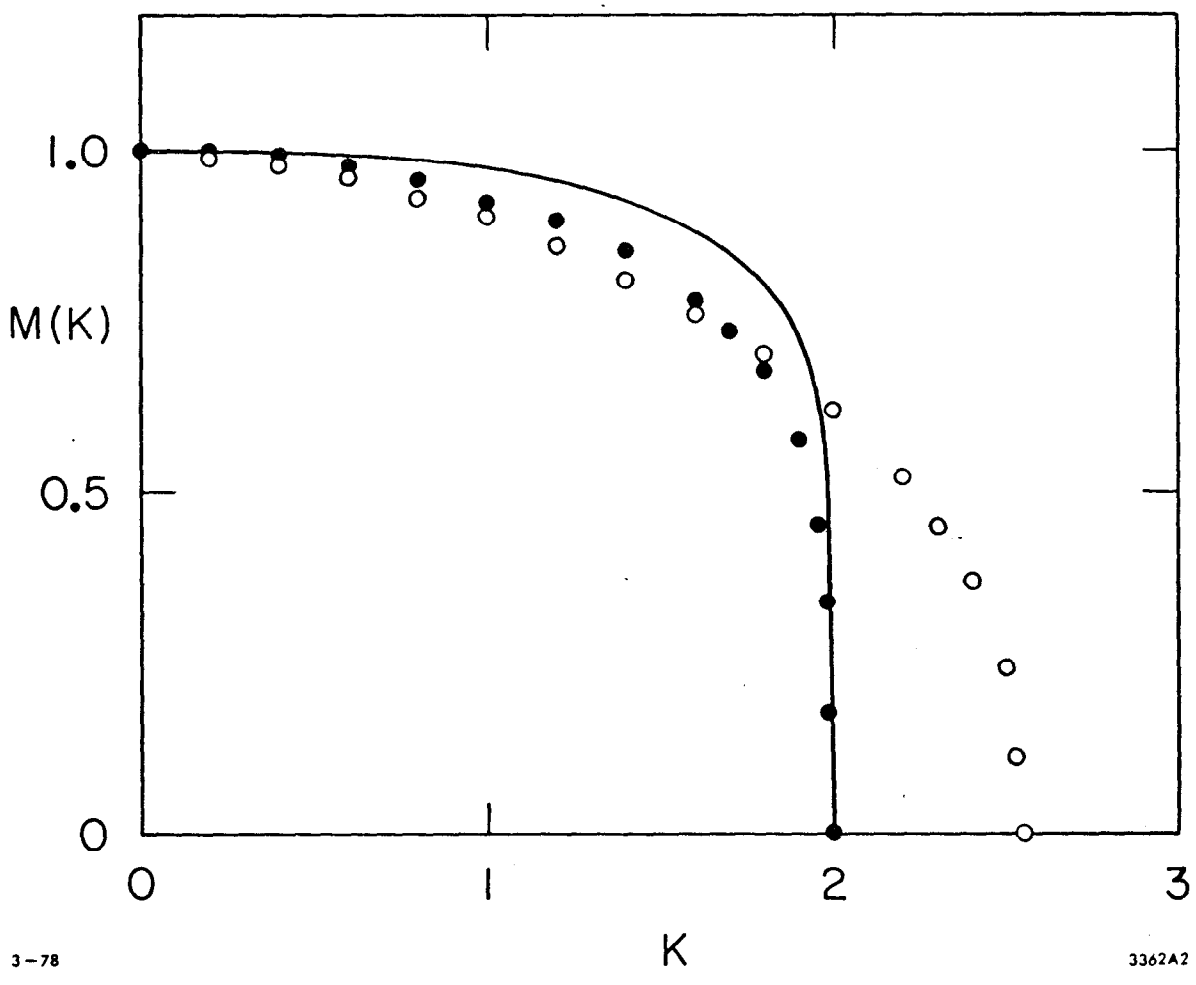

Fig. 2

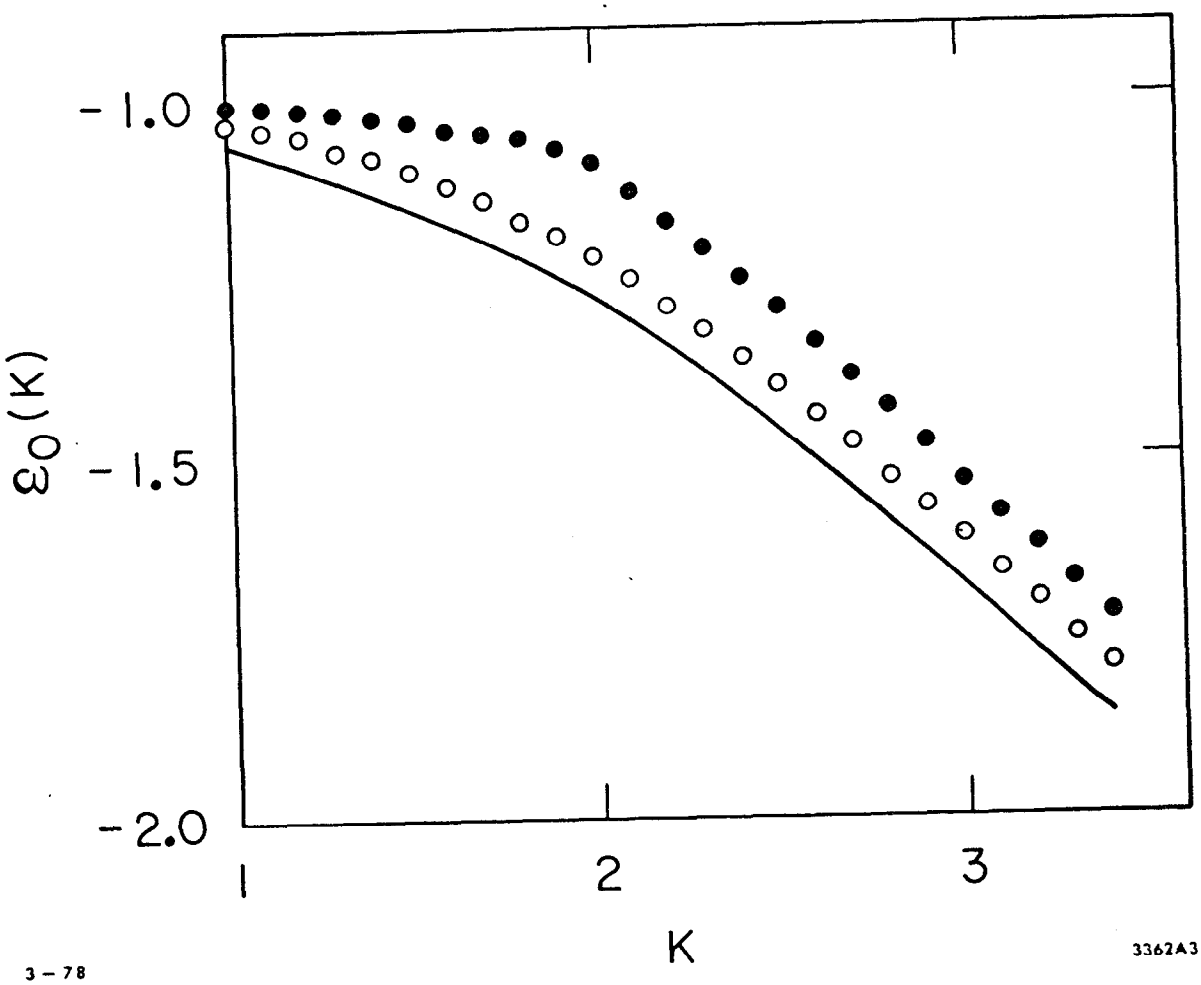

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Fig. 3

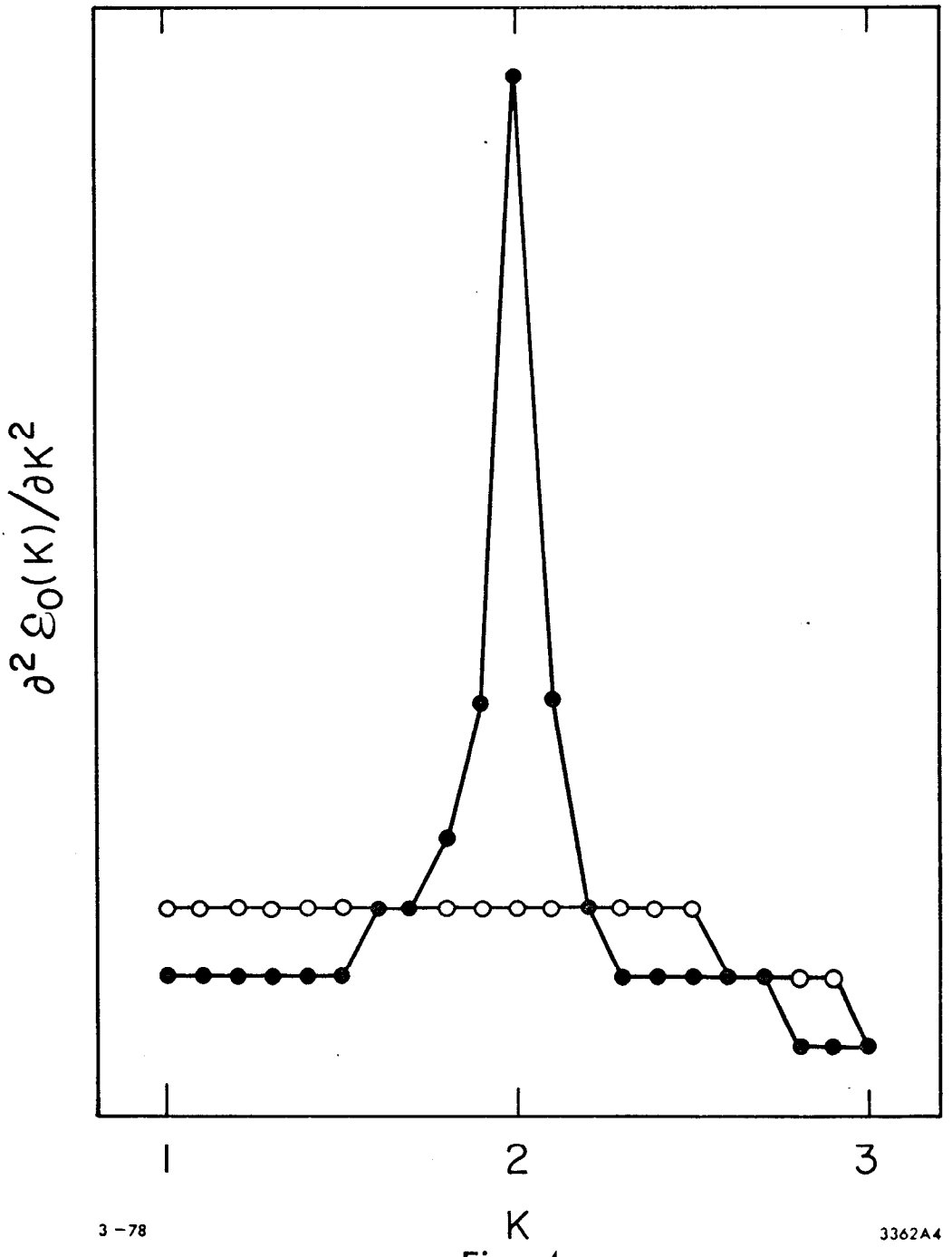

Fig. 4

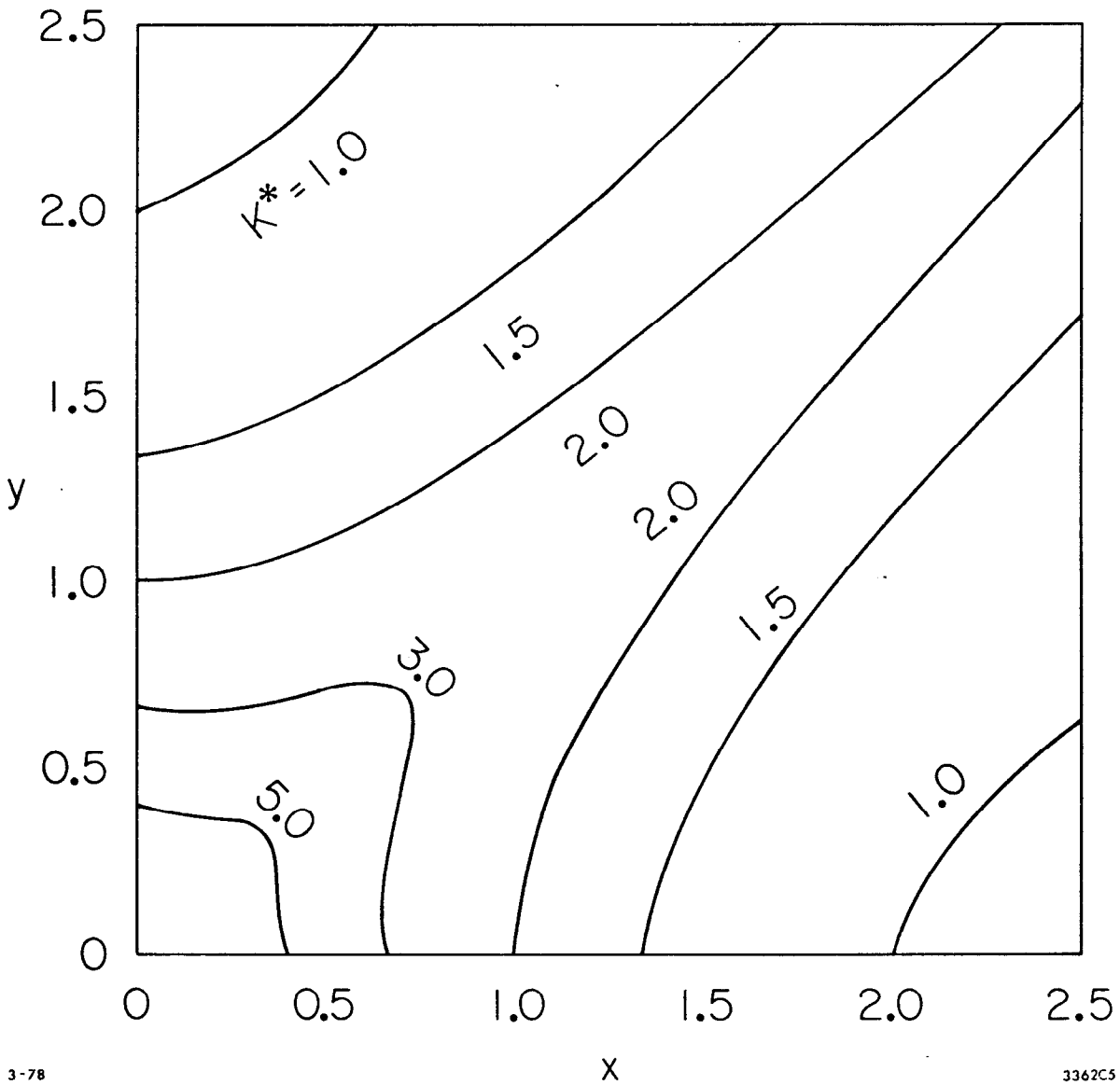
2.5
2.0
1.5
1.0
0.5
0
y
0
0.5
1.0
1.5
2.0
2.5
x
K* = 1.0
1.5
2.0
2.0
1.5
1.0
3.0
5.0
3-78
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Fig. 5