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PHASE TRANSITION IN THE NONLINEAR SU(n) LATTICE FIELD THEORY*

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ABSTRACT

We solve for the phase diagram (phase transition) of the nearest-neighbor coupling SU(n) nonlinear scalar action. We do a one-loop calculation for the effective coupling constant, and find that the system undergoes a phase transition for d>2. For d=2, the theory is asymptotically free.

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I. INTRODUCTION

We use the methods of the lattice gauge theory[1] to study the lattice version of the nonlinear scalar action. We are primarily interested in its critical properties. We have shown[2] that there is an exact mapping from the scalar action to a version of the lattice gauge theory. The way this is done is to form a vector field by considering d(=dimension) uncoupled scalar fields as components of the vector field. A gauge transformation is then defined for the vector field, and the gauge-invariant action is defined by integrating the gauge-transformed vector field over all possible gauge-transformations. Note the gauge-invariant action is invariant by construction. The phase transition properties of the nonlinear scalar action would also be possessed by their mapping into the lattice gauge theory. A phase transition in this theory implies a phase transition separating the confining phase from the asymptotically free phase in the corresponding gauge theory. The theory studied here is also involved in defining the transfer matrix for the lattice gauge theory.[1]

In statistical mechanics the SU(2) lattice nonlinear scalar action is called the O(4) symmetric Heisenberg model. The O(n) Heisenberg model in weak coupling was solved by Polyakov[3], and he showed that the theory possessed a phase transition for $d>2$. His results provide a check for the results of this calculation.

Consider a d-dimensional N^d periodic Euclidean lattice. Let U_n be an SU(n) matrix at the lattice site n. We define the lattice nonlinear scalar action by

$$A = \frac{-1}{g^2} \sum_{n\mu} \mathrm{Tr}(U_n U^+_{n+\mu} + U_{n+\mu} U^+_n) \tag{1.1}$$

$$= \frac{-1}{g^2} \sum_{n\mu} \mathrm{Tr}(\delta_\mu U_n \delta_\mu U^+_n) + \text{constant} \tag{1.1'}$$

where $\delta_\mu f_n = f_{n+\mu} - f_n$, μ the unit basis vectors of the lattice.

The quantum theory is defined by integrating e^A over all possible values for the U_n matrices. Then, the Feynman path integral is defined by

$$Z = \prod_n \int dU_n \, e^{A[U]} = \langle e^A \rangle \qquad (1.2)$$

(where dU_n is the invariant measure).

We will be interested in the theory for $g^2 \simeq 0$. We will make an expansion in g^2. This allows us to make certain approximations; to do so, we introduce the following notation. Let $\{X^a\}$ be the generators of the SU(n) lie algebra; then[4]

$$\begin{aligned} X^a X^b &= \frac{1}{2n}\, \delta^{ab} + \frac{1}{2}\, (d^{ab\alpha} + i\, c^{ab\alpha}) X^\alpha \\ \mathrm{Tr}(X^a X^b) &= \delta^{ab}/2 \end{aligned} \qquad (1.3)$$

where c^{abc}, d^{abc} are respectively the completely anti-symmetric/symmetric tensors. We will need the following formula[4]

$$\begin{aligned} \mathrm{Tr}(X^a X^b X^c X^d) = \frac{1}{8} \Big(&\frac{2}{n}\, \delta^{ab}\, \delta^{cd} - c^{ab\alpha}\, c^{cd\alpha} \\ &+ d^{ab\alpha}\, d^{cd\alpha} - i\, c^{ab\alpha}\, d^{cd\alpha} - i\, d^{ab\alpha}\, c^{cd\alpha} \Big) \end{aligned} \qquad (1.3')$$

Also

$$U_n = \exp\{ i B_n^\alpha X^\alpha \} \qquad (1.4)$$

where $\{B_n^\alpha\}$ is the (compact) scalar quantum field.

For $g^2 \simeq 0$, we can expand the action into a power series of the $\{B_n^\alpha\}$ variables. The measure $dU_n = \mu(B_n) \prod_\alpha dB_n^\alpha$ where $\mu(B) = \exp(-\frac{n}{24} B^2) + 0(B^3)$, gives us

$$Z = \prod_{n\alpha} \int dB_n^\alpha \, \mu(B_n) e^{A[B]} \qquad (1.5)$$

The action supplies a (massless) Gaussian measure for the $\{B_n^\alpha\}$ variables, which allows us to "ignore" the compactness of the B_n^α variable and gives

$$Z \simeq \prod_{n\alpha} \int_{-\infty}^{+\infty} dB_n^\alpha \, \mu(B_n) e^{A[B]} < \infty \qquad (1.6)$$

(where we are ignoring a single variable above which is not bounded by the pure gradient coupling). We call (1.6) the weak coupling approximation for the nonlinear theory. The weak coupling calculation for the critical coupling g_c^2 is self-consistent only if $g_c^2 \ll 1$.

II. EVALUATING THE VERTEX FUNCTION

From Eqs. (1.1) and (1.4) and from the results of Ref. 4, we have, for $g \simeq 0$

$$A = -\frac{1}{2g^2} \sum_{n\mu} (\delta_\mu B_n)^2 + \frac{1}{2g^2} \cdot \frac{1}{12} \sum_{n\mu} (B_n \times B_{n+\mu})^2$$

$$+ \frac{1}{2g^2} \cdot \frac{1}{48} \sum_{n\mu} \left\{ \frac{2}{n} (\delta_\nu B_n)^2 (\delta_\nu B_n)^2 + (\delta_\nu B_n * \delta_\nu B_n)^2 \right\} + O(B^6) \qquad (2.1)$$

$$= A_0 + A_1 + A_2 \qquad (2.1')$$

where

$$a \cdot b = a^\alpha b^\alpha \ , \ (a \times b)^\alpha = c^{\alpha\beta\gamma} a^\beta b^\gamma , \ (a * b)^\alpha = d^{\alpha\beta\gamma} a^\beta b^\gamma \qquad (2.2)$$

The term A_2 in (2.1) is not interesting in this calculation. Firstly, because power counting shows that A_2 cannot affect the critical behavior of the A_1 vertex (at least to one-loop order). Secondly, due to the number of derivatives it carries, it cannot contribute to the quadratic mass divergence. Hence, we will entirely ignore the A_2 term in further discussions.

We are interested in Feynman perturbation theory, and hence we fourier transform the variables. Let

$$B_n^\alpha = \frac{1}{N^d} \prod_\mu \sum_{k_\mu = 0}^{2\pi(N-1)/N} e^{ik_\mu n_\mu} B_k^\alpha \equiv \sum_k e^{ikn} B_k^\alpha$$

$$\delta(k-q) = N^d \prod_\mu \delta_{k_\mu , q_\mu} . \qquad \text{Note } B_k^\alpha = \sum_n e^{-ikn} B_n^\alpha$$

Let

$$r_\mu(q_i, q_j) = e^{iq_{i\mu}} - e^{iq_{j\mu}}$$

$$d_q = \sum_\mu \left| 1 - e^{iq_\mu} \right|^2 \qquad (2.3)$$

Then, with appropriate symmetrization, we have

$$A = -\frac{1}{2g^2}\sum_q d_q B^\alpha_{-q} B^\alpha_q$$

$$+\frac{1}{24g^2}\cdot\frac{1}{4}\, c^{ij\alpha} c^{k\ell\alpha} \sum_{q,\ldots q_4} \delta(\sum_i q_i) r_\mu(q_1,q_2) r_\mu(q_3,q_\mu) B^i_{q_1} B^j_{q_2} B^k_{q_3} B^\ell_{q_4} \tag{2.4}$$

The four-point vertex in (2.4) is symmetric on the first two and last two indices. It has to be further symmetrized to be completely symmetric on all four indices. Define the symmetric vertex function

$$f^{ijk\ell}_{q_i q_j q_k q_\ell} = \frac{1}{12}\Big\{ c^{ij\alpha} c^{k\ell\alpha} r_\mu(q_i,q_j) r_\mu(q_k,q_\ell) + (i\longleftrightarrow k) + (i\longleftrightarrow \ell)\Big\} \tag{2.5}$$

giving

$$A_1 = \frac{1}{24g^2}\sum_{q_1,\ldots q_4} \delta(\sum_i q_i) f^{ijk\ell}_{q_1q_2q_3q_4} B^i_{q_1} B^j_{q_2} B^k_{q_2} B^\ell_{q_4} \tag{2.6}$$

(a) Mass Renormalization

Define

$$D^{\alpha\beta}_q = \frac{1}{N^d}\frac{1}{g^2} \langle B^\alpha_{-q} B^\beta_q e^A\rangle / Z \tag{2.7}$$

$$= \text{Fig. 1} \quad , \tag{2.8}$$

where the last diagram in Fig. 1 comes from the measure. Hence

$$D^{\alpha\beta}_q = \frac{\delta^{\alpha\beta}}{d_q - \pi_q} \tag{2.9}$$

where

$$\delta^{\alpha\beta}\pi_q = g^2 \frac{1}{2}\sum_p f^{\alpha\beta ii}_{q,-q,p,-p}\Big/ d_p - 2g^2 \delta^{\alpha\beta}\frac{n}{24} \tag{2.10}$$

$$\therefore \delta m = \pi_{q=0} = g^2 n \left[\frac{1}{2}\cdot\frac{1}{6}\sum_p \frac{d_p}{d_p} - \frac{1}{12}\right] = 0$$

Hence, there is no mass renormalization and $\pi_q \simeq q^2$ for small q. This implies that the theory preserves its symmetry of being a pure gradient coupling theory. A mass divergence would have altered the calculation for the vertex function.

The symmetry which forbids a mass counterterm is the invariance of the action under $U_n \to U_n V$, $V \in SU(n)$ [which is violated by the mass term $m_0^2 \mathrm{Tr}(U_n + U_n^+)$] .

(b) Coupling Constant Renormalization

Define the effective (four-point) vertex function by

$$\Gamma^{IJKL}_{q_I q_J q_K q_L} = \frac{1}{g^6} \langle B^I_{-q_I} B^J_{-q_J} B^K_{-q_K} B^L_{-q_L} e^A \rangle / Z \tag{2.11}$$

$$= \text{Fig. 2(a)} + \text{Fig. 2(b)} + \text{Fig. 2(c)} \tag{2.12}$$

$$\equiv \frac{\delta(q_I + q_J + q_K + q_L)}{d_{q_I} d_{q_J} d_{q_K} d_{q_L}} \left[\Gamma^{(o)} + \Gamma^{(1)} + \Gamma^{(2)}\right] \tag{2.13}$$

We have defined Γ such that as $g \to 0$, Γ in leading order $\sim 1/g^2$. The symmetrization (for the one-loop graphs) on the external legs plays an important role in combining the graphs. The last graph $\Gamma^{(2)}$ with a tadpole graph given by Fig. 2(c) comes from the $O(B^6)$ terms in the action. This graph gives a momentum independent contribution and enters in the calculation with the opposite sign than that of the one-loop graphs $\Gamma^{(1)}$; it plays an essential role in causing the phase transition since it makes g_c^2 positive. Note

$$\Gamma^{(o)} = \frac{1}{24g^2} f^{IJKL}_{q_I q_J q_K q_L} \tag{2.14}$$

$$\Gamma^{(1)} = \text{Fig. 2(b)} \tag{2.15}$$

On calculating $\Gamma^{(1)}$, we find that it is a linear combination (with appropriate permutations) of the following: $c^{IJ\alpha}c^{KL\alpha}$, $c^{IJ\alpha}d^{KL\alpha}$, $d^{IJ\alpha}d^{KL\alpha}$ and $\delta^{IJ}\delta^{KL}$. We keep terms of the first generic type. The remaining tensors mix with the symmetric piece of the action (the term A_2 in Eq. (2.21)). The vertex defined by $c^{IJ\alpha}c^{KL\alpha}$ contains the phase transition. Keeping only these terms we have (up to a symmetrization on IJKL), after a lot of algebra

$$\Gamma^{(1)} = \frac{c^{IJ\alpha}c^{KL\alpha}}{2}\left(\frac{1}{12}\right)^2 \left\{\frac{9n}{4}\, r_\mu(q_I, q_J)\, \mathcal{J}_{\mu\nu}(q_K+q_L) r_\nu(q_K, q_L) + \tilde{\Gamma}\right\} + \text{the other SU(n) tensors} \tag{2.16}$$

where

$$\mathcal{J}_{\mu\nu}(p) = \sum_q \frac{r_\mu(q,-q-p) r_\nu^*(q,-q-p)}{d_q d_{p+q}} \tag{2.17}$$

and

$$\tilde{\Gamma} = \frac{n}{2}\sum_{q,q'} \frac{1}{d_q d_{q'}}\left\{r_\mu(q_K, q') r_\mu(q_L, q) r_\nu(q_L, -q') r_\nu(q_I, -q)\delta(q_I+q_J+q+q')\right\} - (J \leftrightarrow K) \tag{2.18}$$

Firstly, note that $\tilde{\Gamma}=0$ when all the external legs are set to zero. Secondly, note that $\tilde{\Gamma}$ is not simply a renormalization of the bare vertex function as is the other term in $\Gamma^{(1)}$. It will not enter our calculations and we drop it.

We analyze the function $\mathcal{J}_{\mu\nu}(p)$; it contains information on how the effective coupling constant behaves at momentum of O(p). The long distance property is contained in $p \simeq 0$. Doing an expansion about p=0, we find, for $d > 2$ dimensions

$$\mathcal{J}_{\mu\nu}(p) = \delta_{\mu\nu}(J - \alpha p^{d-2}) + O(p^d) \tag{2.19}$$

where

$$J = \sum_q \left|e^{iq_\mu} - e^{-iq_\mu}\right|^2 / d_q^2 \tag{2.20}$$

and, for $2 < d < 4$

$$\alpha = \frac{5}{4d}\cdot\frac{\Gamma(2-d/2)\Gamma^2(d/2-1)}{(4\pi)^{d/2}\Gamma(d-2)} \tag{2.21}$$

(α is well-defined for all $d > 2$). Therefore, for $p \simeq 0$, making the approximations mentioned gives

$$\Gamma^{(1)} = \frac{c^{IJ\alpha} c^{KL\alpha}}{2(12)^2} \frac{9n}{4} (J - \alpha p^{d-2}) r_\mu(q_I, q_J) r_\mu(q_K, q_L) \tag{2.22}$$

$$+ \text{ 2 permutations}$$

The last remaining graph is the tadpole diagram. This comes from the $O(B_n^6)$ terms of the action. All terms of $O(B_n^6)$ do not contribute; only those contribute which, under one contraction, can give rise to a vertex of the form $(B_n \times B_{n+\mu})^2$. Hence, from the action $\mathrm{Tr}(U_n U^+_{n+\hat{\mu}} + U_{n+\hat{\mu}} U^+_n)$ we keep only the terms of the type $O(B_n^2 B_{n+\mu}^4)$, $O(B_n^4 B_{n+\mu}^2)$ and $O(B_n^3 B_{n+\mu}^3)$. The terms of the type $O(B_n^5 B_{n+\mu})$, $O(B_n B_{n+\mu}^5)$ and $O(B_n^6)$ do not contribute (in lowest order) to the vertex function. Hence

$$A = \frac{1}{g^2} \sum_{n\mu} \mathrm{Tr}(U_n U^+_{n+\mu} + U_{n+\hat{\mu}} U^+_n) \tag{2.23}$$

$$= A_0 + A_1 + A_2 + A_3 \tag{2.24}$$

where

$$A_3 = \frac{i^2}{4g^2} \sum_{n\mu} \left\{ \frac{1}{2!\,4!} (B_n^2 + B_{n+\hat{\mu}}^2) + \frac{1}{3!} B_n \cdot B_{n+\hat{\mu}} \right\} (B_n \times B_{n+\hat{\mu}})^2 \tag{2.25}$$

$$+ \text{ other terms of } O(B^6)$$

Let (suppressing the non-abelian indices)

$$v(q_I, q_J, q_K, q_L) = \frac{c^{IJ\alpha} c^{KL\alpha}}{12} r_\mu(q_I, q_J) r_\mu(q_L, q_K) \tag{2.26}$$

Then

$$\frac{\delta(q_I + \ldots + q_L)\Gamma^{(2)}}{d_{q_I} \ldots d_{q_L}} = \frac{1}{g^6} \langle B^I_{q_I} \ldots B^L_{q_L} A_3 e^{A_0} \rangle / Z \tag{2.27}$$

Therefore

$$\Gamma^{(2)} = \frac{i^2 n}{12} (5I - \frac{1}{d}) v(q_I, \ldots q_L) \tag{2.28}$$

where

$$I = \sum_q \frac{1}{d_q} \tag{2.29}$$

We have computed all the required Feynman diagrams. Collecting the terms which are pertinent to the phase transition, we have from (2.11), (2.14), (2.22) and (2.27)

$$\Gamma^{IJKL}_{q_I \ldots q_L} = \frac{1}{12} \cdot \frac{1}{2g^2(q_K + q_L)} \cdot v(q_I \ldots q_L) \tag{2.30}$$

$$+ \text{ 2 permutations}$$

where

$$\frac{1}{2g^2(p)} = \frac{1}{2g^2} - n\left(5I - \frac{1}{d} - \frac{9J}{8}\right) - \frac{9n}{8} \alpha p^{d-2} \tag{2.31}$$

Anticipating later results, let

$$\frac{1}{2g_c^2} = 5I - \frac{1}{d} - \frac{9J}{8} \tag{2.32}$$

Note that the <u>sign</u> of g_c^2 determines whether the system can go critical or not. For $g_c^2 \geqslant 0$, there is a phase transition; for $g_c^2 < 0$, there is no phase transition. In the Appendix, we show that $g_c^2 > 0$. Hence, to $O(g^2)$, we have ($\beta = -\frac{9n}{4}\alpha$)

$$g^2(p) = \frac{g^2}{1 - ng^2/g_c^2 + g^2 \beta p^{d-2}} \tag{2.33}$$

Note it is important to consider the original equation for $g^2(p)$ as an equation for $1/g^2$ (p) as in (2.31); otherwise the equations are altered. The result above is the SU(n) analog of the result obtained (using different methods) by Polyakov[3] for the O(n) symmetric Heisenberg model.

III. RENORMALIZATION GROUP AND PHASE TRANSITION

Using the results of Wilson[5] and Polyakov[3], we analyze the equation for $g(p)$. To make the analysis more transparent, we first reinterpret the results of the vertex function calculation in the modern renormalization group language. Consider an infinite size lattice. We rewrite Eq. (2.4) doing a rescaling by g of the fields, i.e.

$$A = -\frac{1}{2}\int_q d_q B_{-q} B_q + \frac{g^2}{24}\int_{q_1,\ldots q_4} \delta\left(\sum_i q_i\right) v(q_1,\ldots q_4) B_{q_1}\ldots B_{q_4} \tag{3.1}$$

where we have ignored the non-Abelian indices. For $N\to\infty$, $\sum_q \to \int_q \equiv \int_{-\pi}^{+\pi} \frac{d^d q}{(2\pi)^d}$

Consider the vertex function $\Gamma_{k_1k_2k_3k_4}$ given by our calculation, restricting, however, all the external momenta to take values in the interval $-2^{-\ell}\pi \leqslant k_i \leqslant +2^{-\ell}\pi$ (the 2 in $2^{-\ell}$ is arbitrary). Let

$$\int_k \equiv \int_{-2^{-\ell}\pi}^{+2^{-\ell}\pi} \frac{d^d k}{(2\pi)^d} .$$

We then consider $\Gamma_{k_1k_2k_3k_4}$ to be effective vertex of the effective action that would describe the physics for momentum of $O(2^{-\ell})$. We define, using Eq. (2.29)

$$A_{eff} = -\frac{1}{2}\int_k d_k B_{-k} B_k + \frac{1}{24}\int_{k_1,\ldots k_4} \delta\left(\sum_i k_i\right) g^2(k_3+k_4)\, v(k_1,\ldots k_4) B_{k_1}\ldots B_{k_4} \tag{3.2}$$

Before we go further, we point out that A_{eff} can be considered as having been obtained by a renormalization group transformation from the original action. The original theory has the higher momentum modes (degrees of freedom) running from $2^{-\ell}\pi$ to π in addition to the lower momentum modes. We can break up the variables $\{B_q\}$ into B_k when $|q| \leqslant 2^{-\ell}\pi$ and B_p when $|q| = |p| > 2^{-\ell}\pi$. We then integrate out all the high momentum variables $\{B_p\}$ giving the new action A_{eff}. That is, following Kogut and Wilson,[6] the renormalization group transformation is given by

$$e^{A_{eff}[B_k]} = \prod_p \int_{-\infty}^{+\infty} dB_p \, \mu[B_p] \, e^{A[B_k, B_p]} \tag{3.3}$$

A_{eff} can be thought of as follows. On the original lattice, combine a d-dimensional "block" of variables numbering $2^{\ell d}$ into a single new variable. The new system has 2^{ℓ} lattice spacing between the (new) variables and its behavior is given by A_{eff}.

To reach a non-trivial fixed point, we have to rescale the momenta and field variables of the action A_{eff} $[B_k]$, and bring it into the form of the original action. We define the following change of scale

$$k = 2^{-\ell} q \quad , \quad B_k = \zeta_\ell B_q \quad , \quad \int_k = 2^{-\ell d} \int_q$$

$$\delta(k) = 2^{\ell d} \, \delta(q)$$

We apply the rescaling to functions d_k and $v(k_1, \ldots k_4)$ by expanding to leading order in k, rescaling to q, and rewriting the function. This gives

$$d_k \simeq 2^{-2\ell} d_q \quad ; \quad v(k_1 \ldots k_4) \simeq 2^{-2\ell} \, v(q_1 \ldots q_4) \tag{3.5}$$

We choose ζ_ℓ such that the coefficient of the quadratic term in A_{eff} $[B_q]$ is independent of ℓ. This fixes ζ_ℓ to be

$$\zeta_\ell = 2^{\ell(d+2)/2} \tag{3.6}$$

This then gives ($\epsilon \equiv d-2$)

$$A_{\text{eff}} = -\frac{1}{2}\int_q d_q B_{-q} B_q + \frac{1}{24}\int_{q_1,\ldots q_4} \delta(\Sigma q_i) 2^{-\ell\epsilon} g^2\left(2^{-\ell}(q_3+q_4)\right) v(q_1\ldots q_4) B_{q_1}\ldots B_{q_4} \tag{3.7}$$

Therefore, we find the effective coupling for A_{eff}, which describes physics of momenta $O(2^{-\ell})$, is given by

$$g_\ell^2(p) = 2^{-\ell\epsilon}\, g^2(2^{-\ell}p) \tag{3.8}$$

In the continuum field theory, the bare coupling would be the dimensional coupling constant $g_0^2 = g^2\,\Lambda^{-(d-2)}$, where Λ is some momentum scale. The effective coupling constant is dimensionless, and in the continuum theory is given by the dimensionless function $p^{d-2} g_0^2(p)$, where $g_0^2(p)$ is the dimensional vertex function. On the lattice $g_\ell^2(p)$ is the analog of the dimensionless coupling constant of the continuum theory.

From (2.32), we have

$$g_\ell^2(p) = \frac{g^2\, 2^{-\ell\epsilon}}{1-ng^2/g_c^2 + g^2\beta\, 2^{-\ell\epsilon} p^\epsilon} \tag{3.9}$$

The critical action is given by the fixed point action, for which the system is the same for all scales of momenta, i.e.

$$g_\ell^*(p) = g^*(p) \text{ for all } \ell \tag{3.10}$$

The system also reaches a fixed point action as $\ell \to \infty$. For $ng^2 < g_c$, we have

$$\lim_{\ell\to\infty} g_\ell^2(p) \sim 2^{-\ell\epsilon} \to g^* = 0 \tag{3.11}$$

That is, the theory goes to the trivial massless free-field fixed point. For $ng^2 = g_c^2$, we have the system at the phase transition, since for all ℓ we have

$$g_\ell^2(p) = \frac{1}{\beta} \frac{1}{p^{d-2}} = g^*(p) \tag{3.12}$$

The effective coupling is invariant to the renormalization group hence giving the fixed point coupling $g^*(p)$. This is a non-trivial fixed-point which is given by a four-field interaction. Note that the fixed point action A^* is highly non-local. The interacting piece of A^*, in position space, has the form, for $|n-m| \gg 1$

$$A_I^* \sim \frac{1}{\beta} \sum_{nm\mu\nu} (B_n \times B_{n+\hat{\mu}}) \frac{1}{(n-m)^2} (B_m \times B_{m+\nu})$$

$$\sim \frac{1}{\beta} \sum_{nm\mu\nu} \mathrm{Tr}(X^\alpha U_n U_{n+\hat{\mu}}^+ U_n^+ U_{n+\hat{\mu}}) \frac{1}{(n-m)^2} \mathrm{Tr}(X^\alpha U_m U_{m+\nu}^+ U_m^+ U_{m+\nu}) + O(B^5)$$

Note, from (2.21) $\beta \sim 1/\epsilon$ as $\epsilon \to 0$. This non-local action has no transfer matrix/ Hamiltonian. For $ng^2 \gg g_c^2$, a direct calculation using the compact degrees of freedom shows that the theory, under renormalization, goes to the completely random (disordered) phase. That is,

$$\frac{1}{g_\ell^2} \sim \left(\frac{1}{g^2}\right)^{2\ell}, \quad g^2 \gg 1 \tag{3.13}$$

and $\lim_{\ell \to \infty} g_\ell^2 \to g^* = \infty$. The $g^* = \infty$ fixed point is the completely disordered phase. Hence, we have the following tentative renormalization flow diagram given by Fig. 3. The shaded area is not directly accessible to our calculation. The arrows on the lines show the direction in which g_ℓ changes as the ℓ increases, that is, as it approaches large distances characterized by 2^ℓ. For $d = 2 + \epsilon$, it can be shown that $g_c \simeq \epsilon$; hence the theory is asymptotically free for $\epsilon=0$.

The $\epsilon \to 0$ limit has to be taken carefully. All the lattice constants diverge as $1/\epsilon$. For a well-defined limit, the coefficient of the $1/\epsilon$ term must be exactly zero. Note since $g_c^2 \simeq 1.08$ for d=4, the calculation is self-consistent at best up to d=4.

IV. DISCUSSION

We have deduced the fact that there is a phase transition in the theory for $g^2 n = g_c^2$. To see this, define the propagator

$$D_n = \langle \mathrm{Tr}(U_0 U_n^+) e^A \rangle / Z \tag{4.1}$$

For $g^2 \gg 1$, we have (for $|n| \gg 1$),

$$D_n \sim \left(\frac{1}{g^2}\right)^{|n|} \rightarrow 0 \tag{4.2}$$

and for $g^2 \simeq 0$, we have

$$D_n \sim e^{-\frac{g^2}{2}\left(\frac{1}{|n|^{d-2}}\right)} \sim 1 + O(g^2) \tag{4.3}$$

The phase for $ng^2 < g_c^2$ is the ordered phase, having infinite distance correlation. The phase for $ng^2 \gg g_c^2$ is the disordered phase characterized by finite distance correlation. As shown in Fig. 3, our calculation cannot accurately establish that there are no other phases between the disordered and ordered phase. For the following discussion, however, we assume that there are only two phases.

We find that the $g^*=0, \infty$ fixed points are stable, and that the $g^*=g_c$ fixed point is twice unstable.

From the renormalization flow diagram (Fig. 4) we see that, due to the existence of the fixed point $g^*=g_c$, we can obtain two distinct renormalized continuum field theories in the following way (see Wilson[5]). We can construct[5] one continuum theory by approaching g_c from below. This gives a theory whose dimensional coupling at infinite momentum is given by $\lim_{\Lambda \to \infty} g_0^2 = g_c^2 \Lambda^{-\epsilon}$. As one goes to larger distance, the strength of the coupling decreases (as is the case for QED in perturbation theory). The other continuum theory is obtained by approaching g_c^2 from above. Then, in that theory, the zero distance interaction is also given by g_c^2, but the strength of the coupling increases indefinitely for large distances and is, for large distances, like the quark confinement phase. These two continuum theories are schematically shown in Fig. 4. The theory by itself cannot decide which phase to choose for the physical renormalized system. In a sense, these two phases which give opposite long distance behaviour complement each other since they are simply two different phases of the same underlying field. It may be possible to map one phase into the other using a dual transformation on the field variables.

A remaining question is whether the renormalized theory has the full Euclidean symmetry so that the analytic continuation to real time is relativistic. I have not studied this problem. The results for SU(n) obtained contain no information about the U(1) theory (xy model), since the Abelian theory doesn't contain the vertex studied. However, it is known from statistical mechanics that the xy-model has a phase transition for $d > 2$.

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APPENDIX

From Eq. (2.32) we have

$$\frac{1}{2g_c^2} = 5I - \frac{1}{d} - \frac{9J}{8} \tag{A.1}$$

We show $g_c^2 > 0$. Recall

$$I = \sum_q \frac{1}{d_q} \tag{A.2}$$

$$J = \sum_q |e^{iq_\mu} - e^{-iq_\mu}|^2 / d_q^2 \tag{A.3}$$

Let

$$K = \sum_q \frac{1-\cos q_\mu \cos q_\nu}{d_q^2} \quad (\mu \neq \nu) \tag{A.4}$$

Note I, J, K > 0.

From $\sum_q (d_q^2/d_q^2) = 1$, we have the identify

$$J = 4I - 4(d-1)J - \frac{1}{d} \tag{A.5}$$

Therefore, from (A.1) and (A.5)

$$\frac{1}{g_c^2} = I + 9(d-1)K + \frac{1}{4d} > 0 \tag{A.6}$$

For $\epsilon \simeq 0$, $g_c^2 \simeq \epsilon$ since I, J, $K \sim \frac{1}{\epsilon}$. An accurate calculation for d=4 shows, using (A.6) that

$$g_c^2 \simeq 1.086 \quad , \quad d=4 \tag{A.7}$$

The weak coupling approximation is valid if g_c^2 is small. The weak coupling approximation, optimistically, is valid up to d=4. Note that in the continuum theory, by dimensional analysis, all the finite lattice constants would diverge like Λ^{d-2}, and would make the perturbation theory look divergent.

FIGURE CAPTIONS

1. Feynman diagrams for the propagators

2. (a) Lowest order Feynman diagram for the vertex function.

   (b) The one-loop contribution, with the appropriate symmetrization, to the vertex function.

2. (c) The tadpole diagram contribution to the vertex function.

3. The renormalization group flow diagram for the nonlinear model. The shaded part indicates the domain not directly accessible to our calculation.

4. The two continuum renormalized trajectories obtained from the (lattice) cutoff theory.

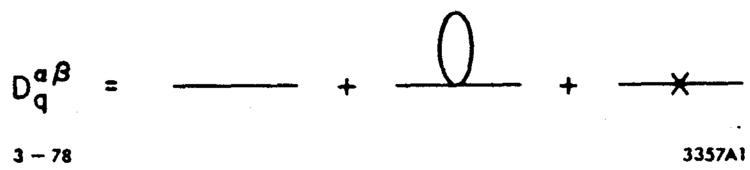


Fig. 1

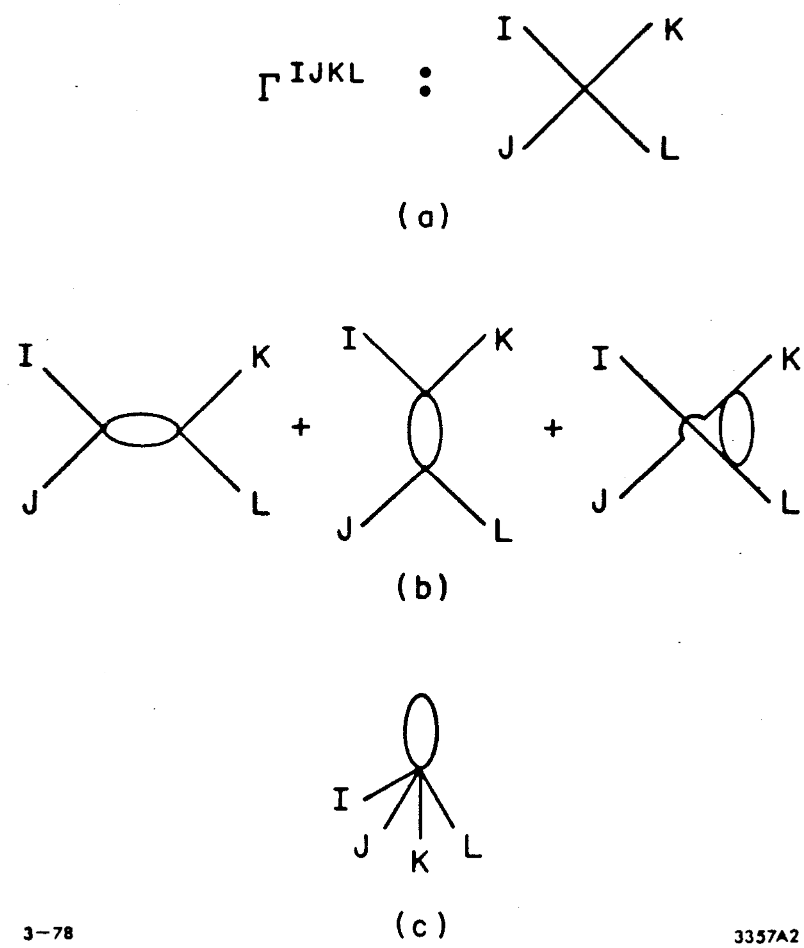


Fig. 2

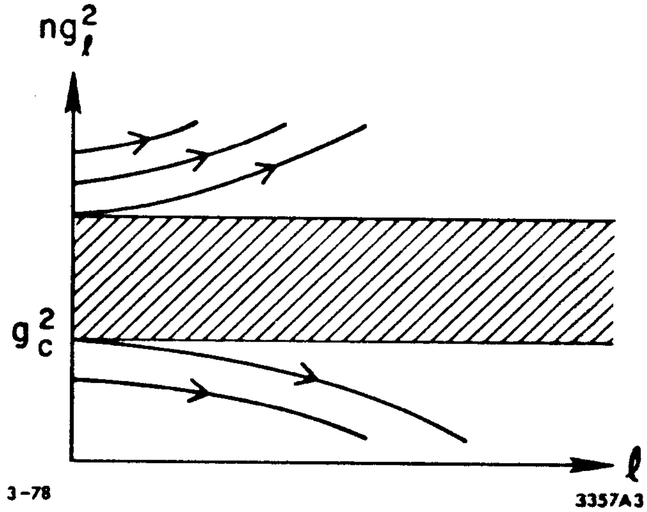
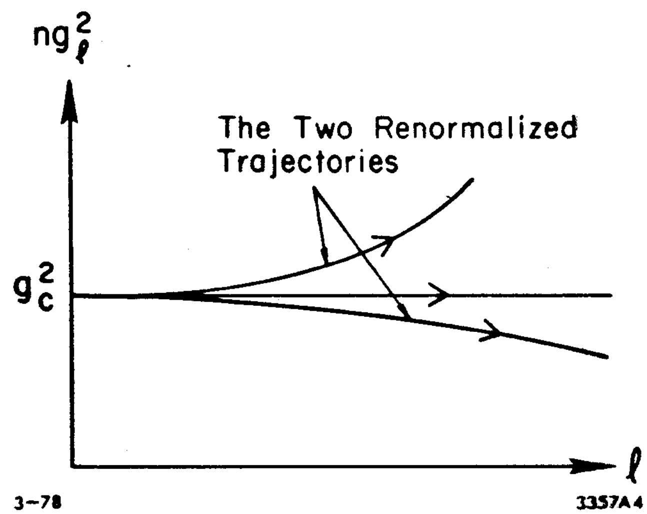


Fig. 3



ng$_\ell^2$
The Two Renormalized Trajectories
g$_c^2$
ℓ
3-78
3357A4

Fig. 4