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DIFFRACTIVE PRODUCTION AND RESCATTERING OF THREE-PARTICLE SYSTEMS*

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ABSTRACT

A new relativistic n-body scattering formalism is introduced, which explicitly satisfies the cluster property, and reproduces the analytic structure of the lowest order Feynman graphs. Applied to a three-particle system, this formalism defines an alternative form of the relativistic Faddeev equation; specific formulas are presented for the case of an s-wave separable interaction. A generalization of this equation is proposed for the purposes of three-particle data analysis, and is shown to provide an exactly unitary description in a form suitable for χ^2-minimalization techniques. This exact description further suggests an approximate formalism, which effectively generalizes the isobar model (including realistic thresholds and some three-body cut structure).

When applied to a four-body system in lowest order, the formalism defines a model for diffractive production of three-body states such as 3π, $K\pi\pi$, $N\pi\pi$, etc. Combined with subsequent rescattering of the three-particle system, this treatment confirms the recent result of Aitchison and Bowler concerning very strong production-resonance interference; this is shown to be related to the difference in the off-shell structure of the corresponding amplitudes. In the particular context of diffractive three-body production, this translates to significant differences in calculated cross sections when the subenergy dependence of the isobar amplitudes are neglected. It is further shown that off-shell (vertex) corrections to the Deck amplitude can produce both strong subenergy dependence and dramatic changes in the cross sections (as a function of three-body mass). These effects are illustrated via an analysis of the 1^+0^+ state of $K\pi\pi$ produced in the reaction $K^+p \rightarrow K^+\pi^+\pi^-p$ at 13 GeV/c. In particular, a Q_2 state of $K^*\pi$ is found at 1.2 GeV (compared to 1.4 GeV in previous analyses), whereas a Q_1 state (coupling predominantly to ρK) is found at 1.3 GeV (in agreement with past analyses). Implications of this result for A_1 production in various reactions are discussed.

I. INTRODUCTION

Despite a long history of effort, experimental candidates for the "missing" 1+ mesons have yet to be definitively established. Thus, data bearing on the enigmatic A_1 remain contradictory, and evidence for the Q mesons is far from decisive. In particular, experiments of the diffractive type have been cited both in support[1] and to refute the existence of the A_1.[2] Under these circumstances it seems clear that neither the production mechanism nor the subsequent rescattering effects are well understood. Although criticism in the past has focused on the latter,[3] a study of these analyses suggests that the form of the production model largely presages subsequent conclusions concerning the resonance(s). It thus seems important to study production (and production-resonance interference) in some detail, and to employ more general models in fitting data. In fact, should some experiment succeed in unequivocally establishing the A_1, such studies will be crucial in reconciling its existence with a mass of apparently negative evidence.[4]

A comprehensive theory of production would be a considerable undertaking, and would include both "direct" resonance production (e.g., coupling of the resonance to the Pomeron) and vector meson exchange in addition to the psuedo-scalar exchanges embodied in the familiar Deck model.[5] However, while such effects may be significant, and even mandatory in order to explain certain features of the data,[6] we shall not consider them in this article. Instead, we shall adopt a simpler hypothesis, and assume that production is dominated by a generalized version of the Deck model. Since the naive Deck amplitude successfully accounts for many features which are characteristic of the diffractive data, this seems a reasonable place to start, and the resulting class of models is relatively well defined. In addition, this approach is well suited for a complementary study of interference effects induced by resonant and nonresonant rescattering in the three-particle final state.

Given a production model, one must correctly sum all subsequent interactions (rescatterings) which involve only the three-particle system (e.g., 3π or $K\pi\pi$). In principle this involves the full 3-to-3 scattering amplitude $T_3(M_3)$, where M_3 is the invariant mass of the three-particle final state. The operator T_3 is in general rather complicated, depending not only on M_3, but also on the invariant pairwise masses $M_{\beta\gamma}$ (in addition to the Euler angles describing the orientation of the plane defined by motion in the three-body C.M.). In particular, T_3 will have poles (on the second sheet) in M_3 corresponding to three-body resonances, as well as right-hand cuts in M_3 and the $M_{\beta\gamma}$ corresponding to the thresholds for three-particle and particle-isobar scattering. Thus, even if one ignores the (still more complicated) left-hand cuts, T_3 possesses a very rich singularity structure in the near physical domain, and may exhibit some rather unusual features when viewed from the standpoint of the more familiar two-body operator. In particular, as this author has pointed out in the past, one may easily generate T_3 amplitudes which possess a resonance pole without an associated phase motion (for physical M_3),[7] or Breit-Wigner-like phase motion without having a pole.[8] Since there may also be strong interference effects between the resonance and production amplitudes,[9] one cannot in general infer the existence (or non-existence) of a three-body resonance from the presence (or absence) of strong phase motion and/or "bumps" in the cross section. Instead, one should fit the data using a representation of the amplitude which is sufficiently general to incorporate all of the important effects, and yet simple enough to explicitly verify the presence or absence of a resonance pole.

In practice the emphasis has been on "simple" rather than "general", and T_3 has almost always been taken in the form $T_3 = \Sigma_\alpha t_\alpha(M^2_{\beta\gamma})f_\alpha(M_3)$, where $\alpha \neq \beta \neq \gamma$ take on the values 1, 2, 3, and we adopt the convention that $t_\alpha(M^2_{\beta\gamma})$ is the (on-shell) two-particle scattering amplitude for particles β and γ. To the extent that isobars dominate the three-body system this has generally been regarded as good physics, and in fact, if f_α is taken instead to be $f_\alpha(M_3, M_{\beta\gamma})$, the decomposition is perfectly general. As justification for neglecting the $M_{\beta\gamma}$ (subenergy) dependence, one would presumably argue that the peaking of $t_\alpha(M^2_{\beta\gamma})$ at the isobar mass ($M_{\beta\gamma} = \mathscr{M}_\alpha$) effectively forces $M_{\beta\gamma}$ to that value; however, this requires f_α to be a <u>smooth</u> function of $M_{\beta\gamma}$. In actuality, as noted by Aaron and Amado,[3] f_α is forced to have a <u>cut</u> in the variable $M_{\beta\gamma}$ by the very general requirements of three-particle unitarity (to be satisfied by T_3), and hence this precondition is false. In practical terms, this means that the neglect of subenergy dependence can never be exact, and should be avoided unless the isobar width is very narrow (on a scale defined by the distance between $\mathscr{M}_\alpha$ and the $M_{\beta\gamma}$ branch point).

Unfortunately, in order to include subenergy dependence in the isobar amplitudes f_α in a fashion consistent with unitarity, one must in general construct them by solving (at the minimum) a one-dimensional integral equation. A prototype for the latter exists in the literature; namely, dynamical three-body equations of the Faddeev type (although relativistic), under the assumption of separable (off-shell) operators t_α which describe the pairwise scattering.[10] However, the free parameters which enter these equations occur in such a complicated way as to preclude their use in data analysis; i.e., one cannot afford to re-solve an integral equation numerically with each variation in the parameters by a χ^2 fitting routine. Although a rather trivial modification can be introduced (in the form of a fictitious "bare" resonance) to permit such a procedure, one then risks predjudicing the conclusions reached in the analysis by the limitations of the model.[11] These considerations prompted this author to introduce a very general alternative to the Faddeev-like equations, which retains the exactly unitary, one-dimensional character, while avoiding the separability assumption.[12] In contrast to the Faddeev equation, the kernel is linearized into a definite (fixed) singular term (necessary for unitarity) and a variable smooth term containing the fitting parameters; in this form it may be manipulated in such a way as to permit rapid numerical solution. Although the resultant equations have been successfully applied to a great variety of problems,[13] they have not achieved commensurate popularity; this disparity apparently reflects the strong preference for potential-like descriptions of the dynamics, as opposed to the generalized boundary conditions upon which the alternative equations are based.

Given this situation, a new alternative has been developed, and is introduced in this article. While again requiring the solution of a one-dimensional equation (as any exact unitarization scheme must), it retains the computational advantages of the former technique (as noted above) in all particulars. Furthermore, it is constructed as a generalization of the separable Faddeev equation (to which it reduces in lowest order), and hence it maintains contact with the t-matrix description. For this reason it lends itself far more readily to physical interpretation than did the former technique, and it is manifestly less complex mathematically. However, it should be emphasized that it does not equal the former in generality; it defines a relatively restricted class of three-body amplitudes. Nevertheless, it is very likely adequate for the type of applications considered here, and is certainly far more general than the isobar approach discussed above. With respect to the latter, its principal flaw is the relative complexity implicit in solving an integral (as opposed to an algebraic) equation, especially since numerical solutions have a tendency to obscure the actual physics. For this reason we also introduce an approximation suggested by the exact three-body treatment, but which can be handled algebraically. This has the character of a correction to the simple isobar model, to which it reduces in the zero width limit. Taken together, these various approaches constitute a rather complete set of tools for handling the final state interaction problem.

By this point it should be obvious that a definitive analysis of diffractive resonance production is a highly nontrivial business, involving a number of independent topics which are far more general in scope. In this article our intention is to embed these topics within a more comprehensive theoretical framework. By so doing we shall not only place the production and resonance terms on the same footing, but will also define a consistent set of rules for calculating non-diffractive processes as well (e.g., $\pi p \rightarrow A_1 \Delta$ and $Kp \rightarrow A_1 \Lambda$). Subsequent articles will refine and explore various aspects of the theory. This being the case, an effort has been made to subdivide the individual sections as an aid to the reader. We now proceed to briefly summarize the contents of these subsections.

The backbone of our approach is stated in Section II.A, in which we develop the formal aspects of relativistic scattering theory (RST) for an n-body system. It is worth noting that the free propagator (G_0) we employ differs from the Blankenbecler-Sugar prescription,[14] and hence is distinct from the form used in all relativistic Faddeev calculations to date. We believe that our reasons for preferring this choice are compelling, but the distinction is unimportant in the particular case of diffractive production. In Section II.B we apply the formalism to derive a result recently noted by Aitchison and Bowler;[9] namely, that one should anticipate large interference effects between the production and resonant contributions to the full amplitude. The result is trivial in this context, being a direct consequence of unitarity. We conclude this part with Section II.C, in which we derive our model for diffractive production.

The next section is devoted to a detailed description of the production model. Thus, Section III.A deals with the form of the Deck singularity in our RST treatment, and Section III.B is concerned with the two-body vertex factor associated with the scattering of each "dissociated" particle off the target (e.g., with πp scattering in A_1 production, and πp and Kp scattering in Q production). From the standpoint of our formalism, one could in principle treat the dissociation in two distinct ways; i.e., one could view the pion as dissociating into a "true" three-pion state, with pairs of pions then interacting to form the ρ; or, one could visualize a direct transition to a π-ρ state (the ρ being quasi-elementary), in which the ρ subsequently decays to two pions. The latter is clearly more in the spirit of the isobar model (and, perhaps more importantly, the quark model), and an interesting feature of our treatment is that the former is totally unacceptable in reproducing the desired energy-dependence of the (diffractive) cross section. Below we shall refer to these as the "simple" and "sequential" models; they are discussed in Section III.C and Section III.D, respectively. In Section IV we describe the partial-wave decomposition of our model; the treatment is essentially identical with that reported by Ascoli, Jones, Weinstein and Wyld for $\pi N \rightarrow (3\pi)N$.[15]

One of our principal motivations in adopting the RST approach was to gain some insight regarding the off-shell structure of the two- and three-particle vertices. In Section V we introduce some simple parametrizations, and report a series of numerical results for the reaction $K^+p \rightarrow (K^+\pi+\pi-)p$ in the dominant 1^+0^+ partial-wave. In particular, we investigate both the subenergy dependence of the production amplitude (which in general is considerable), and the corresponding variation of $d\sigma/dM_3dt$ with M_3 (at t_{min}). Variations associated with the two-body vertex are described in Section V.A; those corresponding to the three-body vertex (sequential model) are given in Section V.B.

Section VI is devoted to our various techniques for constructing a properly unitary amplitude to describe the three-body rescattering. In Section VI. A we introduce our exact three-body treatment; our approximate, pseudo-isobar, treatment is described in Section VI. B, and is illustrated by some numerical examples corresponding to the ρK and $K^*\pi$ channels. We next apply the latter technique to the 1^+0^+ state of $K^+\pi^+\pi^-$, and report a preliminary fit to the SLAC data in Section VII. Although this work is not yet definitive in the sense of satisfactorily explaining all features of the data (other partial-waves, the t-dependence, and the K^- data must also be treated), it provides a striking illustration of the possibilities inherent in more general production models. In particular, the mass of the (predominantly $K^*\pi$) Q_a state comes out to be 1.15 GeV, as opposed to the 1.4 GeV result of previous analyses, whereas the Q_b (ρK) state at 1.3 GeV is in complete agreement with those analyses. Finally, in Section VIII, we summarize and discuss our conclusions. In particular, we note that a reasonably general computer code has been developed along the lines discussed in this article, and will be made available to anyone interested in pursuing this approach.

II. RELATIVISTIC SCATTERING THEORY

A. Formal Framework

The goal of relativistic scattering theory is to define a consistent, relativistically invariant, description with regard to the scattering of mass-shell particles. This means that three-momentum, but not energy, is conserved at each vertex; it is an off-shell theory in the spirit of nonrelativistic potential theory. If one believes that a "proper" description should be based on a field theory; i.e., on an expansion in Feynman graphs in which the particles are off-mass-shell and 4-momentum is conserved, then such an approach must clearly be viewed as an approximation. Although such an objection may be less cogent in an era when hadrons are known to have a discrete "size" associated with an internal (quark) substructure, it is not our purpose to argue the point here. We shall instead justify the use of RST on the premise that it is more important to correctly describe the (very complex) right-hand cut structure associated with three- (or more) particle systems than to impose other constraints, such as crossing. Equivalently, we shall assume that effective off-shell operators can be defined within the context of this formalism, in such a way that one may correctly describe the amplitudes of interest in some domain inclusive of the (s-channel) physical region. As we shall see, it is possible to make a choice which correctly preserves the form of the nearest left-hand singularities; in general, however, one must forgo such luxuries. In fact, there really is no viable alternative for systems of three or more particles.

The key question in constructing such a theory is the choice of a proper relativistic propagator, G_0. In general, if $E=\Sigma_\alpha \epsilon_\alpha$ is the sum of the individual c.m. energies, and $\sqrt{s}$ is the invariant on-shell energy (corresponding to the physical initial or final states), then $G_0 \propto (E-\sqrt{s}-i\epsilon)^{-1}$, where the proportionality factor goes to unity as $E \to \sqrt{s}$ (and is identically equal to unity in the nonrelativistic limit). Any such G_0 will generate an acceptable theory from the standpoint of unitarity; one may exercise the resulting freedom of choice to satisfy other constraints. Almost invariably, the choice adopted in the literature is based on the Blankenbecler-Sugar (BS) prescription for a two-particle system.[14] For an n-body system, this corresponds to $G_0=2E(E^2-s-i\epsilon)^{-1}$, and is "derived" by the requirement that G_0 should have only the minimal singularity structure in s; i.e., the simple pole required for unitarity. However, given our motivations for using RST, and its inherent limitations, the properties of G_0 in the unphysical region are purely academic, and one might better employ one's choice to achieve other ends. In fact, there are serious defects in the standard prescription, and this has led us to propose the alternative discussed below.

Ignoring spin, for simplicity, we characterize an n-body system in terms of the n 4-momenta k_α, where

$$k_\alpha = (\epsilon_\alpha, \vec{k}_\alpha) \quad ,$$
$$\epsilon_\alpha = \left(m_\alpha^2 + \vec{k}_\alpha^2\right)^{\frac{1}{2}} \quad , \tag{2.1}$$

and m_α is the mass of particle α. We thus choose a basis $|k_1 k_2 \ldots k_n\rangle$, normalized such that

$$\langle k_1' k_2' \ldots k_n' | k_1 k_2 \ldots k_n\rangle = \prod_{\alpha=1}^{n} \epsilon_\alpha \delta(\vec{k}_\alpha' - \vec{k}_\alpha) \quad . \tag{2.2}$$

The corresponding completeness relation is

$$1 = \int \prod_{\alpha=1} \frac{d\vec{k}_\alpha}{\epsilon_\alpha} \; |k_1 k_2 \ldots k_n\rangle\langle k_1 k_2 \ldots k_n| \quad . \tag{2.3}$$

On this space we define G_0 as a diagonal operator,

$$\langle k_1'k_2'\ldots k_n'|G_0|k_1k_2\ldots k_n\rangle = \prod_{\alpha=1}^{n} \epsilon_\alpha \delta(\vec{k}_\alpha'-\vec{k}_\alpha)G_0(P,P_0),$$

$$G_0(P,P_0) = 1/P_0 \cdot (P-P_0-i\epsilon) \quad , \tag{2.4}$$

where $P=\Sigma_\alpha k_\alpha$, and P_0 is the total on-shell 4-momentum in the initial (final) state. The linearity of $G_0(P,P_0)$ in P is crucial for our purposes; the corresponding BS choice has $G_0=(P^2-P_0^2-i\epsilon)^{-1}$.

Our motivation for this choice becomes evident when one considers the interactions of an m-body subsystem (m< n). Suppose first that the m particles are isolated, and interact according to some generalized potential V_m as illustrated in Fig. 1a. Since three-momentum is conserved, we would express V_m as an operator on our basis such that

$$\langle k_1'k_2'\ldots k_m'|V_m|k_1k_2\ldots k_m\rangle = E_m^o\delta(\vec{P}_m'-\vec{P}_m)V_m(k_1'k_2'\ldots k_m'|k_1k_2\ldots k_m) \quad ,$$

$$E_m^o = (s_m+\vec{P}_m^2)^{\frac{1}{2}} \quad , \tag{2.5}$$

where $\vec{P}_m=\Sigma_{\alpha=1}^{m}\vec{k}_\alpha$, and $\sqrt{s_m}$ is the invariant energy of the m-body system. We define the m-body t-matrix operator $T_m^{(m)}$ to be the sum of the series implied by Fig. 1a, which we represent by the operator relation

$$T_m^{(m)} = V_m - V_mG_0V_m + V_mG_0V_mG_0V_m - \ldots$$

$$= V_m - V_mG_0T_m^{(m)} \; ; \tag{2.6}$$

here $T_m^{(m)}$ has a similar structure to Eq. (2.5), and $T_m^{(m)}$, G_0, V_m all depend implicitly on s_m. As a consequence of our definitions, we note that G_0 only occurs in the combination

$$E_m^o\delta(\vec{P}_m-\vec{P}_m^o)G_0(P_m,P_m^o) = \delta(\vec{P}_m-\vec{P}_m^o)/(E_m-E_m^o-i\epsilon) \quad , \tag{2.7}$$

where $P_m^o = (E_m^o, \vec{P}_m^o)$ is the initial (on-shell) 4-momentum of the m-body system. The relation stated in Eq. (2.6) serves to define the m-body amplitude $T_m^{(m)}$ $(k_1'k_2'\ldots k_m' | k_1k_2\ldots k_m; s_m)$.

We next consider the situation when n-m additional particles are present but do not interact; this is depicted in Fig. 1b. In this case we define V_m as an operator on the n-body space such that

$$
\begin{aligned}
<k_1'k_2'\ldots k_n' | V_m | k_1k_2\ldots k_n> = E_n^o \delta(\vec{P}_n' - \vec{P}_n) \prod_{\alpha=m+1}^{n} \epsilon_\alpha \delta(\vec{k}_\alpha' - \vec{k}_\alpha) * \\
* V_m(k_1'k_2'\ldots k_m' | k_1k_2\ldots k_m) \quad ,
\end{aligned}
\tag{2.8}
$$

with a similar expression for $T_m^{(n)}$ in which the V_m amplitude is replaced by $T_m^{(n)}$ $(k_1'k_2'\ldots k_m' | k_1k_2\ldots k_m; s_m)$. The series implied by Fig. 1b can then be expressed as the operator expression given in Eq. (2.6), except that $T_m^{(m)}$ is replaced by $T_m^{(n)}$. Now G_0 only occurs in the combination

$$
E_n^o \delta(\vec{P}_n - \vec{P}_n^o) G_0(P_n, P_n^o) = \delta(\vec{P}_n - \vec{P}_n^o)/(E_n - E_n^o - i\epsilon) \quad . \tag{2.9}
$$

When one factors out the (n-m) delta-functions corresponding to the spectator particles, one finds that the amplitude $T_m^{(n)}$ again satisfies the implied integral equation stated in the second line of Eq. (2.6); the only difference is that G_0 now involves $E_n - E_n^o$ instead of $E_m - E_m^o$, and s_m depends both on s_n and the spectator momenta. Explicitly,

$$
s_m \equiv (P_m^o)^2 = (P_n^o - P_{n-m})^2 \quad , \tag{2.10}
$$

$$
P_{n-m} = \sum_{\alpha=m+1}^{n} k_\alpha \quad .
$$

In particular, if we compute s_m in the n-body c.m., we obtain

$$
s_m = s_n + P_{n-m}^2 - 2\sqrt{s_n}\, E_{n-m}^{c.m.} \quad . \tag{2.11}
$$

We have therefore, in principle, defined two distinct amplitudes, $T_m^{(m)}$ and $T_m^{(n)}$. However, since the spectator energies do not change, we clearly have $E_n-E_n^o = E_m-E_m^o$, and hence $T_m^{(m)}=T_m^{(n)} \equiv T_m$. Thus, it makes no difference whether we sum up the m-body interactions in the m-body or the n-body space, we have only to recall that s_m is to be calculated from Eq. (2.10), or Eq. (2.11), as it appears in the amplitude $T_m(k_1'k_2'\ldots k_m' | k_1k_2\ldots k_m;s_m)$. This consistency requirement is sometimes called the cluster property, and it is satisfied automatically in the nonrelativistic theory for the same reason as the above; namely, the propagator involves only the difference $E_n-E_n^o$. However, if one employs instead the BS choice for G_0, one obtains $E_n^2-E_n^{o2}$, and hence $T_m^{(n)} \neq T_m^{(m)}$. This aspect is usually suppressed in, e.g., the literature on relativistic three-body equations,[10] since one normally deals directly with the amplitudes T_m, and does not attempt to derive them from a fundamental interaction (V_m). Nevertheless, such formalisms do not correspond to a consistent RST, and the T_m operators they employ are actually rather odd constructs.

The G_0 operator introduced in Eq. (2.4) is unique in the sense of being the simplest solution to the cluster problem (i.e., one could clearly add pieces depending on $(P-P_0)^2$, non-singular terms, etc.). If one considers G_0 in the c.m., the price one pays is a dependence on $\sqrt{s}$, instead of s, and hence G_0 has a left-hand cut for $s<0$; in practical terms this does not seem a sufficient reason to discard it. Furthermore, while the foregoing might be regarded as a purist's quibble, there is a more compelling reason to make this choice. To see this let us consider how the Feynman graph given in Fig. 2a is reproduced in the RST language (again ignoring spin).

The Feynman amplitude corresponding to Fig. 2a is

$$A_F = g^F_{aec}\, g^F_{bed} \Big/ (t-m_e^2) \quad ,$$
$$t = (k_a-k_c)^2 = (k_b-k_d)^2 \quad . \tag{2.12}$$

In contrast, the RST formalism involves both Fig. 2b and Fig. 2c, since the development is based on sequential scattering, and hence assumes a particular time-ordering of events. Moreover, each vertex in the RST treatment is to be associated with a scattering amplitude (in general off-shell); in particular, the upper vertex (aec) in Fig. 2b is regarded as a special case of Fig. 2d, in which the particles a' and e' are "bound" to produce particle c. The latter applies only in a very formal sense; i.e., if t_{ae} is the invariant a-e elastic amplitude, then

$$t_{ae}(s_{ae}) \longrightarrow g^2_{ae;c} / (s_{ae}-m_c^2) \tag{2.13}$$

as $s_{ae} \rightarrow m_c^2$, and the corresponding <u>on-shell</u> vertex factor at (aec) is $g_{ae;c}/\sqrt{2}$. This rule corresponds precisely to the manner in which one defines the amplitude for scattering from a bound state (e.g., particles a and e could be nucleons, and c the deuteron), but does not necessarily suppose a physical picture of c as an actual a-e bound state. At this point we note that our normalization convention for an invariant two-particle amplitude t_{ae} is such that (in a partial-wave decomposition)

$$I_m t_{ae} = - \frac{\pi k_{ae}}{\sqrt{s_{ae}}} |t_{ae}|^2 \quad , \tag{2.14}$$

where k_{ae} is the (two-body) c.m. momentum; thus t_{ae} is dimensionless.

Applying the above rules to calculate Fig. 2b plus Fig. 2c, and dropping the overall $2E\delta(\vec{P}-\vec{P}_0)$ factor (see, e.g., Eq. 2.5)), the RST amplitude corresponds schematically to $-t_{ae}G_0t_{de}-t_{be}G_0t_{ce}$, and is given by

$$A_{RST} = -\frac{1}{2}\frac{g_{ae;c}\, f_{ae}(k_a k_e) g_{de;b}\, f_{de}(k_d k_e)}{\epsilon_e(\epsilon_e+\epsilon_d-\epsilon_b)} - \frac{1}{2}\frac{g_{be;d}\, f_{be}(k_b k_e) g_{ce;a}\, f_{ce}(k_c k_e)}{\epsilon_e(\epsilon_e+\epsilon_c-\epsilon_a)} \quad . \tag{2.15}$$

Here the f's are vertex form factors which go to unity in the on-shell limit, and $k_e = (\epsilon_e, \vec{k}_e)$ is computed in terms of $\vec{k}_a, \vec{k}_b, \vec{k}_c, \vec{k}_d$ via the conservation of 3-momentum (its sign, but not magnitude, is different in the two terms). In particular, $\epsilon_e = \left[m_e^2 + (\vec{k}_b-\vec{k}_d)^2\right]^{\frac{1}{2}}$. To compare A_{RST} with A_F, we evaluate A_{RST} with the initial and final states on-shell, implying that $\epsilon_a+\epsilon_b = \epsilon_c+\epsilon_d$. We then observe that if

$$g_{ae;c} = g_{ce;a} = g^F_{aec} \quad ,$$
$$g_{de;b} = g_{be;d} = g^F_{bed} \quad , \tag{2.16}$$

we may rewrite A_{RST} in the form

$$A_{RST} = A_F + A^R_{RST} \quad , \tag{2.17}$$

where

$$A^R_{RST} = \frac{g^F_{aec}\, g^F_{bed}}{2\epsilon_e}\left[\frac{1-f_{ae}\, f_{de}}{\epsilon_e+\epsilon_d-\epsilon_b} + \frac{1-f_{be}\, f_{ce}}{\epsilon_e+\epsilon_c-\epsilon_a}\right] \quad , \tag{2.18}$$

and we have used the fact that

$$\epsilon_e^2 - (\epsilon_b - \epsilon_d)^2 = m_e^2 - t \quad . \tag{2.19}$$

For form factors which are analytic in the neighborhood of the on-shell point, one can easily show that A^R_{RST} is regular in the vicinity of $t=m_e^2$; i.e.,

$$f_{ae}(k_a k_e) \longrightarrow 1 + f'_{ae}\left[(k_a + k_e)^2 - m_c^2\right]$$
$$= 1 + f'_{ae}(\epsilon_e+\epsilon_a+\epsilon_c)(\epsilon_e+\epsilon_d-\epsilon_b) \quad , \tag{2.20}$$

so that the vanishing denominators are cancelled exactly. Thus, for residues which satisfy the (standard) relations of Eq. (2.16), A_{RST} has precisely the simple pole at $t=m_e^2$ prescribed by the Feynman rules. In consequence, the RST treatment simply defines an extrapolation away from the pole in terms of the vertex form factors.

In contrast, the BS choice for G_0 defines a different A_{RST} with some undesirable properties. Consider, for example, the case of elastic scattering, with $m_a=m_c$ and $m_b=m_d$. Evaluating Fig. 2b in the c.m. (implying that $\epsilon_c=\epsilon_a$ and $\epsilon_d=\epsilon_b$), the BS prescription corresponds to the replacement

$$\frac{1}{\epsilon_e+\epsilon_c-\epsilon_a} \longrightarrow \frac{2(\epsilon_c+\epsilon_b+\epsilon_e)}{(\epsilon_c+\epsilon_b+\epsilon_e)^2-(\epsilon_a+\epsilon_b)^2} = \frac{\epsilon_e+\sqrt{s_{ab}}}{\epsilon_e/2+\sqrt{s_{ab}}}\,\frac{1}{\epsilon_e} \quad ; \tag{2.21}$$

in this case the contribution from Fig. 2c is exactly equal. If we focus just on the singular term (setting the f's equal to unity), the result is that

$$A_{RST}^{BS} \longrightarrow \frac{\epsilon_e+\sqrt{s_{ab}}}{\epsilon_e/2+\sqrt{s_{ab}}}\,A_F \quad ,$$

$$\epsilon_e = (m_e^2-t)^{\frac{1}{2}} \quad . \tag{2.22}$$

Therefore, the residue of the exchange pole is not only energy-dependent, but also has a square-root branchpoint (due to ϵ_e) exactly at $t=m_e^2$; i.e., the singularity is not a simple pole. If m_e^2 is small, this could be of practical importance in an actual calculation; for example, if one treats NN scattering as an explicit $NN\pi$ problem (in order to correctly describe the effect of pion production above 300 MeV), the corresponding diagram represents OPE, and the t-singularity is very close to the physical region. In fact, the spurious square-root singularity would probably cause some difficulty in describing the higher partial-waves (dominated by OPE). Nevertheless, such a calculation has recently been reported in the literature.[16]

Given the completeness relation in Eq. 2.3), the definition of G_0 in Eq. (2.4), and the definition of an m-body operator on the n-body space in Eq. (2.8), we are now in a position to calculate very simply any process which can be represented as a sequence of subsystem rescatterings. In the next two subsections we consider simple applications to the production problem; three-particle dynamical equations are discussed in Section VI. B.

B. Production-Resonance Interference

In practice, three-particle systems are produced in the laboratory via an inelastic reaction initiated by the beam and target particles. Thus, in all cases where the three-body system is first produced and then undergoes a resonant interaction, there will be a production term as well as a resonance term contributing to the amplitude. The exception to this situation corresponds to direct production of the resonance, with a subsequent decay to the three-body system. In the former case, the tendency has often been to treat the production term independently from the resonance; i.e., as an incoherent background. However, it has recently been noted that this is likely to be a very poor procedure in the case of diffractive production; one should in fact expect rather strong production-resonance interference.[9] This result, which is actually rather more general, emerges in a particularly transparent manner in the RST language, as we shall demonstrate below. In addition, the interference effect is extremely relevant to the results we shall later discuss concerning the $K\pi\pi$ system.

Assume, for simplicity, a two-particle system (described by k_1, k_2) which is produced via some mechanism (T_p) as sketched in Fig. 3a; note that k_f is the total 4-momentum of all other particles in the final state. Once produced, particles 1 and 2 may scatter as illustrated in Fig. 3b; this is described via the elastic amplitude t_{12}. In the 12 c.m. frame, we take $\vec{k}_1=-\vec{k}_2=\vec{k}$, $\vec{k}_1'=-\vec{k}_2'=\vec{k}'$, and write the corresponding amplitude as $t_{12}(\vec{k},\vec{k}';s_{12})$, where

$$s_{12} = (p_1+p_2-k_f)^2 \quad ,$$
$$\omega_{12} \equiv \sqrt{s_{12}} = \epsilon_1+\epsilon_2 \quad . \tag{2.23}$$

In the same frame, we represent the production amplitude by $T_p(\vec{k})$, suppressing the dependence on all other variables in our notation. According to our rules, the sum of Figs. 3a, 3b corresponds to the operator $A = (1-t_{12}G_0)T_p$. In terms of amplitudes,

$$A(\vec{k}) = T_p(\vec{k}) - \int \frac{d\vec{k}'}{\epsilon_1'\epsilon_2'} \frac{t_{12}(\vec{k},\vec{k}';s_{12})}{\epsilon_1' + \epsilon_2' - \omega_{12} - i\epsilon} T_p(\vec{k}') \quad . \tag{2.34}$$

In order to focus on a particular partial-wave, we expand

$$T_p(\vec{k}) = \sum_{\ell m} Y_{\ell m}(\hat{k}) T_p^{\ell m}(k) \quad ,$$
$$t_{12}(\vec{k},\vec{k}';s_{12}) = \sum_{\ell m} Y_{\ell m}(\hat{k}) Y^*_{\ell m}(\hat{k}') t^{\ell}_{12}(k,k';s_{12}) \tag{2.35}$$
$$= \sum_{\ell} \left(\frac{2\ell+1}{4\pi}\right) P_\ell(\hat{k}\cdot\hat{k}') t^{\ell}_{12}(k,k';s_{12}),$$

and obtain

$$A_{\ell m}(k) = T_p^{\ell m}(k) - \int_0^\infty \frac{dk'k'^2}{\epsilon_1'\epsilon_2'} \frac{t^{\ell}_{12}(k,k';s_{12})}{\epsilon_1' + \epsilon_2' - \omega_{12} - i\epsilon} T_p^{\ell m}(k') \quad . \tag{2.36}$$

Under the assumption that the (12) subsystem has a resonance in the neighborhood of $k=k_0$, we take

$$t^{\ell}_{12}(k,k';s_{12}) = g_\ell(k) g_\ell(k')/D_\ell(s_{12}) \quad ,$$
$$D_\ell(s_{12}) = k^2 - k_0^2 + r_\ell(k) + i\pi(k/\omega_{12}) g_\ell^2(k) \quad ,$$
$$r_\ell(k) \equiv \mathrm{PV} \int_0^\infty \frac{dk'k'^2}{\epsilon_1'\epsilon_2'} \frac{g_\ell^2(k')}{\epsilon_1' + \epsilon_2' - \omega_{12}} \quad , \tag{2.37}$$

where $g_\ell(k)$ is the resonance form factor. If k_r is the value of k for which $R_e D_\ell(s_{12})=0$ ($\delta_\ell=90^0$), then $g_\ell^2(k_r)$ is essentially the width (note that in general $g_\ell(k) \propto k^\ell$). Thus, since its value at $k=k_r$ effectively determines the overall normalization, or scale, of $g_\ell^2(k)$ (for a given shape), we see that $r_\ell(k) \to 0$, $k_r \to k_0$ in the zero width limit. The ratio

$$\frac{t_{12}^\ell(k,k';s_{12})}{t_{12}^\ell(k,k;s_{12})} = \frac{g_\ell(k')}{g_\ell(k)} \tag{2.38}$$

is sometimes called the half-off-shell extension function, and is a measure of the shape-dependence of the form factor. In particular, a simple s-wave model might be $g_0(k) = c_0\,(k^2+\mu^2)^{-1}$, where μ is a mass characteristic of particle exchanges in the (12) interaction. From such considerations one infers that the ratio in Eq. (2.38) differs from unity by terms of the order of $(k'^2-k^2)/\mu^2$.

In order to perform the integration in Eq. (2.36), we define a quantity $G_p^{\ell m}(k',k)$ by the relation

$$\frac{T_p^{\ell m}(k')}{T_p^{\ell m}(k)} = \frac{g_\ell(k')}{g_\ell(k)} + G_p^{\ell m}(k',k) \quad ; \tag{2.39}$$

hence $G_p^{\ell m}$ vanishes for $k'=k$, and is a measure of the difference between the half-off-shell extensions of T_p and t_{12}. Using

$$\frac{1}{\epsilon_1'+\epsilon_2'-\omega_{12}-i\epsilon} = \mathrm{PV}\,\frac{1}{\epsilon_1'+\epsilon_2'-\omega_{12}} + \frac{i\pi\epsilon_1'\epsilon_2'}{\omega_{12}k'}\,\delta(k'-k) \quad , \tag{2.40}$$

we obtain

$$\begin{aligned} A_{\ell m}(k) &= \left[k^2-k_0^2-g_\ell(k)\tilde{G}_p^{\ell m}(k)\right] \frac{T_p^{\ell m}(k)}{D_\ell(s_{12})} \quad , \\ \tilde{G}_p^{\ell m}(k) &\equiv \mathrm{PV}\int_0^\infty \frac{dk'k'^2}{\epsilon_1'\epsilon_2'}\,\frac{g_\ell(k')G_p^{\ell m}(k',k)}{\epsilon_1'+\epsilon_2'-\omega_{12}} \quad , \end{aligned} \tag{2.41}$$

(the PV is actually unnecessary since $G_p^{\ell m}(k,k)=0$).

The desired result now follows trivially from the expression for $A_{\ell m}(k)$ in Eq. (2.41). Thus, if $k_r \simeq k_0$ (which is certainly true for a narrow resonance), and the off-shell behavior of T_p and t_{12} is similar ($G_p^{\ell m} \simeq 0$), the full amplitude vanishes near $k=k_r$! This implies that, contrary to one's naive expectations, the signature of a resonance under such circumstances may well be a dip in the cross section (or a shifted peak if the cross section is rising or falling rapidly).[17] The obvious conclusion is that one must be extremely cautious in associating a "bump" with the actual position of a resonance, or with inferring its nonexistence from the absence of a peak; the latter point may be highly relevant in regard to states like the A_1, which appears curiously absent in certain production modes. Thus, it should be reasonably clear that the above argument does not depend critically on the two-particle character of the resonating subsystem. In fact, the crucial cancellation of $I_m D_\ell(s_{12})$ in deriving the multiplicative bracket in Eq. (2.41) is a very general consequence of two-particle unitarity, as guaranteed by the explicit representation of Eq. (2.37). To see this, we note that the unitarity relation takes the form $\Delta t = -t^+ \Delta G_0 t^-$ in our operator notation, and hence (assuming $\Delta T_p=0$)

$$\Delta A = -t_{12}^+ \Delta G_0 A^- \quad , \tag{2.42}$$

which implies that $A=t_{12}A_R$, where A_R does not have the right-hand cut (e.g., is real). Thus, if the right-hand cut structure of t_{12} is contained entirely in the denominator D_{12}, we may write $A=D_{12}^{-1}\tilde{A}$, where $\tilde{A}$ is real. If we repeat this argument for a larger resonating subsystem, with amplitude T_m, the formal proof is identical and we obtain $A=D_m^{-1}\tilde{A}$ (assuming a generalized denominator function D_m, whose zero corresponds to the resonance pole of the subsystem). In general, therefore, one should anticipate considerable interference, unless "direct" production is known to dominate the reaction.

C. Diffractive Production Model

As noted above, one may in principle consider two distinct models of diffractive production. The "sequential" model is illustrated in Fig. 4a, and corresponds to a dissociation of the incoming particle (k_i) directly into a two-particle state, one of which subsequently decays into the observed pair of particles (k_β, k_γ). In contrast, the "simple" model involves dissociation into a true three-body state, with a subsequent interaction of particles β and γ producing the ($\beta\gamma$) isobar; this is shown in Figs. 4c, 4d, respectively. The former is certainly more in the spirit of the isobar model (and, presumably, the quark model), and in fact is consistent with the way in which diffractive production has always been calculated in the literature.[18] However, one might naively assume that it is merely an approximate method for treating the true 1-to-3 vertex. This would be true if the t_α interaction occurred before, rather than after, the t_2 interaction, as in Fig. 4b. Interestingly, in terms of our formalism, this cannot be the case, at least in the sense of leading to the isobar. Thus, the isobar pole occurs as a zero in the denominator function $D_\alpha(s_\alpha)$ of the (off-shell) amplitude t_α. In Fig. 4b, we would calculate

$$s'_\alpha = (P_0 - p_i - k'_\alpha)^2 = (k_i - k'_\alpha)^2 \quad ,$$

or (2.43)

$$s'_\alpha = m_i^2 + m'^2_\alpha - 2m_i\epsilon'_\alpha$$

in the rest frame of particle i. Hence $s'_\alpha \leq (m_i - m'_\alpha)^2$, and this normally restricts it from being anywhere near the isobar $(\text{mass})^2$; e.g., consider $m_i = m'_\alpha = m_\pi$ with respect to the $\pi \to 3\pi$ vertex, in comparison with $\pi \to \rho\pi$. To those familiar with low energy problems this should come as no surprise; for example, if one takes Fig. 4b as the lowest order contribution to pH_e^3 scattering, the $\beta\gamma$ pair cannot emerge as the deuteron.

As a consequence, Fig. 4d is the lowest order (RST) diagram which contains the $\beta\gamma$ isobar in the simple model. Of necessity, Fig. 4c must also be present, which turns out to be crucial in distinguishing the two models. Thus, if one compares Figs. 4c,4d, to Figs. 3a, 3b, one can immediately apply the result of the last subsection, and infer the existence of a large cancellation for $s_\alpha \simeq M_R^2$. Therefore, in contrast to the sequential model, which one expects to peak for $(k_\beta + k_\gamma)^2 \simeq M_R^2$, the simple model will tend to suppressed. In practice, this tendency leads to an unacceptable behav ior of the diffractive cross section $d\sigma/dM_3dt$; i.e., it continues to rise sharply as M_3 increases, instead of peaking just above the isobar threshold. Qualitatively, this is very easy to understand, since the amplitude takes its largest values (for fixed M_3) when $s_\alpha > M_R^2$, and more of that region is kinematically accessible as M_3 increases. Explicit numerical calculations, using the generalized 1-to-3 vertex function discussed in Section III. C, merely confirm this, and have led us to reject the simple model as the mechanism for diffractive production. Physically, this result is rather interesting, since it lends additional credence to the quark model viewpoint; e.g., one must treat the ρ as an elementary object, and not as a resonance in π-π scattering.

We conclude this section by applying our RST rules to calculate Fig. 4a and (for completeness) Fig. 4c; to employ the simple model one would compute Fig. 4d in exactly the manner described in the previous subsection (T_p corresponds to Fig. 4c). In fact, by not specifying the explicit structure of T_3 (considered below), both diagrams can be expressed as

$$T_p(k_\alpha k_\beta k_\gamma p_f | k_i p_i) = - \frac{t_2(k_\alpha p_f | k'_\alpha p_i ; s_2) T_3 (k'_\alpha k_\beta k_\gamma | k_i ; s_3)}{\epsilon'_\alpha (M_{\beta\gamma} + \epsilon'_\alpha - \epsilon_i)} \qquad (2.44)$$

in the $\beta\gamma$ c.m. frame, where

$$\begin{aligned} M_{\beta\gamma} &= \epsilon_\beta + \epsilon_\gamma = \left[\left(k_\beta + k_\gamma\right)^2\right]^{\frac{1}{2}} , \\ \vec{k}'_\alpha &= \vec{k}_i = \vec{k}_\alpha + \vec{p}_f - \vec{p}_i , \\ \epsilon'_\alpha &= \left(m'^{\,2}_\alpha + \vec{k}'^{\,2}_\alpha\right)^{\frac{1}{2}} , \\ \epsilon_i &= \left(m_i^2 + \vec{k}'^{\,2}_\alpha\right)^{\frac{1}{2}} . \end{aligned} \tag{2.45}$$

Here we have allowed for the possibility that $m'_\alpha \neq m_\alpha$; thus T_p will describe processes such as $Kp \rightarrow (3\pi)\Lambda$, as well as the diffractive reaction $\pi p \rightarrow (3\pi)p$, depending on what masses and vertex functions are employed. We note that our convention for the presumed "fast" particles is that $p_i = (E_i, \vec{p}_i)$, $p_f = (E_f, \vec{p}_f)$, corresponding to the masses M_i, M_f (in general, $M_i \neq M_f$). A complete discussion of the vertex factors t_2, T_3 is given in the next section; kinematics and the partial-wave decomposition of T_p are presented in Section IV.

III. DETAILS OF THE PRODUCTION AMPLITUDE

A. Deck Singularity

The general form of our production model is stated in Eq. (2.44). At the simplest level, when T_3 is taken to be constant, and t_2 is set equal to the appropriate on-shell $2 \rightarrow 2$ scattering amplitude (e.g., $t_2 \rightarrow i s_2 \sigma_{tot}$ in the diffractive case), the model differs from the familiar Deck amplitude only by virtue of the denominator. Thus, we recall from Sec. II.A that two RST diagrams are in general necessary to reproduce the corresponding Feynman diagram. In this case, we would require the graph shown in Fig. 5 in order to recover the Deck denominator $(m'^2_\alpha - t_3)/2$, where

$$t_3 = (k_\beta + k_\gamma - k_i)^2 \tag{3.1}$$

is the momentum-transfer at the three-body vertex. However, this second process is typically suppressed in the kinematic region of interest; i.e., for large cross sections we must have the elastic diffractive amplitude t_2 at the vertex involving the large momenta p_i, p_f. This is equivalent to the statement that the t_3 pole at m'^2_α is far away, since the singularity (in this case) arises from Fig. 5, and not Fig. 4a (e.g., if $m'_\alpha = m_i$, the denominator in Eq. (2.44) reduces to $\epsilon'_\alpha M_{\beta\gamma}$). In effect, we have broken the Feynman amplitude into two pieces, and concentrated on that piece which dominates when extrapolated to the diffractive region. Noting that

$$\begin{aligned} t_3 - m'^2_\alpha &= M^2_{\beta\gamma} - 2M_{\beta\gamma}\epsilon_i + m_i^2 - m'^2_\alpha \\ &= (M_{\beta\gamma} - \epsilon_i)^2 - \epsilon'^2_\alpha \\ &= (M_{\beta\gamma} + \epsilon'_\alpha - \epsilon_i)(M_{\beta\gamma} - \epsilon'_\alpha - \epsilon_i)\,, \end{aligned} \tag{3.2}$$

the comparison can be expressed as

$$T_p^{RST} \Big/ T_p^{F} \simeq (M_{\beta\gamma} - \epsilon'_\alpha - \epsilon_i) \Big/ \epsilon'_\alpha \quad , \tag{3.3}$$

which is valid in the special case when $T_3 \simeq$ constant. Thus, even without the introduction of structure at the three-body vertex, our approach leads to subenergy dependence (i.e., on $M_{\beta\gamma}$) relative to the usual Deck model. This distinction becomes unimportant once structure in T_3 is permitted, however, since the ratio in Eq. (3.3) can then essentially be altered at will.

In concluding this subsection, we note that our choice of G_0 is purely academic, insofar as the diffractive amplitude T_p is concerned. Thus, had we chosen the BS propagator in evaluating Fig. 4a, Eq. (2.44) would have been modified by the replacement

$$\frac{1}{M_{\beta\gamma} + \epsilon'_\alpha - \epsilon_i} \longrightarrow \frac{2\,(M_{\beta\gamma} + \epsilon'_\alpha + E_i)}{(M_{\beta\gamma} + \epsilon'_\alpha + \epsilon_i + 2E_i)} \quad \frac{1}{M_{\beta\gamma} + \epsilon'_\alpha - \epsilon_i} \tag{3.4}$$

In practice, however, the multiplicative factor is very close to unity. To see this we introduce the quantities

$$\begin{aligned} s &= (p_i + k_i)^2 \quad , \\ t &= (p_f - p_i)^2 \quad . \end{aligned} \tag{3.5}$$

In the diffractive limit ($s \to \infty$), the cross section is dominated by $t \simeq t_{min}$, and one finds that ϵ'_α, ϵ_i are finite ($M_{\beta\gamma}$ fixed), whereas $E_i \propto s$ (this can be verified directly from the formulas in Sec. IV). The extra factor thus approaches $2E_i / 2E_i$, and hence has no appreciable effect in the relevant kinematic region.

B. Two-Body Vertex

As it stands, the expression for T_P given in Eq. (2.44) is a completely general representation of the process shown in Fig. 4a. In order to apply it to distinct physical situations, one must choose the masses and vertex functions t_2, T_3 accordingly. In general, of course, t_2 and T_3 will depend on spin and isospin (or charge) variables in addition to the relevant 4-momenta. For the purposes of this article, however, we shall specialize to the case of diffractive production, and choose a form for t_2 consistent with high energy elastic scattering ($m'_\alpha = m_\alpha$, $M_i = M_f$). With our normalization convention, the optical theorem is stated as

$$\sigma_{tot} = \frac{-16\pi^3}{\kappa_2\sqrt{s_2}} \, \mathrm{Im}\, t_2(s_2, 0) , \tag{3.6}$$

where $t_2(s_2,t)$ is the on-shell amplitude, and κ_2 is the c.m. momentum corresponding to the energy $\sqrt{s_2}$. Here t is given by Eq. (3.5), and the value of s_2 relevant to Fig. 4a is

$$\begin{aligned} s_2 &= (P_0 - k_\beta - k_\gamma)^2 \\ &= s + M^2_{\beta\gamma} - 2M_{\beta\gamma}(\epsilon_i + E_i) \end{aligned} \tag{3.7}$$

in the $\beta\gamma$ c.m. This on-shell value, s_2, is to be distinguished from the off-shell values

$$\begin{aligned} s_{\alpha f} &= (k_2 + p_f)^2 , \\ s_{\alpha' i} &= (k'_\alpha + p_i)^2 . \end{aligned} \tag{3.8}$$

Of course, if k_α and p_f are the final detected values (i.e., there are no interactions subsequent to that of Fig. 4a), then $s_2 = s_{\alpha f}$.

In accord with Eq. (3.6), we choose

$$t_2(s_2, t) = \frac{-is_2}{32\pi^3} \sigma_{tot} \exp(bt/2) \;, \tag{3.9}$$

using $\kappa_2 \to \sqrt{s_2}/2$; the slope parameter b is to be taken from the high energy behavior $d\sigma_{e\ell}/dt \simeq \sigma_{e\ell}(0) \exp(bt)$. This expression implies that b and σ_{tot} are constant, which is strictly true only in the limit $s_2 \to \infty$. In practice, however, s_2 is finite, and takes on values considerably smaller than s; i.e., the factor $(\epsilon_i + E_i)$ is proportional to s, and the proportionality coefficient varies significantly over the angular range required in a partial-wave projection of T_p. For example, in $Kp \to (K\pi\pi)p$ at $p_L = 13$ GeV/c, $s \simeq 25$ (GeV/c)^2, whereas s_2 can be as small as several (GeV/c)^2. Therefore, since σ_{tot} and b vary appreciably over this range of s_2, it seems advisable to build some energy-dependence into these parameters. For this purpose, a simple parametrization

$$\begin{aligned} b &= \alpha + \beta p_{2,L} \;, \\ \sigma_{tot} &= \sigma_{tot}^{\infty} + \gamma p_{2,L}^{-\frac{1}{2}} \;, \end{aligned} \tag{3.10}$$

was employed in the numerical work described below; here $p_{2,L}$ is the (two-particle) lab momentum corresponding to the invariant energy $\sqrt{s_2}$. Values for $K\pm p$ and $\pi\pm p$ scattering in accord with experimental data in the range 3-10 GeV/c are given in Table I[19]. We note that our expression for $t_2(s_2,t)$ is, strictly speaking, the spin non-flip amplitude, and we have set the spin-flip components equal to zero. Numerically, this is quite reasonable in the present application, but one should in general employ $t_2^{\rho\sigma}(s_2,t)$, where ρ, σ label helicity states.

The Deck amplitude corresponding to Fig. 4a is invariably calculated by simply using $t_2(s_2,t)$ for the t_2 vertex function, ignoring the fact that particle

α' is far off its mass-shell. In our RST formalism, the corresponding statement is that the t_2 scattering amplitude we employ is off the energy-shell $(s_2 \neq s_{\alpha' i})$. We must thus introduce an appropriate off-shell extension of $t_2(s_2, t)$ in the sense of Eq. (2.38). Under most circumstances, a possible advantage of the RST approach is that one may be guided by the analogy of potential scattering (or by approximate relativistic treatments such as the two-particle BS equation). Thus, in a given partial-wave ℓ, one might try

$$t_2^{\ell}(s_2) \rightarrow f_{\ell}(s_{\alpha f}, s_2)\, t_2^{\ell}(s_2)\, f_{\ell}(s_{\alpha' i}, s_2) \quad , \tag{3.11}$$

where $f_{\ell}(s_2', s_2)$ is analogous to the ratio in Eq. (2.38); $f_{\ell}(s_2, s_2) = 1$. Unfortunately, the diffractive amplitude is an exceptional case, and one simply cannot associate it with a credible potential-like mechanism. Furthermore, it is very unnatural to decompose it into partial-waves; one requires an infinite number of ℓ-states to reproduce the characteristic small-angle behavior. Under these circumstances, we have chosen to introduce a purely ad hoc prescription, and to employ it merely as an illustration of the off-shell effects one might anticipate. We thus take

$$\begin{aligned} t_2(k_\alpha p_f | k_\alpha' \, p_i ; s_2) &= f_2(s_{\alpha f}, s_2)\, t_2(s_2, t)\ f_2(s_{\alpha' i}, s_2) \ , \\ f_2(s_2', s_2) &= g_2(s_2')/g_2(s_2) \quad , \end{aligned} \tag{3.12}$$

and consider a variety of simple choices for the function $g_2(s_2)$; e.g., $g_2 = (s_2 + m^2)^{-1}$, where m is some mass. These choices, and the corresponding effects on the subenergy-dependence of T_p, and the M_3-dependence of $d\sigma/dM_3\,dt$, are discussed below in Sec. V.A.

While it is most convenient to label the diffractive amplitude t_2 by the charges of the particles αf (or $\alpha' i$), we shall want T_p amplitudes corresponding to a specified total isospin I for the three-particle $(\alpha\beta\gamma)$ state, and a specified pair isospin I_α for the decaying isobar $(\beta\gamma)$, in discussing the subsequent rescattering. We will thus associate the isospins i_k, and corresponding third components μ_k, with the scattering particles as illustrated in Fig. 6. If we denote the operator t_2 corresponding to Eq. (3.12) by $t_2(\mu_\alpha)$, the required isospin-labeled operator is[20]

$$t_2(I, I_\alpha) = \sum_{\mu_\alpha} C(I_\alpha i_\alpha I; \mu_m - \mu'_\alpha, \mu_\alpha) * \\ * C(I_\alpha i'_\alpha i_m; \mu_m - \mu'_\alpha, \mu'_\alpha) t_2(\mu_\alpha) \quad , \tag{3.13}$$

where μ'_α is calculated in terms of μ_2 and the isospins associated with the external lines; i.e., $\mu'_\alpha = \mu_\alpha + \mu_p - \mu'_p$.

C. <u>Three-Body Vertex (Simple Model)</u>

As noted above, the "simple" model for T_3, in which the incoming particle is viewed as dissociating directly into a three-body state $(\alpha'\beta\gamma)$, is incompatible with the empirical behavior of the diffractive cross section. However, in the course of verifying this fact numerically, we were forced to construct a model for the 1-to-3 vertex which might be of some interest in other applications. This subsection has been included in that spirit, but is not a prerequisite for the remainder of this article.

We are thus concerned with the amplitude $T_3(k'_\alpha k_\beta k_\gamma | k_i ; s_3)$, where

$$s_3 = (P_0 - p_i)^2 = m_i^2 \ . \tag{3.14}$$

This amplitude describes the vertex shown in Fig. 7a, which, in principle, is related by crossing to the 2-to-2 scattering process of Fig. 7b; i.e.,

$$T_3(k'_\alpha k_\beta k_\gamma \big| k_i ; s_3) \rightarrow t_{i\alpha' \rightarrow \beta\gamma}(k_\beta k_\gamma \big| -k'_\alpha k_i ; M^2_{\beta\gamma}) \quad . \tag{3.15}$$

Here we have used the arrow to emphasize that this constraint is really only of use on-shell (4-momentum conserved), and in the sense of an analytic continuation. In practice this means that one might as well vary T_3 freely, unless one considers the very special case $T_3 = t_{i\alpha' \to \beta\gamma}$ = constant (we shall return to this point in the next subsection).

Assuming this, the problem we wish to address is how to construct a T_3 amplitude which has the following characteristics: (a) its decomposition in terms of the $(\beta\gamma)$ pair angular-momentum ℓ_α is trivial; (b) it has the required symmetry properties when two or more of the $\alpha'\beta\gamma$ are identical particles. In particular, we shall assume that i is a pseudo-scalar (PS) meson dissociating into three PS mesons; thus $L=0$, where $L=\vec{\ell}_\alpha + \vec{\lambda}_\alpha$ is the total angular-momentum of the three-body system, λ_α being the angular momentum of particle α' in the $\beta\gamma$ c.m. frame. In general, T_3 is then a function of the three subenergies $(s_1 s_2 s_3)$, $s_{\alpha'} \equiv (k_\beta + k_\gamma)^2$, $\alpha'\beta\gamma$ cyclic permutations of 123; the idea is to choose a form compatible with the above constraints. In the special case $\ell_\alpha = 0$ (e.g., $\pi \to \epsilon(\pi\pi)\pi$) this is trivial, since we may simply take T_3 to be a function of $\bar{M}_3$, where

$$\begin{aligned} \bar{M}_3^2 &= (k'_\alpha + k_\beta + k_\gamma)^2 \\ &= \sum_{\delta=1}^{3} (s_\delta - m_\delta^2) \ , \end{aligned} \tag{3.16}$$

and m_δ is the mass of particle δ. Since $\bar{M}_3$ is symmetric under any permutation, we have, for example,

$$T_3^{(o)}(\pi \to 3\pi) = \psi_{\epsilon\pi}(\bar{M}_3)\left[\phi^{(o)}(123) + \phi^{(o)}(231) + \phi^{(o)}(312)\right] \ , \tag{3.17}$$

where $\phi^{(o)}(123) = C(110;\mu_2\mu_3)$, μ_δ being the third component of isospin for particle δ. Here we have used the label "ϵ" to suggest the decomposition into the s-wave dipion state; there is, however, no reference to dipion isobars in this description (we shall similarly use κ, ρ, K* below). Correspondingly,

$$T_3^{(o)}\,(K\to K\pi\pi) = \psi_{\epsilon K}(\overline{M}_3)\;\phi^{(o)}(123) + \psi_{\kappa\pi}(\overline{M}_3)\left[\phi_2(231) - \phi_3(312)\right] \quad , \tag{3.18}$$

with

$$\begin{aligned}\phi_2(231) &= C(1\tfrac{1}{2}\tfrac{1}{2};\mu_3\mu_1)\;C(\tfrac{1}{2}1\tfrac{1}{2};\mu_3+\mu_1,\mu_2) \quad ,\\ \phi_3(312) &= C(\tfrac{1}{2}1\tfrac{1}{2};\mu_1\mu_2)\;C(\tfrac{1}{2}1\tfrac{1}{2};\mu_1+\mu_2,\mu_3) \quad .\end{aligned} \tag{3.19}$$

Here we have adopted the convention that the non-identical particle (if any) is particle 1; since $\phi^{(o)}(123)\to\phi^{(o)}(123)$ under the permutation P_{23}, and $\phi_2\to-\phi_3$, $\phi_3\to-\phi_2$, $T_3^{(o)}(K\to K\pi\pi)$ is properly symmetric under the interchange of the pion labels. The functions $\psi(\overline{M}_3)$, of course, are entirely arbitrary, and are essentially wave functions for the three-body "bound" state; i.e., one expects them to approach zero as $\overline{M}_3\to\infty$.

However, the case $\ell_\alpha = \lambda_\alpha = 1$ (e.g., $\pi\to\rho(\pi\pi)\pi$) is more complicated, and the solution may be of some interest. Here we would like to take out the explicit factor $\vec{k}_\alpha\cdot\vec{p}_\alpha$, where $\vec{p}_\alpha$ is the momentum of particle β in the $\beta\gamma$ c.m., and $\vec{k}_\alpha$ is the momentum of particle α (in that frame). Unfortunately, we cannot simply write $T_3 = (\vec{k}_\alpha\cdot\vec{p}_\alpha)\psi(\overline{M}_3)$ and still satisfy the symmetry properties. We thus proceed along the following lines, first introducing the functions

$$\begin{aligned}f_\alpha(s_1s_2s_3) &= 4\,s_\alpha(\vec{k}_\alpha\cdot\vec{p}_\alpha)\\ &= s_\alpha(s_\beta - s_\gamma) + (m_\beta^2 - m_\gamma^2)(\overline{M}_3^2 - m_\alpha^2) \quad ,\end{aligned} \tag{3.20}$$

$\alpha\beta\gamma$ cyclic. One may easily verify that

$$\begin{aligned}P_{\beta\gamma}f_\alpha &= -f_\alpha \quad ,\\ P_{\alpha\beta}f_\alpha &= -f_\beta \quad ,\\ P_{\alpha\gamma}f_\alpha &= -f_\gamma \quad .\end{aligned} \tag{3.21}$$

We may then construct

$$T_3^{(1)}(\pi \to 3\pi) = \psi_{\rho\pi}(\bar{M}_3)\left[f_1\phi^{(1)}(123) + f_2\phi^{(1)}(231) + f_3\phi^{(1)}(312)\right] , \tag{3.22}$$

where

$$\begin{aligned} &\phi^{(1)}(123) \equiv C(111;\mu_2\mu_3)\ C(111;\mu_2+\mu_3,\mu_1) , \\ &P_{23}\phi^{(1)}(123) = -\phi^{(1)}(123) , \\ &P_{23}\phi^{(1)}(231) = -\phi^{(1)}(312) . \end{aligned} \tag{3.23}$$

We note that by evaluating $s_1s_2s_3$, e.g., in the $\alpha\gamma$ c.m. in terms of $\vec{k}_\beta \cdot \vec{p}_\beta$, one may express f_α as a quadratic in $(\vec{k}_\beta \cdot \vec{p}_\beta)$. Thus, as a necessary consequence of symmetry, $T_3^{(1)}$ contains both s- and d- wave components in the variable $\cos\Theta_\alpha = \hat{k}_\alpha \cdot \hat{p}_\alpha$, as well as the assumed p-wave. The algebra for isolating these components is straightforward, and hence Eq. (3.22) solves the problem. The corresponding solution for $K \to K\pi\pi$ is

$$\begin{aligned} T_3^{(1)}(K \to K\pi\pi) &= \psi_{\rho K}(\bar{M}_3)\ f_1\phi_1(123) \\ &\quad + \psi_{K^*\pi}(\bar{M}_3)\ \ f_2\phi_2(231) + f_3\phi_3(312) , \end{aligned} \tag{3.24}$$

with

$$\phi_1(123) \equiv C(111;\mu_2+\mu_3)\ C(1\tfrac{1}{2}\tfrac{1}{2};\mu_2+\mu_3,\mu_1) . \tag{3.25}$$

D. Three-Body Vertex (Sequential Model)

In the sequential model T_3 corresponds to the diagram shown in Fig. 8a. Thus, we look for T_3 in the form

$$T_3(k'_\alpha k_\beta k_\gamma | k_i; s_3) = \frac{g^F_{r\beta\gamma}}{M^2_{\beta\gamma} - m^2_\alpha}\ c_3 f_{r\alpha'}(k_{\beta\gamma} k'_\alpha) \tag{3.26}$$

where $k_{\beta\gamma} \equiv k_\beta + k_\gamma$, m_α is the mass of the isobar, c_3 is some constant, and

$f_{r\alpha'}$ is a vertex form factor such that $f_{r\alpha'} = 1$ if $k^2_{\beta\gamma} = m^2_\alpha$ and $(k_{\beta\gamma} + k'_\alpha)^2 = s_3 = m^2_i$ (see Eq. (3.14)). In order to determine c_3, we note that T_3 is related by crossing to the amplitude for Fig. 8b; i.e., on-shell,

$$T_3(-k'_\alpha k_\beta k_\gamma | k_i; s_3) = t_{i\alpha' \to \beta\gamma}(k_i k'_\alpha | k_\beta k_\gamma; M^2_{\beta\gamma}) \qquad (3.27)$$

$$\longrightarrow P_{\ell_\alpha}(\hat{p}_\alpha \cdot \hat{k}_i)\, g^F_{r\beta\gamma}\, g^F_{ri\alpha'} \Big/ (M^2_{\beta\gamma} - m^2_\alpha) \,.$$

Here ℓ_α is the spin of the isobar, and we have introduced $\vec{p}_\alpha$ as the value of $\vec{k}_\beta$ in the $\beta\gamma$ c.m. frame ($\alpha\beta\gamma$ cyclic); thus c_3 is just $g^F_{ri\alpha'}$ times the Legendre polynomial. In practice, however, we wish to explicitly extract the phase space factors $|\vec{p}_\alpha|^{\ell_\alpha}$, $|\vec{k}_i|^{\ell_\alpha}$. Noting that the on-shell amplitudes have the ratio

$$\frac{t_{i\alpha' \to \beta\gamma}(M^2_{\beta\gamma})}{t_{\beta\gamma \to \beta\gamma}(M^2_{\beta\gamma})} \xrightarrow[M_{\beta\gamma} \to m_\alpha]{} \frac{g^F_{ri\alpha'}}{g^F_{r\beta\gamma}} \,, \qquad (3.28)$$

whereas

$$t_{\beta\gamma \to \beta\gamma}(M^2_{\beta\gamma}) \to \frac{(2\ell_\alpha + 1)}{4\pi} P_{\ell_\alpha}(\hat{p}_\alpha \cdot \hat{k}_i)\, t^{\ell\alpha}_\alpha(M^2_{\beta\gamma}) \,, \qquad (3.29)$$

we are led to the form

$$T_3(k'_\alpha k_\beta k_\gamma | k_i; s_3) = \frac{(2\ell_\alpha + 1)}{4\pi} P_{\ell_\alpha}(\hat{p}_\alpha \cdot \hat{k}_i)\,(|\vec{k}_i| / |\vec{p}_\alpha|)^{\ell\alpha} \,* \qquad (3.30)$$

$$* \frac{g^F_{ri\alpha'}}{g^F_{r\beta\gamma}}\, t^{\ell\alpha}_\alpha(M^2_{\beta\gamma})\, \tilde{f}_{r\alpha'}(\bar{M}^2_3) \,.$$

Here we have employed the notation $t^{\ell\alpha}_\alpha(M^2_{\beta\gamma})$ for the elastic $\beta\gamma$ scattering amplitude ($\alpha\beta\gamma$ cyclic) in partial-wave ℓ_α; clearly $t^{\ell\alpha}_\alpha(M^2_{\beta\gamma}) \propto |\vec{p}_\alpha|^{2\ell\alpha}$. We have also introduced the quantity

$$\bar{M}^2_3 = (k_r + k'_\alpha)^2 = m^2_\alpha + m'^2_\alpha + 2m_\alpha \epsilon'_\alpha \qquad (3.31)$$

in the $\beta\gamma$ c.m., and added an off-shell vertex factor $\tilde{f}_{r\alpha'}$ such that $\tilde{f}_{r\alpha'}(m_i^2)=1$. To the extent that the g^F couplings are specified, $\tilde{f}_{r\alpha'}$ is the only unknown in our expression for T_3, since $t_\alpha^{\ell\alpha}$ is empirically determined (e.g., in terms of the phase shift δ_{ℓ_α}). We note that for applications to $\pi \to (3\pi)$ the factor $g^F_{ri\alpha'}/g^F_{r\beta\gamma}$ is always unity; whereas in $K \to K\pi\pi$, the graph where α' is the K leads to the ratio $g^F_{\rho KK}/g^F_{\rho\pi\pi}$ (in the numerical work below we adopt the SU(3) value of 1/2 for this ratio).

Although the result given in Eq. (3.30) is a very reasonable parametrization of the 1-to-3 vertex, it should be clear that only the limit of Eq. (3.27) in the unphysical region is uniquely defined; e.g., one might append a factor such as $M_{\beta\gamma}/m_\alpha$ which would affect our numerical results, but leave that limit invariant. As a consequence, one must recognize that the absolute normalization of our model is necessarily approximate, and anticipate the introduction of some scaling factors in fitting actual data. Equivalently, the on-shell condition $\tilde{f}_{r\alpha'}=1$ does not uniquely specify the normalization except in the very special case $\tilde{f}_{r\alpha'}\equiv 1$. The parametrization of that function, and the corresponding effects with respect to the amplitude subenergy dependence and the M_3 — dependence of the cross section, are discussed in Sec. V.B.

IV. PARTIAL-WAVE DECOMPOSITION AND KINEMATICS

In this section we deal with the specifics of calculating T_p in a partial-wave projection, and present the formulas for computing $d\sigma/dM_3\,dt$ in terms of the partial-wave amplitudes (PWA). Our development parallels very closely that of Ascoli, Jones, Weinstein and Wyld, who dealt specifically with $\pi^- p \to (\pi^+\pi^-\pi^-)p$.[15] We shall, however, consider the process of Fig. 4a in more generality, allowing arbitrary masses and isospins for all of the particles involved. The application of the resultant formulas to distinct cases is then largely a matter of inserting appropriate expressions for the functions t_2 and T_3 discussed above.

In practice we shall wish to write T_p as the sum (over α) of the individual exchange graphs represented by the amplitude of Eq. (2.44); we shall thus add an index (α) to that expression, and introduce the expansion

$$T_p^{\alpha}(k_\alpha k_\beta k_\gamma p_f \,|\, k_i p_i) = \sum_{LM\ell_\alpha\lambda_\alpha} \tau_p^{\alpha}(LM\ell_\alpha\lambda_\alpha) \quad \mathscr{Y}^{LM}_{\ell_\alpha\lambda_\alpha}(\hat{p}_\alpha, \hat{k}_\alpha) \tag{4.1}$$

where $LM\ell\alpha\lambda\alpha$ are the total angular momentum, its projection along $\hat{k}_\alpha$, the isobar spin, and the angular momentum of particle α in the $\beta\gamma$ (isobar) c.m. frame, respectively. Here $\vec{p}_\alpha$ is the value of $\vec{k}_\beta$ in the $\beta\gamma$ c.m. ($\alpha\beta\gamma$ cyclic), $\hat{k}_\alpha$ is the direction of $\vec{k}_\alpha$ in the three-body ($\alpha\beta\gamma$) c.m., and

$$\mathscr{Y}^{LM}_{\ell_\alpha\lambda_\alpha}(\hat{p}_\alpha, \hat{k}_\alpha) = \sum_m C(\ell_\alpha\lambda_\alpha L;\, m, M-m)\; Y_{\ell_\alpha m}(\hat{p}_\alpha)\; Y_{\lambda_\alpha\, M-m}(\hat{k}_\alpha)\,. \tag{4.2}$$

We note that $\vec{k}_\alpha$ has the same direction $\hat{k}_\alpha$ (but not magnitude) in the $\beta\gamma$ c.m., and that $\mathscr{Y}^{LM}_{\ell\lambda}$ has the alternate expression

$$\mathscr{Y}^{LM}_{\ell\lambda}(\hat{p}_\alpha, \hat{k}_\alpha) = \frac{1}{4\pi}\left[(2\ell+1)(2\lambda+1)\right]^{\frac{1}{2}} \sum_\mu C(\ell\lambda L;\mu\, 0) \; * \tag{4.3}$$

$$* \; D^{L}_{M\mu}{}^{*}(\omega_\alpha)\; d^{\ell}_{\mu 0}(\tilde{\Theta}_\alpha)\,,$$

where $\cos \tilde{\Theta}_\alpha = \hat{p}_\alpha \cdot \hat{k}_\alpha$ ($0 \le \tilde{\Theta}_\alpha \le \pi$), and ω_α represents the Euler angles which define the rotation R_α of the "α" coordinate system, shown in Fig. 9b, into the "fixed" coordinate system, shown in Fig. 9a. In particular, it is useful to employ a standard reference configuration for the three-body final state, which we take to be Fig. 9b with $\alpha = 1$. The assignment of the integers (123) to the three-particle system is clearly arbitrary; in $K\pi\pi$ we shall take particle 1 to be the K. In general, we take $(\Theta_\alpha, \phi_\alpha)$ to be the spherical coordinates of $\hat{k}_\alpha$ in the fixed coordinate system; then $\omega_\alpha = (\phi_\alpha, \Theta_\alpha, 0)$, and

$$D^L_{M\mu}(\omega_\alpha) = \exp(-iM\phi_\alpha)\, d^L_{M\mu}(\Theta_\alpha) \quad . \tag{4.4}$$

As defined in Eq. (4.2), the functions $\mathscr{Y}^{LM}_{\ell\lambda}$ are orthonormal on the angular space $\hat{p}_\alpha$, $\hat{k}_\alpha$, and hence

$$\tau^\alpha_p(LM\ell_\alpha\lambda_\alpha) = \int d\hat{p}_\alpha d\hat{k}_\alpha \mathscr{Y}^{LM*}_{\ell_\alpha\lambda_\alpha}(\hat{p}_\alpha, \hat{k}_\alpha)\; T^\alpha_p(k_\alpha k_\beta k_\gamma p_f | k_i p_i) \quad . \tag{4.5}$$

In order to perform the integration we must determine the dependence of T^α_p on the angles. To do so it appears easiest to evaluate certain quantities in the $\beta\gamma$ c.m., whereas others are simplest in the three-body c.m. We shall thus adopt the convention that energies, etc. in the three-body c.m. are distinguished by a bar overhead; e.g., $\bar{\epsilon}_\alpha$ is the energy of particle α in the c.m. frame. In particular

$$\begin{aligned} \bar{E}_f &= (s - M_f^2 - M_3^2)/2M_3 \quad , \\ \bar{E}_i - \bar{E}_f &= (M_3^2 + t - m_i^2)/2M_3 \quad , \\ \left|\overline{\vec{p}_i - \vec{p}_f}\right| &= [(\bar{E}_i - \bar{E}_f)^2 - t]^{\frac{1}{2}} \quad . \end{aligned} \tag{4.6}$$

We shall also require $\overline{\hat{k}_\alpha \cdot \hat{p}_f} \equiv \cos\Theta_{\alpha f}$ (all angles except $\tilde{\Theta}_\alpha$ are evaluated in the three-body c.m.) This is evaluated via the expressions

$$\cos\Theta_{\alpha f} = \cos\Theta_\alpha \cos\Theta_f + \sin\Theta_\alpha \sin\Theta_f \cos\phi_\alpha \quad , \tag{4.7}$$

$$\cos\Theta_f = \frac{2\overline{E}_f(\overline{E}_f - \overline{E}_i) + M_i^2 - M_f^2 - t}{2(\overline{E}_f^2 - M_f^2)^{\frac{1}{2}}\left[(\overline{E}_i - \overline{E}_f)^2 - t\right]^{\frac{1}{2}}} \quad ,$$

where we have used $\vec{k}_i = \vec{p}_f - \vec{p}_i$ to evaluate $\cos\Theta_f = \overline{\hat{p}_f \cdot \hat{k}_i}$. The energy of particle α in the two frames is given by

$$\begin{aligned} \epsilon_\alpha &= (M_3^2 - M_{\beta\gamma}^2 - m_\alpha^2)\big/2M_{\beta\gamma} \quad , \\ \overline{\epsilon}_\alpha &= (M_{\beta\gamma}\epsilon_\alpha + m_\alpha^2)\big/M_3 \quad . \end{aligned} \tag{4.8}$$

In terms of the above, it is straightforward to derive the following expressions for the $\beta\gamma$ frame quantities (e.g., by considering invariants such as $k_\alpha \cdot (p_i - p_f)$ in both frames),

$$\begin{aligned} M_{\beta\gamma} E_f &= (M_3 - \overline{\epsilon}_\alpha)\overline{E}_f + \cos\Theta_{\alpha f}(\overline{\epsilon}_\alpha^2 - m_\alpha^2)^{\frac{1}{2}}(\overline{E}_f^2 - M_f^2)^{\frac{1}{2}} \quad , \\ M_{\beta\gamma}(E_i - E_f) &= (M_3 - \overline{\epsilon}_\alpha)(\overline{E}_i - \overline{E}_f) - \cos\Theta_\alpha(\overline{\epsilon}_\alpha^2 - m_\alpha^2)^{\frac{1}{2}}\left[(\overline{E}_i - \overline{E}_f)^2 - t\right]^{\frac{1}{2}} \quad , \\ \epsilon_i &= M_{\beta\gamma} + \epsilon_\alpha + E_f - E_i \quad , \\ \epsilon'_\alpha &= (\epsilon_i^2 - m_i^2 + m_\alpha'^2)^{\frac{1}{2}} \quad . \end{aligned} \tag{4.9}$$

Similarly, the pair energies required for the t_2 factor are

$$\begin{aligned} s_2 &= s + M_{\beta\gamma}^2 - 2M_{\beta\gamma}(\epsilon_i + E_i) \quad , \\ s_{\alpha' i} &= s + m_\alpha'^2 - m_i^2 + 2(\epsilon'_\alpha - \epsilon_i)E_i \quad , \\ s_{\alpha f} &= s + m_\alpha^2 - M_3^2 - 2M_{\beta\gamma}E_f \quad . \end{aligned} \tag{4.10}$$

Finally, we will also need

$$\vec{k}_\alpha \cdot (\vec{p}_f - \vec{p}_i) = (M_{\beta\gamma} + \epsilon_\alpha)(E_f - E_i) + M_3(\overline{E}_i - E_f) \quad . \tag{4.11}$$

Recalling Eq. (3.30), we observe that the direction $\hat{p}_\alpha$ only occurs in $P_{\ell_\alpha}(\hat{p}_\alpha \cdot \hat{k}_i)$, and

$$\int d\hat{p}_\alpha \mathscr{Y}^{LM*}_{\ell_\alpha \lambda_\alpha}(\hat{p}_\alpha, \hat{k}_\alpha)\, P_{\ell'_\alpha}(\hat{p}_\alpha \cdot \hat{k}_i) = \delta_{\ell_\alpha \ell'_\alpha} \frac{4\pi}{2\ell\alpha + 1} \mathscr{Y}^{LM}_{\ell_\alpha \lambda_\alpha}(\hat{k}_i, \hat{k}_\alpha) \quad . \tag{4.12}$$

Combining Eqs. (2.44), (3.12), (3.30), and the above, we obtain the result

$$\tau^\alpha_p(LM\ell_\alpha\lambda_\alpha) = \frac{g^F_{ri\alpha'}}{g^F_{r\beta\gamma}} \frac{t^{\ell_\alpha}_\alpha(M^2_{\beta\gamma})}{2\pi} \left[(2\ell_\alpha + 1)(2\lambda_\alpha + 1)\right]^{\frac{1}{2}} * \tag{4.13}$$

$$* \sum_\mu C(\ell_\alpha \lambda_\alpha L; \mu 0) \int_{-1}^{1} d\cos\Theta_\alpha\, d^{\ell_\alpha}_{\mu o}(\tilde{\Theta}_{i\alpha})\, d^L_{M\mu}(\Theta_\alpha)\, F^\alpha_p(M\ell_\alpha\Theta_\alpha) \quad ,$$

where

$$F^\alpha_p(M\ell_\alpha\Theta_\alpha) = -\left(|\vec{k}_i| / |\vec{p}_\alpha|\right)^{\ell_\alpha} \frac{\tilde{f}_{r\alpha'}(\overline{M}^2_3)}{\epsilon'_\alpha(M_{\beta\gamma} + \epsilon'_\alpha - \epsilon_i)} * \tag{4.14}$$

$$* \int_o^\pi d\phi_\alpha \cos(M\phi_\alpha)\, f_2(s_{\alpha f}, s_2)\, t_2(s_2, t)\, f_2(s_{\alpha' i}, s_2) \quad .$$

In Eq. (4.13) we have used $\cos\tilde{\Theta}_{i\alpha} = \hat{k}_i \cdot \hat{k}_\alpha$; this is computed from the relation

$$\cos\tilde{\Theta}_{i\alpha} = \frac{\epsilon^2_\alpha - m^2_\alpha + (M_{\beta\gamma} + \epsilon_\alpha)(E_f - E_i) + M_3(\overline{E}_i - \overline{E}_f)}{(\epsilon^2_i - m^2_i)^{\frac{1}{2}} (\epsilon^2_\alpha - m^2_\alpha)^{\frac{1}{2}}} \quad , \tag{4.15}$$

where we have used Eqs. (2.45) and (4.11). We note that $-\pi \leq \tilde{\Theta}_{i\alpha} \leq 0$; this can be seen from the fact that $|\tilde{\Theta}_{i\alpha}| = |\Theta_\alpha|$ in the nonrelativistic (NR) limit, whereas

Θ_α is the angle $\hat{k}_\alpha$ makes with respect to $\hat{k}_i$ in the three-body c.m., and $\tilde{\Theta}_{i\alpha}$ is the angle $\hat{k}_i$ makes with respect to $\hat{k}_\alpha$ (as the z axis) in the $\beta\gamma$ c.m. As a consequence, using standard properties of the rotation functions, one deduces that

$$\begin{aligned}&\sum_\mu C(\ell_\alpha \lambda_\alpha L;\mu 0)\, d^{\ell\alpha}_{\mu o}(\tilde{\Theta}_{i\alpha})\, d^L_{M\mu}(\Theta_\alpha)\\ &\xrightarrow[\text{NR}]{} C(\ell_\alpha \lambda_\alpha L;0,M)\; d^{\lambda\alpha}_{0M}(\Theta_\alpha) \quad .\end{aligned} \tag{4.16}$$

From this one may infer the usual dominance of the $L=1$, $M=0$ $(1+0+)$ state for M_3 near threshold (at $t \simeq t_{min}$, the dependence of F^α_p on ϕ_α, Θ_α is very weak). We also note that the $1+1+$ state arises predominantly from $\lambda_\alpha=1$, $\ell_\alpha=0$ (irregardless of t); e.g., from $\pi \to \epsilon\pi$ or $K \to \epsilon K$.

The above equations provide the necessary information to calculate the PWA $\tau^\alpha_p(LM\ell_\alpha\lambda_\alpha)$, which is a function of $M_{\beta\gamma}$, s and t. Below we shall require the corresponding amplitude in an isospin basis; we denote this by $\tau^\alpha_p(ILM\ell_\alpha\lambda_\alpha I\;)$, and compute it by interpreting $t^{\ell\alpha}_\alpha(M^2_{\beta\gamma})$ as the appropriate isospin (I_α) elastic amplitude, and by replacing $f_2 t_2 f_2$ in Eq. (4.14) by t_2 (I, I_α), as defined in Eq. (3.13). The standard Deck amplitude can be recovered by setting $f_2(s_2', s_2) = \tilde{f}_{r\alpha'}(\bar{M}^2_3) \equiv 1$, and by replacing $\epsilon'_\alpha(M_{\beta\gamma} + \epsilon'_\alpha - \epsilon_i)$ by $m^2_i - t$. We next consider the relation between the PWA's ($\alpha = 1,2,3$) and the differential cross section. We first introduce the channel helicity amplitudes ($J=L$ for our spinless three-body system)

$$\begin{aligned}f^{IJM}_{\alpha M' I_\alpha}(M_3, M^2_{\beta\gamma}, s, t) &= \sum_{\ell_\alpha \lambda_\alpha} \left[\frac{(2\ell_\alpha+1)(2\lambda_\alpha+1)}{2\,(2J+1)}\right]^{\frac{1}{2}} C(\ell_\alpha\lambda_\alpha J; M'0) \,*\\ &* \; d^{\ell_\alpha}_{M'0}(\tilde{\Theta}_\alpha)\; \tau_\alpha(IJM\ell_\alpha\lambda_\alpha I_\alpha) \quad ,\end{aligned} \tag{4.17}$$

where $\tau_\alpha = \tau^\alpha_p$ + rescattering terms (i.e., the full PWA). The isospin wave functions ϕ_α are defined by

$$\phi_\alpha(I_\alpha I;\mu_1\mu_2\mu_3) = C(I_\alpha i_\alpha I;\mu_\beta+\mu_\gamma,\mu_\alpha) * \\ * C(i_\beta i_\gamma I_\alpha;\mu_\beta\mu_\gamma) , \tag{4.18}$$

with $\alpha\beta\gamma$ cyclic, and the isospin conventions of Fig. 6. Employing the three subenergies $s_\alpha \equiv M^2_{\beta\gamma}$, we next form a total helicity amplitude

$$F^{IJM}_{M'}(s_1s_2s_3;\mu_1\mu_2\mu_3;st) = \sum_{I_1} \phi_1(I_1 I;\mu_1\mu_2\mu_3)\, f^{IJM}_{1M'I_1}(M_3, s_1, s, t) \\ + \sum_{I_3, M''} \phi_3(I_3 I;\mu_1\mu_2\mu_3)\, d^J_{M'M''}(-\Theta_{13})\, f^{IJM}_{3M''I_3}(M_3, s_3, s, t) \\ + \sum_{I_2, M''} \phi_2(I_2 I;\mu_1\mu_2\mu_3)\, d^J_{M'M''}(\Theta_{12})\, f^{IJM}_{2M''I_2}(M_3, s_2, s, t) . \tag{4.19}$$

Here $\cos\Theta_{\alpha\beta} = \overline{\hat{k}_\alpha \cdot \hat{k}_\beta}$, and we have put the (-) in the α=3 term so that Θ_{12}, Θ_{13} both lie between 0 and π. These angles may be computed in terms of s_1, s_2, s_3 via the relations

$$\epsilon_\alpha = (M_3^2 + m_\alpha^2 - s_\alpha)/2M_3 , \\ \cos\Theta_{\beta\gamma} = \frac{2\,\epsilon_\beta\,\epsilon_\gamma - s_\alpha + m_\beta^2 + m_\gamma^2}{2(\epsilon_\beta^2 - m_\beta^2)^{\frac{1}{2}}(\epsilon_\gamma^2 - m_\gamma^2)^{\frac{1}{2}}} . \tag{4.20}$$

The angle $\tilde{\Theta}_\alpha$ required in Eq. (4.17) is determined (in the range 0 to π) by

$$\cos\tilde{\Theta}_\alpha = \frac{s_\beta - s_\gamma + (m_\beta^2 - m_\gamma^2)(M_3^2 - m_\alpha^2)/s_\alpha}{4\,|\vec{p}_\alpha|(\epsilon_\alpha^2 - m_\alpha^2)^{\frac{1}{2}}} , \tag{4.21}$$

where we have used Eq. (3.20). Finally, the total invariant amplitude T is given by

$$T = \sum_{IJM} T^{IJM}_{\mu_1\mu_2\mu_3} ,$$

$$T^{IJM}_{\mu_1\mu_2\mu_3}(k_1 k_2 k_3 p_f \,|\, k_i p_i) = \sum_{M'} \left(\frac{2J+1}{8\pi^2}\right)^{\frac{1}{2}} D^{J\,*}_{MM'}(\omega_1)\, F^{IJM}_{M'}\; * \tag{4.22}$$
$$* \,(s_1 s_2 s_3 ; \mu_1 \mu_2 \mu_3 ; st) \;.$$

One may verify this formula for the $\alpha = 1$ part of $F^{IJM}_{M'}$ by comparing Eq. (4.1) with Eqs. (4.17), (4.19), (4.21) and (4.22), using the alternate expression for $\mathscr{Y}^{LM}_{\ell\lambda}$ in Eq. (4.3); the $\alpha \neq 1$ contributions then follow as a simple consequence of rotating from the "α" coordinate system of Fig. 9b to the reference configuration $\alpha = 1$.

In general, the invariant amplitudes calculated in our RST formalism are normalized such that the cross section σ is given by

$$\sigma = \int \prod_{j=1}^{n} \left(\frac{d\vec{k}_j}{\epsilon_j}\right) (2\pi)^4\, \delta(P_f - P_o)\, \frac{|T|^2}{[(p_1 \cdot p_2)^2 - p_1^2 p_2^2]^{\frac{1}{2}}} \;, \tag{4.23}$$

where $P_f = \Sigma k_j$, and the incoming two particles have 4-momenta p_1, p_2 ($P_o = p_1 + p_2$). In the present case the phase space is

$$\int \frac{d\vec{p}_f}{E_f} \prod_{\alpha=1}^{3} \frac{d\vec{k}_\alpha}{\epsilon_\alpha}\, \delta(P_f - p_i - k_i) = \frac{\pi}{4 M_i p_L} \int d\omega_1 ds_1 ds_2 dt \frac{dM_3}{M_3} \;. \tag{4.24}$$

Inserting the expression for T in Eq. (4.22) into the above, we obtain

$$\frac{d\sigma}{dM_3 dt} = \frac{(2\pi)^5}{8\,(M_i p_L)^2 M_3} \int ds_1 ds_2 \sum_{JMM'} \left|\overline{F}^{J}_{MM'}\right|^2 \;, \tag{4.25}$$

$$\overline{F}^{J}_{MM'} = \sum_{I} F^{IJM}_{M'}(s_1 s_2 s_3 ; \mu_1 \mu_2 \mu_3 ; st) \;.$$

We conclude this section with several observations regarding the above formulae. We first note that one may separate the amplitudes further according to parity (P) by dividing the sum over $\ell_\alpha, \lambda_\alpha$ in Eq. (4.17) into $\ell_\alpha + \lambda_\alpha =$ even vs.

$\ell_\alpha + \lambda_\alpha =$ odd; the corresponding symmetries guarantee that P is conserved; i. e., there is no interference term in Eq. (4.25). Secondly, if one takes into account the target helicity states, all amplitudes acquire appropriate indices ρ, σ as noted in Sec. III. B, and

$$\sum_{JMM'} \left| \overline{F}^{J}_{MM'} \right|^2 \rightarrow \frac{1}{2s_T+1} \sum_{\rho\sigma} \sum_{JMM'} \left| \overline{F}^{J;\rho\sigma}_{MM'} \right|^2 .$$

Also, if there are n_I identical particles in the final state, $d\sigma$ acquires the additional factor $(n_I!)^{-1}$. Finally, if all three particles are identical our PWA's τ_α satisfy $\tau_1 = \tau_2 = \tau_3$; whereas, if particles 2 and 3 are identical, $\tau_3 = (-)^N \tau_2$, where N depends on the channel quantum numbers. Specifically,

$$N = I_1 + I_3 + \ell_1 + \ell_3 + i_1 + i_3 + S \quad , \tag{4.27}$$

where S is the total strangeness of the three-body system, $I_3 \ell_3 i_3$ correspond to the particular quantum numbers common to $\alpha = 2$ and $\alpha = 3$, and the sum $I_1 + \ell_1 + i_1$ is the same for any $\alpha = 1$ channel state.

V. NUMERICAL STUDIES OF THE PRODUCTION AMPLITUDE

A. Two-body Vertex Factor

In Section III. B we introduced a simple off-shell extension of the two-particle elastic amplitude in the special case of diffractive scattering (small t, large s_2); this extension corresponds to the vertex function $f_2(s_2', s_2)$ which appears in Eq. (4.14). In order to gauge the effects of such a factor, we now consider a specific example. Thus, by applying the formulas of the preceding section to the reaction $K^+p \rightarrow (K^+ \pi^+ \pi^-)p$, one may readily obtain numerical results for a variety of simple choices for f_2. In what follows we discuss a representative sampling of such results for the dominant 1^+0^+ partial-wave at $p_L = 13$ GeV/c, $t = -.02$ (GeV/c)^2.

In order to perform an explicit calculation one must choose a particular representation for each of the on-shell amplitudes $t_\alpha^{\ell\alpha}(M_{\beta\gamma}^{\;2})$ and $t_2(s_2, t)$. The choice of the latter (for Kp and πp elastic scattering) is discussed in Section III. B; for the former we consider the two p-wave (ℓ_α=1) channels corresponding to ρK and $K^*\pi$, and define

$$N_\alpha^{\ell_\alpha}(M_{\beta\gamma}^2) = \bar{\gamma}_\alpha = \gamma_\alpha \left(\kappa_\alpha / \kappa_{\alpha,r}\right)^{2\ell_\alpha} \quad ,$$

$$D_\alpha^{\ell_\alpha}(M_{\beta\gamma}^2) = (\kappa_\alpha - \kappa_{\alpha,r} + i\bar{\gamma}_\alpha/2)(\kappa_\alpha + \kappa_{\alpha,r} + i\bar{\gamma}_\alpha/2) \quad , \tag{5.1}$$

where κ_α is the magnitude of the $\beta\gamma$ c.m. momentum, and $\kappa_{\alpha,r}$, γ_α are adjusted to reproduce the isobar mass and width (these parameters clearly depend on the state ℓ_α as well). Numerically, one has $(\kappa_{\alpha,r}, \gamma_\alpha) = (.359, .0857)$ and $(.289, .0356)$ for the ρ and K^* states, respectively (in GeV/c units); these correspond to $\mathcal{M}_r = .770$, $\Gamma_r = .160$ and $\mathcal{M}_r = .892$, $\Gamma_r = .050$. According to our conventions, the corresponding invariant scattering amplitude is given by $t_\alpha^{\ell_\alpha}(M_{\beta\gamma}^2) = (M_{\beta\gamma}/\pi) N_\alpha^{\ell_\alpha} / D_\alpha^{\ell_\alpha}$, and this is the form one should employ in treating the rescattering corrections. However, it is more in the spirit of the sequential (or standard isobar) model to replace the multiplicative factor $M_{\beta\gamma}$ by $\mathcal{M}_\alpha$ (the isobar mass), and we shall thus take ($\kappa_\alpha = |\vec{p}_\alpha|$)

$$\left(|\vec{k}_i| / |\vec{p}_\alpha|\right)^{\ell_\alpha} t_\alpha^{\ell_\alpha}(M^2_{\beta\gamma}) = \frac{\mathcal{M}\alpha}{\pi} \frac{\gamma_\alpha}{\kappa^2_{\alpha,r}} \frac{\left|\vec{p}_\alpha\right|^{\ell_\alpha}\left|\vec{k}_i\right|^{\ell_\alpha}}{D_\alpha^{\ell_\alpha}\left(M^2_{\beta\gamma}\right)} \qquad (5.2)$$

in computing $\tau_p^\alpha(LM\ell_\alpha\lambda_\alpha)$ via Eqs. (4.13), (4.14); i.e., the dependence on $M_{\beta\gamma}$ appears only in the isobar "propagator", $D_\alpha^{\ell\alpha}$. Our motivation for choosing the latter [e.g., instead of simply $M^2_{\beta\gamma} - (\mathcal{M}_\alpha - i\Gamma_\alpha/2)^2$] is that we shall later require a form suitable for analytic continuation below the pair ($M_{\beta\gamma}$) threshold. Although one could clearly employ more sophisticated representations in place of Eq. (5.1), the precise form should not be a critical factor in the present application.

With regard to f_2, we consider the simple parametrization

$$f_2(s_2', s_2) = \left(\frac{s_2 + \mu_2{}^2}{s_2' + \mu_2{}^2}\right)^{n_g} , \qquad (5.3)$$

where n_g is an integer, and μ_2 is some mass defining the scale of the off-shell variation. In a low energy problem, one might estimate μ_2 to be of the order of several pion masses, in which case $\mu_2{}^2$ would be totally negligible in comparison with s_2 or s_2' under the conditions of interest ($s_2, s_2' \propto s$). In fact, μ_2 would have to be several GeV in order to have even a slight effect on the value of f_2. For the purpose of our very qualitative investigation, we thus assume that $s_2, s_2' \gg \mu_2{}^2$, and hence our class of models is defined purely by the value of n_g (for definiteness, we take μ_2=2m_π). Following the usual conventions, we define an "isobar" amplitude $\tilde{\tau}_p^\alpha$ via the relation

$$\tau_p^{\ \alpha}(LM\ell_\alpha\lambda_\alpha) = \frac{\kappa_\alpha^{\ell\alpha}\, Q_\alpha^{\lambda\alpha}}{D_\alpha^{\ell\alpha}(M^2_{\beta\gamma})}\ \tilde{\tau}_p^{\alpha}(LM\ell_\alpha\lambda_\alpha) \quad , \tag{5.4}$$

where $Q_\alpha = |\vec{k}_\alpha|$ in the three-body c.m. (Q_α is the c.m. spectator momentum). The behavior of $\tilde{\tau}_p^\alpha$ as a function of the subenergy $M_{\beta\gamma}$ (for fixed M_3=1.5 GeV) is shown in Figs. 10a, 10b for the choices n_g= -1, 0, 1; the curves are normalized arbitrarily to the value 1.0 at .77 GeV for ρK, and 1.0 at .89 GeV for $K^*\pi$ (to focus on the $\tilde{f}_2$ sensitivity, the calculation was performed with $\tilde{f}_{r\alpha'} \equiv 1$).

Although the specific choice $n_g = -1$ produces an amplitude $\tilde{\tau}_p^\alpha$ whose subenergy dependence is very weak, it is clear that the variation can easily be on the order of 50% or more (of the value at $M_{\beta\gamma} = \mathscr{M}_\alpha$) for permissable definitions of f_2. In fact, the usual (Deck) choice corresponds to n_g=0, and one would normally consider form factors with $n_g > 0$ as more reasonable (corresponding to $f_2 \to 0$ with increasing s_2', fixed s_2). Thus, even without the $M_{\beta\gamma}$ branchpoint noted by Aaron and Amado[3] (which appears in the rescattering terms), it is quite possible for the isobar amplitude to exhibit a considerable subenergy dependence. On the other hand, similar calculations at other values of M_3 yield plots which are virtually identical to those shown (to the order of 10% or so), and hence the dependence is not as complicated as it might be; i.e., $\tilde{\tau}_p^\alpha \simeq A(M_{\beta\gamma})\, B(M_3)$. Moreover, it appears that a linear approximation $A(M_{\beta\gamma}) \simeq A_0 + A_1 M_{\beta\gamma}$ might well account for 90% of the effects. To a lesser extent, this also appears true of the $\tilde{f}_{r\alpha'}$ variations considered below.

Inasmuch as the computation of $\tilde{\tau}_p^\alpha$ requires a double numerical integration (over θ_α and ϕ_α), it becomes rather time-consuming to evaluate the cross section by repetitively evaluating τ_p^α at each requisite combination of s_1, s_2 in Eq. (4.25). This can be avoided by expanding $\tilde{\tau}_p^\alpha$ in a complete orthonormal set on the physically allowed interval of $M_{\beta\gamma}$ (determined by M_3); thus

$$\tilde{\tau}_p^{\alpha}(LM\ell_\alpha\lambda_\alpha) = \sum_{k=1}^{n} b_k^{\alpha;p}(M_3)\phi_k^{\alpha}(M_{\beta\gamma}, M_3) \quad ,$$

$$b_k^{\alpha;p}(M_3) = \int_{m_\beta+m_\gamma}^{M_3-m_\alpha} dM_{\beta\gamma}\, \phi_k^{\alpha}(M_{\beta\gamma}, M_3)\tilde{\tau}_p^{\alpha}(LM\ell_\alpha\lambda_\alpha) \quad . \tag{5.5}$$

The numerical results above (and those below) suggest that the sum might well be truncated at n=2 or 3; all of the calculations reported in this article were performed with n=4, using the explicit choice

$$\phi_k^{\alpha}(M_{\beta\gamma}, M_3) = \left[\frac{2k-1}{M_3-m_1-m_2-m_3}\right]^{\frac{1}{2}} P_{k-1}(x_\alpha) \quad ,$$

$$x_\alpha = \frac{2M_{\beta\gamma}+m_\alpha-m_\beta-m_\gamma-M_3}{M_3-m_1-m_2-m_3} \quad , \tag{5.6}$$

where P_k is the Legendre polynomial (x_α is in the range -1 to +1). Employing this technique, the differential cross sections corresponding to the above f_2 parametrization (n_g = -1, 0, 1) are shown in Figs. 11a, 11b. Although clearly negligible in the ρK channel, the vertex effects are rather substantial for $K^*\pi$ (on the order of 20%). However, it turns out that even the latter are small compared to the variations with $\tilde{f}_{r\alpha'}$, and hence it seems most reasonable to simply choose n_g=0, and to concentrate on the effects of the three-body vertex. Unless otherwise stated, we shall adopt this course in what follows.

B. Three-Body Vertex Factor

Off-shell effects from the three-body vertex are specified in terms of the function $\tilde{f}_{r\alpha'}(\overline{M}_3^2)$, which is essentially a form factor for the dissociation of the incoming particle into a quasi-two-body system (the isobar plus particle α'). From Eq. (3.31), we see that it is effectively a function of ϵ'_α (the energy of α' in the isobar rest frame), and it thus appears simplest to parametrize it in the form

$$\tilde{f}_{r\alpha'}(\overline{M}_3^{\,2}) = g_{r\alpha'}(\epsilon'_\alpha)/g_{r\alpha'}(\hat{\epsilon}'_\alpha) \quad ,$$
$$\hat{\epsilon}'_\alpha = (m_i^2 - m'^2_\alpha - \mathcal{M}^2_\alpha)/2\mathcal{M}_\alpha \quad . \tag{5.7}$$

This guarantees the crossing constraint $\tilde{f}_{r\alpha'}(m_i^2)=1$ (typically $\hat{\epsilon}'_\alpha < 0$). It should be noted, however, that this constraint is very weak in terms of specifying the absolute normalization of the resulting cross section. For example, one might modify any given $\tilde{f}_{r\alpha'}$ by the multiplicative factor $(\epsilon'_\alpha + \epsilon_0)/(\hat{\epsilon}'_\alpha + \epsilon_0)$. Since $\hat{\epsilon}'_\alpha$ is a negative constant, one may clearly choose $\epsilon_0 \simeq |\hat{\epsilon}'_\alpha|$ and hence create an arbitrarily large enhancement in the overall normalization. Thus, once one permits $\tilde{f}_{r\alpha'} \neq$ constant, the absolute normalization cannot be predicted with confidence, and must be taken as a free parameter. Although this introduces a certain latitude in calculating a particular diagram, the same vertex function <u>and</u> its associated normalization factor will also occur in a great variety of other processes, and hence the unknown factor will ultimately be constrained.

For our numerical examples, we again consider a simple parmetrization

$$g_{r\alpha'}(\epsilon'_\alpha) = (\epsilon'^{\,2}_\alpha + \mu^2_{r\alpha'})^{-n_f} \quad , \tag{5.8}$$

where n_f is an integer. In this case the mass parameter $\mu_{r\alpha'}$ can play a role, and it is useful to make some estimates as to its magnitude. In practical terms, ϵ'_α will range from m'_α to a value perhaps several times that via the kinematical relations in Section IV, and hence, to first order, it is reasonable to consider the corresponding nonrelativistic problem. Thus, for a two-particle system $(\mathcal{M}_\alpha, m'_\alpha)$ in its c.m., the partial-wave amplitude is a function $t_\ell(Q, Q';E_2)$ of the energy E_2, and the off-shell momenta Q, Q'. For fixed E_2, Q', t_ℓ is an analytic function of Q, with a singularity structure implied by the associated potential, $V_\ell(Q, Q')$. In particular, if the latter is a simple Yukawa potential corresponding to the exchange of a particle of mass m_e, the singularities of t_ℓ lie outside the strip $|I_m Q| < m_e$.[21] Although the singularities are actually cuts and rather complicated, for values Q on the real axis the qualitative aspects can be represented by the simple form $t_\alpha \propto (Q^2+\bar{m}_e^2)^{-1}$, where $\bar{m}_e \simeq m_e$. Transforming to the rest frame of $\mathcal{M}_\alpha$, one has

$$\epsilon'^2_\alpha + \mu^2_{r\alpha'} \xrightarrow[\text{NR}]{} \left(1 + \frac{2m'_\alpha}{\mathcal{M}_\alpha}\right)Q^2 + m'^2_\alpha + \mu^2_{r\alpha'} \quad . \tag{5.9}$$

Thus, in order to reproduce the same singularity structure, we deduce that

$$\mu^2_{r\alpha'} \simeq \left(1+ \frac{2m'_\alpha}{\mathcal{M}_\alpha}\right) \bar{m}_e^2 - m'^2_\alpha \quad . \tag{5.10}$$

Using $\bar{m}_e \simeq m_\epsilon \simeq .6$, this leads to $\mu_{\rho K} \simeq .76$ and $\mu_{K^*\pi} \simeq .67$ GeV. On the other hand, one might view the situation as a true three-body problem. In this case Q would represent the momentum of α' in the three-body c.m., and $\bar{m}_e$ would correspond to the lightest (important) exchange between any of the three pairs; $(\pi\pi)$ and $(K\pi)$. Transforming to the pair c.m., the corresponding relation is

$$\mu^2_{r\alpha'} \simeq m'^2_\alpha \left(\frac{\bar{m}_e^2}{\nu^2_{\alpha'}} - 1\right) \quad ,$$
$$\nu_\alpha^{-1} = m_\alpha^{-1} + (m_\beta + m_\gamma)^{-1} \quad . \tag{5.11}$$

Taking $\bar{m}_e \simeq 2m_\pi = .28$, this gives $\mu_{\rho K} \simeq .58$ and $\mu_{K^*\pi} \simeq .32$ GeV.

In the numerical work, it turns out that the specific value of $\mu_{r\alpha'}$ (in the range suggested by the above) is far less interesting than the integer n_f which determines the suppression at large ϵ'_α. For purposes of illustration, we thus present results based on the specific choice $\mu_{\rho K} = .6$ and $\mu_{K^*\pi} = .3$ GeV. Taking $n_g=0$ for the two-body vertex, the calculated subenergy dependence for $n_f = 0,1,2$ is shown in Figs. 12a, 12b (varying $\mu_{r\alpha'}$ tends to interpolate between these curves;

e.g., $\mu_{r\alpha'} \rightarrow \infty$ is clearly equivalent to $n_f \rightarrow 0$). The variations displayed again indicate that a substantial dependence on subenergy should be anticipated. Similar calculations at other energies also reveal the relative independence of the shape as a function of M_3; again, $\tilde{\tau}_p^{\alpha} \simeq A(M_{\beta\gamma})B(M_3)$. On the other hand, $A(M_{\beta\gamma})$ can only be crudely represented as a linear function; it appears that a quadratic would be required.

The related cross section plots are given in Figs. 13a, 13b. Here the effect of varying n_f is far more dramatic. Thus, increasing n_f transforms a rising cross section into one which flattens out and begins to decline at large M_3; further increases in n_f narrow the corresponding peak, moving it to lower M_3, and cause the high-mass tail of the spectrum to fall more rapidly. It is obvious that the resulting behavior might well be interpreted as a "resonance" if seen in an experimental situation. Here, of course, the physics of the model is quite different; the curves simply reflect the diffractive "edge" of the dissociating particle. In practice, one would attempt to distinguish the two interpretations (in a given experiment) by extracting the phase behavior of the associated isobar amplitude $\tilde{\tau}^{\alpha}$ (e.g., relative to some other, presumably featureless, amplitude). In this case the phase motion corresponding to Figs. 13a, 13b is absolutely flat, since $\tilde{\tau}_p^{\alpha}$ just carries the phase (-i) of $t_2(s_2, t)$. If a resonance (defined, for our purposes, as a pole on the appropriate sheet of the S-matrix) always corresponded uniquely to classical Breit-Wigner phase motion, making such a distinction would therefore be easy. However, for systems of three (or more) particles this is certainly not the case, and the task becomes considerably more delicate.[7,8]

We therefore conclude that vertex corrections can play a crucial role in the M_3 dependence, and cannot arbitrarily be neglected in interpreting experimental results. This point is well illustrated by the $K\pi\pi$ analysis (in the Q region) described below. The other aspect to be considered concerns the possible consequences of non-negligible subenergy dependence. This question is more subtle, and the answer depends to a large extent on the precise situation involved. For example, the individual "isobar" cross sections discussed above may be recomputed using $\mathscr{M}_\alpha$ in place of $M_{\beta\gamma}$ in calculating $\tilde{\tau}_p^{\alpha}$ via Eq. (5.5); this procedure effectively uses that expansion to define an analytic continuation when $M_{\beta\gamma} = \mathscr{M}_\alpha$ is not kinematically accessible (for small M_3). The resulting curves differ only fractionally from those shown (perhaps 10% variation or less), except at small M_3, where the difference can be a factor of 2 or 3 (but the cross section itself is quite small). This is easy to understand from Figs. 12a, 12b, since the decline in $\tilde{\tau}_p^{\alpha}$ with $M_{\beta\gamma} < \mathscr{M}_\alpha$ only tends to enhance the peaking produced by the factor $|D_\alpha|^{-2}$ (in $d\sigma$). Although the effect in the cross sections does increase when the widths of the ρ, K^* are increased (being roughly proportional), the subenergy corrections seem unlikely to be of importance unless the widths become very broad indeed (say $\Gamma > 400$ MeV).

On the other hand, the consequences are more substantial when one considers interference effects. Thus, by keeping both ρK and $K^*\pi$ components in Eq. (4.19), one may compute the difference $\sigma(\rho K + K^*\pi) - \sigma(\rho K) - \sigma(K^*\pi)$, and compare the results for the exact formula vs. $M_{\beta\gamma} \rightarrow \mathscr{M}_\alpha$. In this case the overlap differs by 20% at $M_3 = 1.3$ GeV, and by 30% at $M_3 = 1.2$ GeV (again, for broader isobars, this behavior will be enhanced). In experimental data analysis (into isobar channels) this effect might conceivably lead to some misidentification regarding the content of individual channels. However, the most important effect might well be seen in the details of production-resonance interference. In that case the $M_{\beta\gamma}$-dependence of the production and resonant terms plays the role of the off-shell extrapolation discussed in Section II. B. In particular, we recall from Eq. (2.39) that the difference between the production mechanism and resonance in that respect determines whether the net outcome is a dip or a bump (and all that lies between); we shall return to this point in Section VII. Finally, in the Discussion (Section VIII) we present a practical recipe for incorporating such effects in data analysis.

VI. UNITARY DESCRIPTION OF THREE-BODY RESCATTERING

A. Exact Three-Body Treatment

As noted in the Introduction, a properly unitary treatment of the three-body rescattering necessarily involves the solution of an integral equation. In order to understand this physically, we represent the production process schematically by Fig. 14a, and production-plus-rescattering by Fig. 14b. In both cases, the label α indicates that the pair $\beta\gamma$ are the last to interact; i.e., Fig. 14b corresponds to the amplitude that one would experimentally define as the "isobar amplitude." The operator T_3 represents the full 3-to-3 scattering amplitude, and may clearly be decomposed in the form $T_3 = \Sigma_{\alpha, \alpha_0} \tau_{\alpha\, \alpha_0}$, according to which pair interacts first or last. The nature of $\tau_{\alpha\alpha_0}$ is expressed by the implicit integral equation in Fig. 14c; if one iterates this expression one simply obtains the full multiple scattering series in diagrammatic form. In writing Fig. 14b, we have explicitly assumed that particle f does not interact subsequent to the production mechanism. This may be justified in terms of our RST formalism by noting that the large energy s (carried by line f in the three-body c.m.) would otherwise appear in intermediate states (via G_0), and give rise to corresponding factors of s^{-1}. Such diagrams are thus suppressed relative to Fig. 14b, in which $E_f(s)$ simply cancels as long as f does not interact (see Section II.A).

The origin of the three-body integral equation is thus clear; it arises as the means of summing the full infinite series of sequential pair interactions (represented by off-shell scattering amplitudes t_α). It is perhaps less obvious that the equation cannot be substantially simplified (e.g., reduced to quadrature) without destroying unitarity; to see this one must recognize the complexity of the associated singularity structure.[22] In our operator notation, the equation of Fig. 14c may be expressed as

$$\tau_{\alpha\alpha_0} = \delta_{\alpha\alpha_0} t_\alpha - \sum_{\beta\neq\alpha} t_\alpha G_0 \tau_{\beta\alpha_0} \quad . \tag{6.1}$$

In order to express this in terms of amplitudes, we introduce the notation

$$\begin{aligned} \vec{p}_\alpha &= \mu_\alpha(\vec{k}_\beta/m_\beta - \vec{k}_\gamma/m_\gamma) \quad , \\ \mu_\alpha^{-1} &= m_\beta^{-1} + m_\gamma^{-1} \quad , \end{aligned} \tag{6.2}$$

where the vectors $\vec{k}_\alpha, \vec{k}_\beta, \vec{k}_\gamma$ are taken to be in the three-body c.m. We may then choose $\vec{k}_\alpha, \vec{p}_\alpha$ as independent variables, and note that

$$\int d\vec{k}_1\, d\vec{k}_2\, d\vec{k}_3\, \delta(\vec{k}_1+\vec{k}_2+\vec{k}_3) = \int d\vec{k}_\alpha\, d\vec{p}_\alpha \tag{6.3}$$

for any choice of α (α=1, 2, 3). It is convenient to express our operators on the c.m. basis $|\vec{k}_\alpha\vec{p}_\alpha>$; e.g.,

$$<\vec{k}'_\alpha\vec{p}'_\alpha|t_\alpha|\vec{k}_\alpha\vec{p}_\alpha> = \epsilon_\alpha\delta(\vec{k}'_\alpha-\vec{k}_\alpha)t_\alpha(\vec{k}_\alpha\vec{p}'_\alpha|\vec{k}_\alpha\vec{p}_\alpha;s_\alpha) \quad ,$$

where $t_\alpha(\vec{k}_\alpha\vec{p}'_\alpha|\vec{k}_\alpha\vec{p}_\alpha;s_\alpha) \equiv t_\alpha(k'_\beta k'_\gamma|k_\beta k_\gamma;s_\alpha)$ is the invariant $\beta\gamma$ scattering amplitude, and

$$\begin{aligned} s_\alpha &= (P_0^3 - k_\alpha)^2 \\ &= M_3^2 + m_\alpha^2 - 2M_3\,\epsilon_\alpha \quad . \end{aligned} \tag{6.4}$$

The completeness relation now becomes

$$1 = \int \frac{d\vec{k}_\alpha d\vec{p}_\alpha}{\epsilon_1 \epsilon_2 \epsilon_3} \; |\vec{k}_\alpha \vec{p}_\alpha> <\vec{k}_\alpha \vec{p}_\alpha| \qquad . \tag{6.5}$$

In practice, one employs the three alternative bases (α=1, 2, 3) simultaneously, since t_α is far more simply expressed in terms of $\vec{k}_\alpha, \vec{p}_\alpha$ than $\vec{k}_\beta, \vec{p}_\beta$. One thus requires the transformation brackets

$$<\vec{k}'_\alpha \vec{p}'_\alpha | \vec{k}_\beta \vec{p}_\beta> = \epsilon_1 \epsilon_2 \epsilon_3 \delta\left(\vec{p}_\beta + \frac{\mu_\beta}{m_\gamma} \vec{p}'_\alpha \pm \frac{\mu_\beta}{\nu_\alpha} \vec{k}'_\alpha\right) \delta\left(\vec{k}_\beta \mp \vec{p}'_\alpha + \frac{\mu_\alpha}{m_\gamma} \vec{k}'_\alpha\right) \quad , \tag{6.6}$$

where $\nu_\alpha^{-1} = (m_\beta + m_\gamma)^{-1} + m_\alpha^{-1}$, and the upper (lower) signs correspond to $\alpha\beta\gamma$ cyclic (anticyclic). Employing our RST rules, Eq. (6.1) can now be expressed as

$$\tau_{\alpha\alpha_0}\left(\vec{k}'_\alpha \vec{p}'_\alpha | \vec{k}_{\alpha_0} \vec{p}_{\alpha_0}; M_3\right) = \delta_{\alpha\alpha_0} \epsilon'_\alpha \delta\left(\vec{k}'_\alpha - \vec{k}_{\alpha_0}\right) t_\alpha \left(\vec{k}'_\alpha \vec{p}'_\alpha | \vec{k}'_\alpha \vec{p}_{\alpha_0}; s'_\alpha\right)$$

$$- \sum_{\beta \neq \alpha} \int \frac{d\vec{k}_\beta}{\epsilon_\beta \epsilon_\gamma} \; \frac{t_\alpha(\vec{k}'_\alpha \vec{p}'_\alpha | \vec{k}'_\alpha \vec{P}_{\alpha\beta}; s'_\alpha)}{\epsilon'_\alpha + \epsilon_\beta + \epsilon_\gamma - M_3 - i\epsilon} \; \tau_{\beta\alpha_0}\left(\vec{k}_\beta \vec{P}'_{\beta\alpha} | \vec{k}_{\alpha_0} \vec{p}_{\alpha_0}; M_3\right) \quad , \tag{6.7}$$

where

$$\begin{aligned}
\vec{P}_{\alpha\beta} &= \pm \left(\vec{k}_\beta + \frac{\mu_\alpha}{m_\gamma} \vec{k}'_\alpha\right) \quad , \\
\vec{P}'_{\beta\alpha} &= \mp \left(\vec{k}'_\alpha + \frac{\mu_\beta}{m_\gamma} \vec{k}_\beta\right) \quad , \\
\epsilon'_\alpha &= \left(m_\alpha^2 + \vec{k}'^2_\alpha\right)^{\frac{1}{2}} \quad , \\
\epsilon_\beta &= \left(m_\beta^2 + \vec{k}_\beta^2\right)^{\frac{1}{2}} \quad , \\
\epsilon_\gamma &= \left[m_\gamma^2 + \left(\vec{k}'_\alpha + \vec{k}_\beta\right)^2\right]^{\frac{1}{2}} \quad .
\end{aligned} \tag{6.8}$$

Although only a single (vector) integration is involved in Eq. (6.7), it is nevertheless an integral equation on the full $\vec{k}_\alpha \vec{p}_\alpha$ space, since the argument of $\tau_{\beta\alpha_0}$ depends on $\vec{k}'_\alpha$ (via $\vec{P}'_{\beta\alpha}$). However, under the special assumption that t_α

is separable in each partial-wave, it can ultimately be reduced to a one-dimensionable integral equation (in an angular-momentum decomposition). For our present purposes we shall confine our discussion to the simplest possible case, corresponding to a purely s-wave two-particle amplitude.[23] We thus take

$$t_\alpha(\vec{k}'_\alpha \vec{p}'_\alpha | \vec{k}'_\alpha \vec{p}_\alpha; s_\alpha) = \frac{1}{4\pi} \frac{g_\alpha(\bar{s}'_\alpha) g_\alpha(\bar{s}_\alpha)}{D_\alpha(s'_\alpha)} ,$$

$$\bar{s}'_\alpha = (k'_\beta + k'_\gamma)^2 = (\epsilon'_\beta + \epsilon'_\gamma)^2 - \vec{k}'^{\,2}_\alpha ,$$

$$\epsilon'_\beta = \left[m_\beta^2 + \left(\vec{p}'_\alpha \mp \frac{\mu_\alpha}{m_\gamma} k'_\alpha\right)^2\right]^{\frac{1}{2}} , \tag{6.9}$$

$$\epsilon'_\gamma = \left[m_\gamma^2 + \left(\vec{p}'_\alpha \pm \frac{\mu_\alpha}{m_\beta} \vec{k}'_\alpha\right)^2\right]^{\frac{1}{2}} ;$$

$\bar{s}_\alpha$ is defined similarly with $\epsilon'_\beta \to \epsilon_\beta$, $\epsilon'_\gamma \to \epsilon_\gamma$ computed with $\vec{p}'_\alpha \to \vec{p}_\alpha$. Defining a reduced amplitude $\hat{\tau}_{\alpha\alpha_0}$ via the relation

$$\tau_{\alpha\alpha_0}\left(\vec{k}'_\alpha \vec{p}'_\alpha | \vec{k}_{\alpha_0} \vec{p}_{\alpha_0}; M_3\right) = \frac{g_\alpha(\bar{s}'_\alpha)}{D_\alpha(s'_\alpha)} \hat{\tau}_{\alpha\alpha_0}\left(\vec{k}'_\alpha, \vec{k}_{\alpha_0}; M_3\right) g_{\alpha_0}\left(\bar{s}_{\alpha_0}\right) , \tag{6.10}$$

substitution into Eq. (6.7) yields the equation

$$\hat{\tau}_{\alpha\alpha_0}\left(\vec{k}'_\alpha, \vec{k}_{\alpha_0}; M_3\right) = \delta_{\alpha\alpha_0} \epsilon'_\alpha \delta\left(\vec{k}'_\alpha - \vec{k}_{\alpha_0}\right) / 4\pi$$

$$- \frac{1}{4\pi} \sum_{\beta \neq \alpha} \int \frac{d\vec{k}_\beta}{\epsilon_\beta \epsilon_\gamma} \frac{g_\alpha(\bar{s}_{\alpha\beta}) g_\beta(\bar{s}_{\beta\alpha})}{\epsilon'_\alpha + \epsilon_\beta + \epsilon_\gamma - M_3 - i\epsilon} \frac{\hat{\tau}_{\beta\alpha_0}\left(\vec{k}_\beta, \vec{k}_{\alpha_0}; M_3\right)}{D_\beta(s_\beta)} , \tag{6.11}$$

$$\bar{s}_{\alpha\beta} = (\epsilon_\beta + \epsilon_\gamma)^2 - \vec{k}'^{\,2}_\alpha ,$$

$$\bar{s}_{\beta\alpha} = (\epsilon'_\alpha + \epsilon_\gamma)^2 - \vec{k}_\beta^{\,2} ,$$

with ϵ'_α, ϵ_β, ϵ_γ given by Eq. (6.8). If we now consider the partial-wave decomposition of Eq. (4.1), we have assumed $\ell_\alpha=0$, $\ell_{\alpha_0}=0$, and hence $\lambda_\alpha=\lambda_{\alpha_0}=L$; thus

$$\tau_{\alpha\alpha_0}\left(\hat{k}'_\alpha\vec{p}'_\alpha \,|\, \hat{k}_{\alpha_0}\vec{p}_{\alpha_0};M_3\right) = \frac{1}{4\pi} \sum_L \frac{(2L+1)}{4\pi} P_L\left(\hat{k}'_\alpha \cdot \hat{k}_{\alpha_0}\right)\tau^L_{\alpha\alpha_0}\left(\bar{s}'_\alpha k'_\alpha \,|\, \bar{s}_{\alpha_0}k_{\alpha_0};M_3\right) . \tag{6.12}$$

Here we recall that the direction $\hat{p}_\alpha$ specified in Eq. (4.1) refers to the $\beta\gamma$ c.m. value of $\hat{k}_\beta$ (or, equivalently, the direction of $\vec{p}_\alpha$ as defined in Eq. (6.2) but evaluated in the $\beta\gamma$ c.m.); whereas $\bar{s}_{\alpha'}$, etc. are independent of that variable. We then obtain

$$\tau^L_{\alpha\alpha_0}\left(\bar{s}'_\alpha k'_\alpha \,|\, \bar{s}_{\alpha_0}k_{\alpha_0};M_3\right) = \frac{g_\alpha(\bar{s}'_\alpha)}{D_\alpha(s'_\alpha)}\, \hat{\tau}^L_{\alpha\alpha_0}\left(k'_\alpha, k_{\alpha_0};M_3\right) g_{\alpha_0}\left(\bar{s}_{\alpha_0}\right) , \tag{6.13}$$

where $\hat{\tau}^L_{\alpha\alpha_0}$ satisfies the one-dimensional equation

$$\hat{\tau}^L_{\alpha\alpha_0}\left(k'_\alpha, k_{\alpha_0};M_3\right) = \delta_{\alpha\alpha_0}\epsilon'_\alpha \delta\left(k'_\alpha - k_{\alpha_0}\right)\Big/ k'^2_\alpha$$

$$+ \sum_{\beta\neq\alpha} \int_0^{K_\beta} \frac{dk_\beta k_\beta^2}{\epsilon_\beta} K^L_{\alpha\beta}\left(k'_\alpha, k_\beta;M_3\right) \frac{\hat{\tau}^L_{\beta\alpha_0}\left(k_\beta, k_{\alpha_0};M_3\right)}{D_\beta(s_\beta)} ; \tag{6.14}$$

$$K^L_{\alpha\beta}(k'_\alpha, k_\beta;M_3) = -\frac{1}{2}\int_{-1}^{1} dz_{\alpha\beta} \frac{g_\alpha(\bar{s}_{\alpha\beta})g_\beta(\bar{s}_{\beta\alpha})P_L(z_{\alpha\beta})}{\epsilon_\gamma\,(\epsilon'_\alpha+\epsilon_\beta+\epsilon_\gamma-M_3-i\epsilon)} .$$

Here the dependence on $z_{\alpha\beta} \equiv \hat{k}'_\alpha \cdot \hat{k}_\beta$ enters entirely through ϵ_γ, and hence the $z_{\alpha\beta}$ integration can be performed analytically for simple choices of g_α, g_β.

To those familiar with the literature on relativistic Faddeev equations, the result expressed in Eq. (6.14) is unusual only in two aspects. The most obvious distinction is a simple consequence of the propagator (G_0) choice discussed in

Section II. A; i.e., setting $E=\epsilon'_\alpha+\epsilon_\beta+\epsilon_\gamma$, we have $(E-M_3)^{-1}$ instead of $2E/E^2-M_3^2)$. In addition, however, the k_β integration has been cut off at a finite value K_β, instead of running to infinity as in the BS-type equations. In order to understand the origin of this cut-off, we first note that the two-particle amplitude satisfies the equation $t_2=V_2-V_2G_0t_2$ (see Eq. (2.6)), and hence our choice in Eq. (6.9) implies that $V_2=g_\alpha(\bar{s}_\alpha')g_\alpha(\bar{s}_\alpha)/4\pi$, and

$$D_\alpha(s_\alpha) = 1 + \int_0^\infty \frac{dpp^2}{\epsilon_\beta\epsilon_\gamma} \cdot \frac{g_\alpha^2(M_{\beta\gamma}^2)}{M_{\beta\gamma}-\sqrt{s_\alpha}-i\epsilon} \quad , \tag{6.15}$$

where p is the $(\beta\gamma)$ c.m. momentum, $M_{\beta\gamma} \equiv \epsilon_\beta+\epsilon_\gamma$. We thus observe that $D_\alpha(s_\alpha)$ has a left-hand cut for $s_\alpha \leq 0$ as a consequence of the linear dependence on $\sqrt{s_\alpha}$. This is no problem at the two-particle level, since $s_\alpha < 0$ is typically far from the physical region. On the other hand, s_α is a variable in the three-particle problem, and we see from Eq. (6.4) that it varies to $-\infty$ if we let $k_\alpha \to \infty$. Correspondingly, the kernel of Eq. (6.14) would develop an associated imaginary part, and the solution $\hat{\tau}$ would not possess the desired properties; i.e., an imaginary part generated by the right-hand cut structure. To prevent this we must require $s_\alpha \geqslant 0$, which is equivalent to $k_\alpha \leq K_a$, with

$$K_\alpha = (M_3^2 - m_\alpha^2)/2M_3 \quad . \tag{6.16}$$

Superficially, this might appear rather odd, especially since $k_\beta \to \infty$ in the non-relativistic Faddeev equation. In fact, however, this behavior is entirely consistent with the physics of the problem. To see this, we appeal once again to the cluster property, which requires that a two-particle subsystem, in the presence of n-2 non-interacting additional particles, is precisely the same as a completely isolated two-particle system. In other words, if we view the two-body system in its c.m. frame, the other particles may have any momenta whatsoever $(0,\infty)$

without changing any characteristic of the subsystem (except the relation of the pair energy to the total n-body energy; see, e.g., Eq. (2.11) and the subsequent discussion). In particular, if we calculate s_α in the $\beta\gamma$ c.m. frame, we have

$$\sqrt{s_\alpha} = (M_3^2+\tilde{k}_\alpha^2)^{\frac{1}{2}} - (m_\alpha^2+\tilde{k}_\alpha^2)^{\frac{1}{2}} \quad , \tag{6.17}$$

where $\tilde{k}_\alpha$ is the magnitude of $\tilde{\vec{k}}_\alpha$ in that frame. Thus, as $\tilde{k}_\alpha$ varies from 0 to ∞, $\sqrt{s_\alpha}$ varies from M_3-m_α down to 0; i.e., it has precisely the variation implied by the cut-off discussed above. With this understanding, the finite upper limit K_β appears as a simple consequence of formulating the dynamical problem in the three-body c.m. In practical terms this is very convenient, because it insures that Eq. (6.14) is of the Fredholm type (unless the form factors g_α are chosen to be pathological).

As it stands, Eq. (6.14) corresponds to the purely academic situation of 3-to-3 scattering, whereas our goal is to add rescattering corrections to a postulated production mechanism. This, however, merely requires that we replace the driving term by the corresponding projection of the production amplitude. Thus (in the special case ℓ_α=0), we want $\tau^L_{\alpha\alpha_0}$ to become $\tau_p^\alpha(LM\ell_\alpha\lambda_\alpha)$ to lowest order, which is equivalent to dropping the integral term in Eq. (6.14). The necessary substitution is therefore

$$\begin{aligned} &\delta_{\alpha\alpha_0}\epsilon'_\alpha\delta\left(k'_\alpha-k_{\alpha_0}\right)\Big/k'^{\,2}_\alpha \longrightarrow \hat{\tau}^L_{p;\alpha}(k'_\alpha;M_3) \quad , \\ &\hat{\tau}^L_{p;\alpha}(k_\alpha;M_3) = \frac{D_\alpha(M^2_{\beta\gamma})}{g_\alpha(M^2_{\beta\gamma})}\ \tau_p^\alpha(LM\ell_\alpha\lambda_\alpha) \quad , \end{aligned} \tag{6.18}$$

and the full PWA is given by

$$\tau_\alpha(LM\ell_\alpha\lambda_\alpha) = \frac{g_\alpha(M^2_{\beta\gamma})}{D_\alpha(M^2_{\beta\gamma})}\ \hat{\tau}^L_\alpha\ (k_\alpha;M_3) \quad . \tag{6.19}$$

Here $\hat{\tau}^L_\alpha$ is the solution of Eq. (6.14) using the driving term $\hat{\tau}^L_{p;\alpha}$; the latter is to be computed from Eq. (6.18) by expressing the right-hand side as a function of $M_{\beta\gamma}$ and M_3, and then using the relation

$$M^2_{\beta\gamma} = M^2_3 + m^2_\alpha - 2M_3\epsilon_\alpha \tag{6.20}$$

to replace $M^2_{\beta\gamma}$ by $k_\alpha = (\epsilon^2_\alpha - m^2_\alpha)^{\frac{1}{2}}$ as the independent variable.

Given any real-valued functions g_α, the structure of Eq. (6.14) guarantees that the corresponding channel amplitudes $\hat{\tau}^L_\alpha$ will be properly related by unitarity. Specifically, if we denote the production-plus-rescattering operator by τ_α^{3p}, and the 3-to-3 operator by $\tau^{33}_{\alpha\beta}$, the unitarity relation can be expressed as

$$\begin{aligned}\Delta\tau^{3p}_\alpha = -\,2\pi i \sum_{\beta,\beta'} \int \frac{d\vec{k}_1 d\vec{k}_2 d\vec{k}_3}{\epsilon_1\epsilon_2\epsilon_3}\ \delta(\vec{k}_1+\vec{k}_2+\vec{k}_3)\ * \\ * \ \tau^{33}_{\alpha\beta} | \vec{k}_1\vec{k}_2\vec{k}_3 \rangle \delta(\epsilon_1+\epsilon_2+\epsilon_3-M_3) \langle \vec{k}_1\vec{k}_2\vec{k}_3 | \tau^{3p}_{\beta'} \quad ,\end{aligned} \tag{6.21}$$

where $\Delta\tau$ is the discontinuity of τ across the three-particle scattering cut (if we factor out the overall phase factor arising from the production term, $\Delta\tau = 2i\mathrm{Im}\,\tau$). We note that the overlap terms $\beta \neq \beta'$ are neglected in the isobar approximation.

The rescattering corrections so defined, however, correspond to a very specific dynamics; i.e., to a very special pairwise interaction (separable) expressed in terms of a particular form factor, $g_\alpha(\bar{s}_\alpha)$. The latter, moreover, cannot be chosen purely at random, since it must generate the two-particle denominator function $D_\alpha(s_\alpha)$ via Eq. (6.15). The question then arises as to whether such a unitarization scheme is sufficiently flexible to adequately represent the expected dynamical effects. In particular, a resonance can only arise as a (complex) zero in the Fredholm determinant $|1-\widetilde{K}|$ corresponding to the kernel $\widetilde{K}=K/D$ of Eq. (6.14). This will happen only if the g_α are precisely right, which is rather unlikely to be the case. Thus, although one can gain some flexibility in modifying g_α by replacing the "1" in Eq. (6.15) by a real function $c(s_\alpha)$ (without disturbing unitarity), the possibilities are clearly limited. Moreover, the quark model suggests that pairwise forces may actually be irrelevant in generating the resonance; such a dynamics would best be represented by an effective three-body force. As an aside, however, we note that there can be a very extensive trade-off between a true three-body force and so-called off-shell effects arising from the pair interactions,[24] providing that one is not restricted to a very specialized class (e.g., separable). As an example, this author has performed a number of calculations based on a relativistic boundary condition model (i.e., a very singular "potential"),[13] which produces a much stronger effective three-body interaction than corresponding separable models. Nevertheless, the unitarizing equation derived above must be generalized in some respect if it is to provide the basis for a workable scheme of data analysis.

For this purpose we propose a very simple modification of Eq. (6.14). Thus, we observe that the replacement $K^L_{\alpha\beta} \rightarrow K^L_{\alpha\beta} + A^L_{\alpha\beta}$, where $A^L_{\alpha\beta}(k'_\alpha, k_\beta; M_3^2)$ is a real-valued (nonsingular) function, leaves the unitarity properties intact. This replacement, therefore, generates a large class of unitary rescattering corrections, since $A^L_{\alpha\beta}$ may be chosen with complete freedom. Physically, the minimal ($A^L_{\alpha\beta} \equiv 0$) equation corresponds to particle exchange between isobars (e.g., to π exchange between $\rho\pi$ and $\epsilon\pi$ configurations of the 3π system); this presumably accounts for the important long-range dynamics. The addition of $A^L_{\alpha\beta}$ can then be viewed as an effective representation of the short-range effects (e.g., quark dynamics). Furthermore, the nonsingular nature of this term, as opposed to $K^L_{\alpha\beta}$ (which contains the G_0 pole singularity), permits us to manipulate Eq. (6.14) into a form which is suitable for data analysis. Thus, even if one could achieve sufficient generality by simply modifying g_α, it would be necessary to re-solve Eq. (6.14) numerically each time the parameters of g_α were altered. In practical terms, this is incompatible with a χ^2 fitting procedure. However, in the modified equation we hold the g_α fixed, and take advantage of the smooth nature of $A^L_{\alpha\beta}$ to expand it in some convenient complete set:

$$A^L_{\alpha\beta}(k'_\alpha, k_\beta; M_3) = \sum_n A^{L;n}_{\alpha\beta}(M_3)\psi^L_{\alpha n}(k'_\alpha)\psi^L_{\beta n}(k_\beta) \quad ; \qquad (6.22)$$

the function $A^L_{\alpha\beta}$ is symmetric, and hence the coefficients satisfy $A^{L;n}_{\beta\alpha} = A^{L;n}_{\alpha\beta}$. The functions $\psi^L_{\alpha n}$ may be chosen at our convenience, but should incorporate the threshold properties ($\propto k'^L_\alpha$) and fall off as $k'_\alpha \to \infty$; e.g.,

$$\psi^L_{\alpha n}(k'_\alpha) = k'^L_\alpha\left(k'^2_\alpha + m_n^2\right)^{-L-1} \quad , \qquad (6.23)$$

when the m_n are some increasing sequence of masses (m_0, $2m_0$, ...). If we then truncate the expansion at $n \leq N$, the resulting generalization of Eq. (6.14) can be easily manipulated into a set of M linear <u>algebraic</u> equations,[25] where M=N times the number of channels (α index). These algebraic equations (which require one to solve and store certain moments of the minimal (A=0) integral equation), determine the generalized amplitudes τ_α^{3p} in terms of the $A^{L;n}_{\alpha\beta}$ coefficients. Hence, by taking the latter as the fitting parameters, the problem reduces to a finite matrix calculation; this can be handled numerically with sufficient speed so as to permit its direct inclusion in a χ^2 routine. When combined with the techniques for handling the subenergy dependence described in the next subsection, the result is easily competitive with the techniques presently employed (in terms of practicality), while also guaranteeing an exact solution of the unitarity problem. Stated somewhat differently, the latter implies that the effect of three-body cut structure, as well as isobar cut structure, is properly incorporated in the model amplitude. As noted in the Introduction, this may well be essential in obtaining meaningful conclusions in many situations of experimental interest.

B. Approximate Three-Body Treatment

Although the approach described above constitutes an exact solution to the rescattering problem, and hence is definitely to be preferred, it is nevertheless useful to explore more approximate techniques in which the physical interpretation is more transparent. In particular, the isobar model, in which the resonant pair is treated as a stable elementary particle (with a real mass), embodies a good deal of the relevant physics and is relatively easy to employ. It does, however, completely neglect the three-body cut structure, and it tends (correspondingly) to introduce an artificially abrupt behavior at the isobar threholds.[26] Such problems become exacerbated, of course, when the "resonance" is actually a very broad object such as the ϵ. These facts suggest that some compromise between the literal isobar model and the exact three-body treatment could provide a very useful alternative.

In this spirit we introduce the following approach. We first note that the factor which forces one to an integral equation in the exact treatment corresponds to the overlap $\beta \neq \beta'$ terms in the unitarity relation, Eq. (6.21). In the isobar approximation, one argues that such terms may be neglected in comparison with the diagonal $\beta = \beta'$ term. Thus, introducing the on-shell amplitude

$$\tau^{L}_{\alpha\alpha_0}\left(k'_\alpha | k_{\alpha_0};M_3\right) \equiv \tau^{L}_{\alpha\alpha_0}\left(M^2_{\beta\gamma} k'_\alpha | M^2_{\beta_0\gamma_0} k_{\alpha_0};M_3\right) \quad , \tag{6.24}$$

with $M^2_{\beta\gamma}$ $\left(M^2_{\beta_0\gamma_0}\right)$ given in terms of $M_3{}^2$ and $\epsilon'_\alpha\left(\epsilon_{\alpha_0}\right)$ by Eq. (6.20), the unitarity relation reduces to

$$\Delta\tau^{L}_{\alpha\alpha_0}\left(k'_\alpha | k_{\alpha_0};M_3\right) = -2\pi i \sum_\beta \int_0^{k_\beta^M} \frac{dk_\beta k_\beta^2 \kappa_\beta}{\epsilon_\beta \sqrt{s_\beta}} \tau^{L}_{\alpha\beta}(k'_\alpha | k_\beta;M_3) * \tag{6.25}$$

$$* \tau^{L\,*}_{\beta\alpha_0}\left(k_\beta | k_{\alpha_0};M_3\right) \quad .$$

Here κ_β is pair c.m. momentum of particles $\alpha'\gamma'$ $(\alpha'\neq\gamma'\neq\beta)$, whose invariant energy is given by

$$s_\beta = M_3^{\,2} + m_\beta^{\,2} - 2M_3\epsilon_\beta \quad . \tag{6.26}$$

In Eq. (6.25) the upper limit k_β^M corresponds to the kinematic maximum for the spectator motion; i.e., $\epsilon_\beta^M = \left(k_\beta^{M2} + m_\beta^{\,2}\right)^{\frac{1}{2}}$ satisfies Eq. (6.26) with $\sqrt{s_\beta} = m'_\alpha + m'_\gamma$. We now observe that as a consequence of Eq. (6.15), we may express Eq. (6.25) in the form

$$\Delta\tau^{L}_{\alpha\alpha_0}\left(k'_\alpha | k_{\alpha_0};M_3\right) = -\sum_\beta \int_0^{K_\beta} \frac{dk_\beta k_\beta^{\,2}}{\epsilon_\beta} \frac{\Delta D_\beta(s_\beta)}{g^2_\beta(s_\beta)} \tau^{L}_{\alpha\beta}(k'_\alpha | k_\beta;M_3) * \tag{6.27}$$

$$* \tau^{L\,*}_{\beta\alpha_0}\left(k_\beta | k_{\alpha_0};M_3\right) \quad ,$$

where ΔD_β vanishes for $k_\beta > k_\beta^{\,M}$. Referring to Eq. (6.13), we note that the product $\tau^{L}_{\alpha\beta}\tau^{L\,*}_{\beta\alpha_0}$ implies that the integrand contains the factor

$$\frac{\Delta D_\beta(s_\beta)}{|D_\beta(s_\beta)|^2} \xrightarrow[\Gamma_\beta \to 0]{} \frac{2\pi i\, \delta(s_\beta - \mathcal{M}_\beta^{\,2})}{\left(dD_\beta/ds_\beta\right)_{\mathcal{M}_\beta^{\,2}}} \tag{6.28}$$

in the zero width limit. The singular nature of the diagonal term in this limit is one rationale for ignoring the $\beta\neq\beta'$ contributions, and hence one may "justify" the isobar model in the narrow width approximation.

In the realistic case of finite (and possibly large) widths, however, we shall define a generalized isobar description along the following lines. Thus, we look for $\tau^{L}_{\alpha\alpha_0}$ in the form

$$\tau^L_{\alpha\alpha_0}\left(k'_\alpha | k_{\alpha_0}; M_3\right) = \delta_{\alpha\alpha_0}\epsilon'_\alpha \frac{\delta\left(k'_\alpha - k_{\alpha_0}\right)}{k'^2_\alpha} t_\alpha(s'_\alpha)$$

$$+ h^L_\alpha(k'_\alpha) t_\alpha(s'_\alpha) X^L_{\alpha\alpha_0}(M_3) t_{\alpha_0}\left(s_{\alpha_0}\right) h^L_{\alpha_0}\left(k_{\alpha_0}\right) \quad , \tag{6.29}$$

where $h_\alpha(k'_\alpha)$ is a real-valued function. This is motivated by the form of Eqs. (6.13) and (6.14), and the fact that resonant amplitudes factorize, at least in the neighborhood $M_3 \simeq M_R$. The second term will thus be realistic if a three-body resonance is present, which is usually the situation of interest (at least potentially). In the isobar limit, $h_\alpha(k_\alpha)$ is essentially a form factor describing the coupling of the isobar to the resonance (more precisely, $g_\alpha h_\alpha$ is that form factor), and $X^L_{\alpha\alpha_0}$ is an isobar-to-isobar scattering amplitude. The latter is restricted by requiring that this expression satisfy the (approximate) unitarity constraint of Eq. (6.27). Via straight forward substitution, one obtains the condition

$$\Delta X^L_{\alpha\beta} = \sum_\gamma X^L_{\alpha\gamma} \Delta\rho^L_\gamma X^{L\,*}_{\gamma\beta} \quad ,$$

$$\Delta\rho^L_\alpha = \int_0^{K_\alpha} \frac{dk_\alpha k^2_\alpha}{\epsilon_\alpha} h^L_\alpha(k_\alpha)^2 \Delta t_\alpha(s_\alpha) \quad . \tag{6.30}$$

This equation has a familiar form; one simply takes $X^L_{\alpha\beta}(M_3)$ as the solution of

$$X^L_{\alpha\beta}(M_3) = \lambda^L_{\alpha\beta}(M_3) + \sum_\gamma \lambda^L_{\alpha\gamma}(M_3)\rho^L_\gamma(M_3) X^L_{\gamma\beta}(M_3) \quad , \tag{6.31}$$

where $\lambda^L_{\alpha\beta}$ is a real-valued function of M_3^2. In particular, one might choose

$$\lambda^L_{\alpha\beta}(M_3) = \lambda^{(o)}_{\alpha\beta}/(M_3^2 - s_{\alpha\beta}) \quad ; \tag{6.32}$$

i.e., a K-matrix parametrization. In the case of a single uncoupled channel α, one then obtains

$$X^L_{\alpha\alpha}(M_3) = \lambda^L_{\alpha\alpha}(M_3) / \left[1-\lambda^L_{\alpha\alpha}(M_3)\rho^L_\alpha(M_3)\right]$$
$$= \lambda^{(o)}_{\alpha\alpha} / \left[M_3^{\,2} - \lambda^{(o)}_{\alpha\alpha}\rho^L_\alpha(M_3)\right] \quad . \tag{6.33}$$

The simple choice expressed in Eq. (6.32) thus leads to a Breit-Wigner parametrization of $X^L_{\alpha\alpha}$.

Of course, since the function $\rho^L_\alpha(M_3)$ is restricted only by its discontinuity in Eq. (6.30), it is not uniquely determined. One possibility is to adopt a dispersive definition

$$\rho^L_\alpha(M_3) = \frac{1}{2\pi i} \int_{M_t}^{\infty} \frac{dM_3'^{\,2}\, \Delta\rho^L_\alpha(M_3')}{M_3'^{\,2} - M_3^{\,2} - i\epsilon} \quad , \tag{6.34}$$

where $M_t \equiv \Sigma_\alpha m_\alpha$ is the three-body threshold, and $\Delta\rho^L_\alpha(M_3')$ is calculated via Eq. (6.30). This particular choice has the asymptotic property $\rho^L_\alpha \to 0$ as $M_3^{\,2} \to \infty$, but such a constraint is purely ad hoc; one could just as well add a constant, or a polynomial in $M_3^{\,2}$, to the definition in Eq. (6.34). A different kind of possibility is suggested by the appearance of the exact three-body equation. We thus define the alternative choice

$$\rho^L_\alpha(M_3) = \int_0^{K\alpha} \frac{dk_\alpha k_\alpha^{\,2}}{\epsilon_\alpha}\; h^L_\alpha(k_\alpha)^2 t_\alpha(s_\alpha) \quad ; \tag{6.35}$$

this corresponds more closely to the way in which the denominator $D^{-1}_\alpha(s_\alpha)$ gives rise to singularities in $\hat{\tau}^L_{\alpha\alpha_0}$ via Eq. (6.14), and has the additional virtue that ρ^L_α can be expressed analytically for simple choices of h^L_α, t_α. As an example, we have calculated ρ^L_α via Eq. (6.35) for the ρK and $K^*\pi$ channels using

$$t_\alpha = (\sqrt{s_\alpha}/\pi) N^1_\alpha(s_\alpha)/D^1_\alpha(s_\alpha) \quad , \quad \text{and} \quad (L{=}1)$$
$$h^L_\alpha(k_\alpha) = c\left[N^1_\alpha(s_\alpha)\right]^{-\frac{1}{2}} / (k_\alpha^{\,2} + 4m_\pi^{\,2})^{-1} \quad . \tag{6.36}$$

Here $N_\alpha^{\ell\alpha}$, $D_\alpha^{\ell\alpha}$ (ℓ_α=1) are the explicit parametrizations of Eq. (5.1), and c=0.088 $(\text{GeV/c})^{3/2}$. The corresponding (dimensionless) values for Re ρ_α^L, Imρ_α^L are shown in Figs. 15a, 15b. A comparision of the two sets of curves illustrates the effect of narrowing the isobar width; i.e., $\rho_\alpha^L(K^*\pi)$ exhibits a considerably more rapid behavior near the isobar threshold (1.03 GeV) than does $\rho_\alpha^L(\rho K)$ (1.22 GeV). In order to discuss the zero width limit one takes $h_\alpha^L(k_\alpha) = \sqrt{2}\,\tilde{h}_\alpha^L(k_\alpha)/g_\alpha^F$, where $\tilde{h}_\alpha^L$ is normalized to unity at the isobar mass ($k_\alpha = k_\alpha^R$); with this convention $X_{\alpha\beta}^L(M_3)$ is precisely the spectator-isobar invariant PWA. In the zero width limit one then has

$$\Delta\,\rho_\alpha^L \to 2\int_0^{K_\alpha} \frac{dk_\alpha k_\alpha^2}{\epsilon_\alpha}\; 2\pi i\,\delta(s_\alpha - \mathcal{M}_\alpha^2) \qquad (6.37)$$

$$= 2\pi i\, k_\alpha^R / M_3 \quad ;$$

hence the formalism expressed in Eq. (6.31) becomes just the usual isobar model (with our convention for the phase space factor). However, for finite widths, our generalization avoids the cusp-like behavior at the isobar thresholds, and produces a more realistic-looking amplitude.

Given the form of Eq. (6.29), the procedure for constructing the amplitude of interest is trivial. Thus,

$$\tau_\alpha(LM\ell_\alpha\lambda_\alpha) = \sum_\beta \int_0^{K_\beta} \frac{dk_\beta k_\beta^2}{\epsilon_\beta}\; \frac{\tau_{\alpha\beta}^L(k_\alpha | k_\beta; M_3)}{t_\beta^{\ell_\beta}(s_\beta)}\; \tau_p^\beta(LM\ell_\beta\lambda_\beta) \qquad (6.38)$$

$$= \tau_p^\alpha(LM\ell_\alpha\lambda_\alpha) + h_\alpha^L(k_\alpha)\, t_\alpha^{\ell_\alpha}(s_\alpha) \sum_\beta X_{\alpha\beta}^L\, I_\beta(LM\ell_\beta\lambda_\beta) \quad ,$$

where

$$I_\beta(LM\ell_\beta\lambda_\beta) = \int_0^{K_\beta} \frac{dk_\beta k_\beta^2}{\epsilon_\beta}\; h_\beta^L(k_\beta)\,\tau_p^\beta(LM\ell_\beta\lambda_\beta) \quad . \qquad (6.39)$$

Here we recall that we simplified above to the case of a single channel for each β ($\ell_\beta = 0$, $\lambda_\beta = L$); in general, of course, the amplitude $X^L_{\alpha\beta}$ is a matrix on the (discrete) space labeled by ($\beta\,\ell_\beta\,\lambda_\beta$), $h^L_\beta(k_\beta) \rightarrow h^L_{\beta\ell_\beta\lambda_\beta}(k_\beta)$, and $\Sigma_\beta \rightarrow \Sigma_{\beta\ell_\beta\lambda_\beta}$. For practical manipulations, one may introduce the isobar amplitude $\tilde{\tau}_\alpha$ corresponding to Eq. (5.4), and expand the latter as in Eq. (5.5); i.e., with expansion coefficients $b^\alpha_k(M_3)$ referring to the full PWA. With this understanding, Eq. (6.38) can be expressed as

$$b^\alpha_k = b^{\alpha;p}_k + C^\alpha_k \sum_\beta X_{\alpha\beta} I_\beta \quad . \tag{6.40}$$

Here, for simplicity, we have employed α, β as general labels for the set of discrete indices, and it is understood that b^α_k, $b^{\alpha;p}_k$, C^α_k, I_β, $X_{\alpha\beta}$ are functions of M_3 in a given state of total L, M. The coefficient C^α_k corresponds to the expansion of $h^L_\alpha(k_\alpha)$; specifically,

$$C^\alpha_k = \int_{m_\beta + m_\gamma}^{M_3 - m_\alpha} dM_{\beta\gamma}\, \phi^\alpha_k(M_{\beta\gamma}, M_3)\, \frac{h^L_{\alpha\ell_\alpha\lambda_\alpha}(k_\alpha)}{\kappa_\alpha^{\ell_\alpha} k_\alpha^{\lambda_\alpha}}\, g^2_\alpha(M^2_{\beta\gamma}) \quad . \tag{6.41}$$

As noted above, the combination $(h^L_\alpha g_\alpha)$ is a vertex form factor coupling the isobar to the "resonance" described by $X^L_{\alpha\beta}$; thus $(h^L_\alpha g_\alpha)/k_\alpha^{\lambda_\alpha}$ is a function of k^2_α (the threshold behavior cancels). Similarly, $g_\alpha(M^2_{\beta\gamma})/\kappa_\alpha^{\ell_\alpha}$ is a function of κ^2_α. As a consequence, the integrand is in fact a smooth function of $M_{\beta\gamma}$ for reasonable choices of h^L_α.

Given Eq. (6.40), the cross sections including the effects of rescattering (with or without resonances, depending on the parametrization of $X_{\alpha\beta}$) can now be calculated in precisely the same fashion as we employed in Section V for the purely production term. In fact, the degrees of freedom have all been expressed in discrete form, and the resulting formalism is comparable in simplicity to the familiar isobar model. Nevertheless, the proposed generalization incorporates both subenergy dependence and (some) effects of the three-particle cut structure, and thus represents a far more comprehensive description of the production-rescattering problem. It should also be noted that the primary effect of our approximate treatment is the ease with which $X_{\alpha\beta}$ can be calculated; our exact three-body scheme can also be reduced to the form of Eq. (6.40), but $X_{\alpha\beta}$ must then be calculated via the set of algebraic equations discussed above.

In concluding this section, it is instructive to reconsider the question of production-resonance interference within the context of Eq. (6.40). Suppose, for example, that the production isobar amplitude $\tilde{\tau}_p^{\alpha}$ of Eq. (5.4) is independent of the subenergy. Recalling Eq. (5.5), we thus have $\tilde{\tau}_p^{\alpha} = b_1^{\alpha;p}\,\phi_1^{\alpha}$, where ϕ_1^{α} is just a constant (in subenergy) by Eq. (5.6). Suppose also that the combination $(h_{\alpha}^{L} g_{\alpha}^{2}/\kappa_{\alpha}^{\ell_{\alpha}} k_{\alpha}^{\lambda_{\alpha}})$ occurring in Eq. (6.41) is a constant; then $C_k^{\alpha} = \delta_{1k} C_1^{\alpha}$. Combining these assumptions with Eqs. (5.4) and (6.39), we find that

$$b_k^{\alpha} = \delta_{1k} b_1^{\alpha} \quad ,$$

and

$$b_1^{\alpha} = b_1^{\alpha;p} + C_1^{\alpha} \sum_{\beta} X_{\alpha\beta}\rho_{\beta}\, b_1^{\beta;p}/C_1^{\beta} \quad , \tag{6.42}$$

where we have used Eq. (6.35) as our definition of ρ_{β}. In the case of uncoupled channels ($\lambda_{\alpha\beta} = 0$, $\alpha \neq \beta$), this reduces to

$$\begin{aligned} b_1^{\alpha} &= (1+X_{\alpha\alpha}\rho_{\alpha}) b_1^{\alpha;p} \\ &= b_1^{\alpha;p}/(1-\lambda_{\alpha\alpha}\rho_{\alpha}) \qquad (6.43) \\ &= (M_3^2 - s_{\alpha\alpha})\, b_1^{\alpha;p}/(M_3^2 - s_{\alpha\alpha} - \lambda_{\alpha\alpha}^{(0)}\rho_{\alpha}) \quad , \end{aligned}$$

the last line following from the particular parametrization of Eq. (6.32). We have thus reproduced the result stated in Eq. (2.41); i.e., the full amplitude $\tilde{\tau}_\alpha = b_1^\alpha \phi_1^\alpha$ has a zero at $s_{\alpha\alpha}$ (the position of the K-matrix pole). Conversely, the difference in the subenergy behavior of $\tilde{\tau}_p^\alpha$ and $(h_\alpha^L g_\alpha^2/\kappa_\alpha^{\ell_\alpha} k_\alpha^{\lambda_\alpha})$ weakens the interference effect, and leads in practice to smoother, more realistic cross sections. A similar smoothing takes place as the result of having two or more coupled channels in Eq. (6.42); in fact, this was noted by Basdevant and Berger in their unitarized isobar analysis of the A_1 (they included coupling to $K^*\bar{K}$).[1] Thus, in practical terms, the effect of subenergy dependence is to add additional "channels" to the isobar model. The sensitivity of the differential cross sections to such interference is well illustrated by the examples discussed in the next section, and indicates that ad hoc neglect of the subenergy variations is a very questionable procedure.

VII. APPLICATION TO $K^+p \to K^+\pi^+\pi^-p$

As an illustration of our formalism, and of the various questions concerning subenergy dependence and vertex corrections discussed above, we report in this section some results recently obtained for the 1^+0^+ state of $K^+\pi^+\pi^-$. These results must be regarded as preliminary, in that only the parameters describing the resonant rescattering ($X_{\alpha\beta}$) were varied in the fitting procedure. Thus, the fits obtained are within the context of a given model of production, and the particular choice for $h_\alpha(k_a)$ stated in Eq. (6.36). In addition, a definitive treatment of the $K\pi\pi$ system would require the consideration of other JM states, a fit to the K^- data as well, the possible inclusion of direct production, and an adequate treatment of nonresonant features such as the t-dependence of $d\sigma/dM_3\,dt$. All of these aspects are presently being studied, and will be presented in a paper devoted exclusively to the $K\pi\pi$ system. Despite the preliminary status of this work, however, it appears very likely that the present conclusions will hold up, unless direct production is very significant. In any case, our general points concerning the utility of this approach, and the likely importance of vertex and subenergy effects, are independent of the specifics of this reaction (e.g., the masses and widths of the Q mesons).

We shall deal specifically with the SLAC experiment at 13 GeV/c.[27] The relevant data have been analyzed by several independent groups, and the present consensus is that two Q mesons exist in the mass range 1.2 to 1.5 GeV.[28] Thus, the experiment appears to require a state Q_2 (1.39) decaying almost entirely to $K^*\pi$ ($\Gamma_2 \simeq 160$ MeV), and a state Q(1.30) coupling principally to ρK ($\Gamma \simeq 200$ MeV). This explanation invokes both direct and Deck-like production, and is based on an isobar-model treatment of the rescattering effects. In particular, subenergy dependence is entirely neglected, as is the effect of structure functions at the Deck vertices.

For our purposes, the relevant experimental facts consist of the isobar "cross sections" $d\sigma(K^*\pi)/dM_3dt$ and $d\sigma(\rho K)/dM_3 dt$ (i.e., the cross sections which correspond to including only a single isobar channel in the total amplitude), and the "relative phase" $\Delta\phi \equiv \phi(\rho K) - \phi(K^*\pi)$ between the respective isobar amplitudes. This set of "data" is convenient to work with, and affords an easy comparison between our results and those quoted above. However, our formalism should really be applied directly to the data, and a comparison at the present level cannot be entirely precise. The reason is simply that the original fitting procedure from which these "data" derived explicitly assumed zero subenergy dependence, and the "cross section" $d\sigma(K^*\pi)/dM_3 dt$ is actually calculated using the fitted isobar amplitudes. Thus, if our treatment were applied to the data, the resulting "cross section" we would compute would be somewhat different. In the present application, since the ρ and K^* are relatively narrow, this should not be too important, but a comparison with ϵ and κ might cause some problems. Similarly, the definition of "relative phase" in our treatment is ambiguous, since our "isobar" amplitude is actually a function of subenergy. Relying again on the narrowness of the isobars in this instance, we adopt the convention that $\Delta\phi$ is computed by simply setting $M_{\beta\gamma} = m_\alpha$, but this would not be satisfactory in general.

With this understanding, we have applied the approximate three-body treatment of Sec. VI. B to the production model with $n_g = 0$, various choices of n_f, $\mu_{\rho K} = .6$, $\mu_{K^*\pi} = .3$ in the conventions of Sec. V. Thus, the cross sections in the absence of rescatterings (resonances) are those shown in ($n_f = 2$) Figs.

13a, 13b. Employing the simple parametrization of Eq. (6.32), this leads to a seven parameter model, including normalizations $N_{\rho K}$, $N_{K^*\pi}$ of the production term. Specifically, since we expect some coupling of Q (1.30) to $K^*\pi$, we take (channel $1=\rho K$, channel $2=K^*\pi$)

$$s_{12} = s_{21} = s_{11} \quad ,$$
$$\lambda_{12}^{(o)} = \lambda_{21}^{(o)} = -\sqrt{\lambda_{11}^{(o)}\beta_2} \quad , \tag{7.1}$$
$$\lambda_{22}(M_3^2) = \lambda_{22}^{(o)} \Big/ (M_3^2 - s_{22}) + \beta_2 \Big/ (M_3^2 - s_{11}) \quad ,$$

with λ_{11}, λ_{12}, λ_{21} given by Eq. (6.32). The five parameters determining $X_{\alpha\beta}$ are thus s_{11}, s_{22}, $\lambda_{11}^{(o)}$, $\lambda_{22}^{(o)}$, β_2. The full seven parameters are determined by a standard χ^2 fitting routine using the three "data" sets discussed above. As a technical aid, we note that the differential cross sections can be expressed in the form

$$\frac{d\sigma(\alpha)}{dM_3\, dt} = \sum_{k,k'} c_{kk'}^{\alpha}(M_3^2)\, b_k^{\alpha}\, b_{k'}^{\alpha *} \quad , \tag{7.2}$$

where $c_{kk'}^{\alpha}$ is an Hermitian matrix, and is computed using the subenergy expansion set ϕ_k^{α} of Eq. (5.6), and the relevant denominator function $D_{\alpha}(s_{\alpha})$. This array is stored, and hence $d\sigma$ can be rapidly computed given the b_k^{α} from Eq. (6.40).

Initially, an attempt was made to reproduce the Q_1, Q_2 fit described in the literature. Thus, the choice $n_f = 1$ produces a "Deck" amplitude which is virtually identical to the isobar version; this model was employed along with the $X_{\alpha\beta}$ parametrization of Eq. (7.1). The success of this fit depends on production-resonance interference to suppress the (relatively flat) Deck term in $K^*\pi$ at $M_3 \gtrsim 1.4$, together with a significant <u>direct</u> coupling to the Q_2

resonance at 1.4 GeV. However, in the absence of a direct production term, such a fit could not be obtained with our model; the $K^*\pi$ intensity and $\Delta\phi$ constraints were basically incompatible. On the other hand, the inclusion of vertex corrections in our approach permits us to obtain a totally different type of fit. Thus, by using a three-body vertex function which suppresses the large M_3 tail of the $K^*\pi$ distribution, the correct behavior of $\Delta\phi$ can be obtained via the "natural" trend for $\phi(\rho K)$, with $\phi(K^*\pi)$ comparatively small and flat (the cross section for ρK is always easy to fit). The only drawback to this alternative is that such vertex functions also move the "Deck" peak to lower M_3, destroying the agreement of $\sigma(K^*\pi)$ in that region (see, e.g., Fig. 13.b). This fact was noted by Bowler in his attempts to construct a one-resonance (Q_1) fit to the data;[28] he subsequently concluded that two Q's were essential.

In our case, we chose $n_f = 2$, and performed a χ^2 fit using all but the two highest mass points in the $K^*\pi$ intensity (we return to this point below). This actually produced a rather good fit, but turned out to be physically unacceptable; i.e., the parameter $\lambda_{22}^{(o)}$ came out to be <u>negative</u> ($\lambda_{22}^{(o)} \simeq -2.2\ \mathrm{GeV}^2$), producing a pole on the wrong sheet ($s = s_r + i\Gamma/2$). The origin of this problem is that we require a <u>suppression</u> of the low M_3 $K^*\pi$ cross section to compensate for the effect noted above. Since $\mathrm{Re}\rho_\alpha\ (K^*\pi) < 0$ in that region, we see from Eq. (6.43) that λ_{22} must be <u>positive</u> for M_3 in the range 1.0 - 1.1 GeV ($M_3^2 < s_{22} \simeq$ 1.28 in the fit); thus $\lambda_{22}^{(o)} < 0$.

On the other hand, there is no cogent reason <u>a priori</u> to restrict the $X_{\alpha\beta}$ parametrization to the K-matrix form; i.e., to impose a simple Breit-Wigner pole. In fact, the present author has pointed out several examples of more complicated effects which are naturally indigenous to three-particle systems.[7,8] In particular, it was recently proposed that an unusual singularity at $M_3 \simeq 1.18$

in $K^*\pi$ could be connected with a Q_2 resonance (the corresponding A_1 effect is at 1.1 GeV);[29] this would not correspond to a simple pole. Fits incorporating that singularity will be reported elsewhere; we restrict ourselves here to the simpler choice

$$\lambda_{22}(M_3^2) = -\lambda_{22}^{(o)}(M_3^2 - s_{22}) + \beta_2 \Big/ (M_3^2 - s_{11}) \quad , \tag{7.3}$$

with $\lambda_{22}^{(o)} > 0$. If one ignores the (small) coupling β_2, this leads to

$$X_{22}(K^*\pi) \simeq -\frac{\lambda_{22}^{(o)}(M_3^2 - s_{22})}{1 + \lambda_{22}^{(o)}(M_3^2 - s_{22})\,\rho_2(K^*\pi)} \quad . \tag{7.4}$$

Although this does produce an acceptable pole on the correct sheet, it clearly is not of the Breit-Wigner type. In fact, if s_{22} is not far from the pole position, the behavior of $X_{22}(K^*\pi)$ is similar to that of b_1^{α} in Eq. (6.43), with a nearby zero on the real axis. We emphasize that there is no fundamental reason to reject this form. Moreover, its unusual properties are highly suggestive with respect to some mysteries concerning the A_1. In particular, an A_1 pole of this type would not result in a peak in $\rho\pi$ scattering, and hence the absence of a signal in processes initiated by ρ-exchange (e.g., $\pi\rho \rightarrow A_1\Delta$) would be automatic.[30]

In the present case, the use of Eq. (7.3) immediately solves the problem, and the resulting fit is illustrated in Figs. 16a - 16c (dashed lines). The corresponding parameters are given in Table II. The fit to the ρK intensity is clearly excellent, but does not represent much of a feat; almost any production model attempted yields a virtually unique curve, with almost identical results for the $\lambda_{11}^{(o)}$, s_{11} fit parameters. The $K^*\pi$ intensity is generally superior in the range $M_3 \leq 1.45$, and even manages a number of wiggles which may or may not be realistic in the data. Its principal failure is clearly at high M_3 , where it cannot reproduce the very rapid drop of the "data". Since the corresponding $K^*\pi$ amplitude is virtually pure production in that region, another calculation was performed using a Gaussian vertex function. Recalling Eq. (5.7), this fit uses

$$g_{r\alpha'}(\epsilon'_\alpha) = \exp\left(-\epsilon'^2_\alpha \big/ \mu^2_{r\alpha'}\right) \ , \qquad (7.5)$$

with $\mu_{\rho K} \simeq .52$, $\mu_{K^*\pi} \simeq .46$ GeV, and is represented by the solid line in Figs. 16a-c. With respect to $K^*\pi$, this clearly yields a significant improvement at large M_3 without producing much change anywhere else. Taken together with the variations with n_f shown in Fig. 13b, this suggests that a vertex shape could be found to fit the data precisely. Physically, the $K^*\pi$ distribution would then be a direct reflection of the structure function for $K(K\pi\pi)$, and not the signature of a resonance as interpreted in the Q_2 (1.4) fits. At this point it does not seem worthwhile to explicitly construct such a function, particularly since the underlying isobar analysis could be misplacing the content of the various isobar channels in this mass region. In this context we note that the total 1^+0^+ cross section exhibits a break above 1.45 GeV very much like that of our curves, with the corresponding cross section difference attributed to ϵK.[31]

With respect to the $\Delta\phi$'s produced by our fits, the agreement is comparable in quality to the published analyses, and such discrepancies as exist are insignificant given the above-noted ambiguities in defining the relative phase. In addition, however, the curves exhibit a small-M_3 behavior which has not previously been obtained in such fits. In fact, past analyses have simply thrown away the values of $\Delta\phi$ for $M_3 < 1.2$ GeV, on the grounds that the very small ρK cross section in that region makes $\Delta\phi$ intrinsically unreliable.[28] Although this argument certainly has merit, it is nevertheless true that all of the experimental fits reported exhibit the trend displayed in Fig. 16c. Unfortunately, this behavior is very difficult, if not impossible, to obtain within the context of the isobar model, and certainly not using simple Breit-Wigner poles. In our case the essential ingredient is the inclusion of the subenergy dependence; i.e., retaining terms beyond $k=1$ in the expansion of Eq. (5.5). Indeed, if we calculate $\Delta\phi$ using the same fit parameters, but keep only b_1^{α}, we obtain the dashed curve in Fig. 17 (shown with the usual case $k \leq 4$ for comparison). The possibility of obtaining this behavior is therefore linked to the subenergy dependence. In addition, of course, the fitting parameters have to have a certain character. Thus, the appearance of this trend in our fits is a direct consequence of the low mass pole we obtain in the $K^*\pi$ channel. Specifically, we have used the above formalism to define an analytic continuation of $X_{\alpha\beta}$ to the second sheet, and by this means established the pole positions conclusively. Using the notation $Q(M_R, \Gamma_R)$, where the pole is at $\sqrt{s_R} = M_R - i\Gamma_R/2$, our current results yield the values $Q_1(1.30,\ 0.16)$ and $Q_2(1.15,\ 0.24)$. The former is clearly in good accord with the results of the past analyses; the latter has a very different character.

We conclude this section with an example which displays the sensitivity of production-resonance interference to postulated subenergy dependence. Thus, in Fig. 18, we note the effect of replacing $M_{\beta\gamma}$ in the ρK channel (solid curve) with $m_\alpha = M_\rho$ (dashed curve); all parameters have been held fixed. This substitution clearly produces a dramatic effect in the normalization; if this is scaled down there is still an appreciable difference in the shape. A similar calculation for $K^*\pi$ yields a much smaller effect, on the order of 10-15%; in part this is due to the narrower K* width, but the most important factor is apparently the nature of the interplay between the production mechanism and the resonance.

VIII. SUMMARY AND DISCUSSION

In the preceding sections we have dealt with a great many topics relevant to our central theme of diffractive production and rescattering. In fact, we have gone considerably beyond that particular application, introducing both a formal framework for calculating a large class of production models, and a varied set of mathematical techniques for analyzing any three-body final state. Our intention in this final section is to briefly summarize our principal results and conclusions, and to conclude with some specific suggestions of a practical nature for experimental analysis. In particular, we wish to stress the following points:

(1) A consistent set of rules has been introduced for the relativistic n-body scattering problem. Expressed in a manifestly covarient form, the theory satisfies the cluster property explicitly; this property is violated by virtually all relativistic equations in the literature, including all forms of the relativistic Faddeev (three-body) equations. In addition, the relativistic free propagator thus proposed guarantees that the singularity structure arising from one boson exchange will be exactly reproduced. Again, this is not the case for current theories, and could be an important feature for applications at intermediate energies; e.g., to describe $NN \rightarrow NN$ and $NN \rightarrow NN\pi$ in the context of an explicit three-body ($NN\pi$) theory.

(2) When applied to a three-particle system, our scattering formalism defines an alternative to the current relativistic Faddeev equations. In addition to the propagator difference noted above, the equation possesses an automatic

and unique cut-off; this eliminates the need for ad hoc procedures and/or form factors in obtaining convergence (the equation is manifestly of the Fredholm type). As an example, we have presented explicit equations for the case of a separable s-wave interaction.

(3) A modification of our relativistic three-body equation has been introduced for the purpose of data analysis. The resulting formalism permits one to construct exactly unitary rescattering corrections for any three-body final state, and in such a manner that the technique can be utilized within a χ^2 fitting program. Although less general than a comparable technique previously proposed,[12] the new proposal is considerably easier to apply and to interpret physically.

(4) As a simple application of our formalism, we have considered the question of production-resonance interference. Our results confirm those of Aitchison and Bowler,[9] and indicate that dramatic interference effects are more the rule than the exception. In our treatment the controlling factor can be isolated, and is identified as the difference in off-shell behavior between the production and decay amplitudes. In the special case of diffractive three-body production, this translates to the difference in the subenergy dependence of the Deck and resonant amplitudes. The importance of this effect has been illustrated via our numerical results for the $K\pi\pi$ system. At the two-particle level, the result is best summarized by noting that the total amplitude behaves more like $\cos(\delta) \exp(i\delta)$ than $\sin(\delta) \exp(i\delta)$. The nature of our scattering treatment is such that this behavior tends to emerge automatically.

(5) Our formalism has been applied to construct a generalization of the Deck model for diffractive production, including vertex form factors at the two- and three-particle vertices. In particular, a complete set of formulas for calcula-

ting such models and the corresponding cross sections have been presented in detail. Via an obvious modification of the vertex amplitudes, these formulas can be immediately applied to non-diffractive processes such as $\pi p \to A_1 \Delta$, $Kp \to A_1 \Lambda$, etc. In fact, the unknown parameters introduced by our generalized description will ultimately be over-constrained by successive applications to a great variety of reactions governed by the same rules (and parameters). Thus, the definitive test of our approach will be its ability to simultaneously account for a heterogenous assortment of data; given our current experience with the A_1, this is no small challenge.

(6) Within the context of our production model, we have investigated the nature and importance of subenergy dependence, and the effect of vertex structure on the differential cross section. In particular, we found that introducing structure at either vertex implied a considerable dependence of the isobar amplitude on the subenergy. Taken in conjunction with the dependence (implied by unitarity) which arises from rescattering,[3] this result re-emphasizes the need for properly taking this degree of freedom into account. In fact, even for the relatively narrow isobar states (K^*, ρ) considered in our numerical examples, it was found that significant effects can occur, particularly with respect to the rather subtle interplay between the production and resonance terms (see, e.g., Fig. 18).

It was also found that the introduction of a form factor describing the dissociation of the incoming meson into the three-body state can have dramatic consequences for the differential cross section. In fact, one may easily simulate a resonance-like peak when this is combined with a sharply rising phase space. This effect can thus lead to the misidentification or misplacement of a resonance, a fact which is well illustrated by our numerical treatment of the

$1^+0^+ K\pi\pi$ system.

(7) Motivated by the simplicity and physical appeal of the isobar concept (as opposed to a true three-body approach), a generalization of the isobar technique has been introduced for use in three-particle data analysis. The proposed form includes some of the true three-body cut structure, and avoids the cusp-like behavior associated with the artificial isobar "thresholds". Thus, while rigorously approaching the isobar model in the zero width limit, the generalized model results in smoother, more realistic cross sections, and is much better suited for applications to broad "isobars" such as the ϵ. At the same time, it is equal to the naive isobar model in practicality and simplicity. The utility of this approach is evident in our $K\pi\pi$ analysis.

(8) By employing strong vertex corrections associated with kaon dissociation, our approximate (isobar) treatment achieves a rather good fit to the 1^+0^+ state of $K\pi\pi$ generated by the reaction $K + p \to K^+\pi^+\pi^- p$ at 13 GeV/c. In contrast to the conclusions reached in previous analyses,[28] the state Q_2 (coupling predominantly to $K^*\pi$) is found to occur at 1.15 GeV, and not at 1.40 GeV. It is furthermore associated with an amplitude suggested by some recent theoretical work[30]; this does not correspond to a simple Breit-Wigner representation. Assuming a similar effect in the A_1 system, this would translate to an A_1 of 1070-1100 MeV in terms of its pole location, but would <u>not</u> give rise to a peak in $\rho\pi$ scattering. It would thus not be seen in processes dependent on ρ-exchange (e.g., $\pi\rho \to A_1\Delta$), but could coincide very nicely with the effect recently seen in τ-decay.[33] This result thus has some rather appealing features, but is as yet premature in the sense that a description of all the relevant $K\pi\pi$ data has not been attempted. The requisite work is in progress, and a paper dealing specifically with the Q mesons should be forthcoming in the

near future. With this understanding, we quote the results to date as M_{Q_1} = 1.30, $\Gamma_{Q_1} = .16$; and $M_{Q_2} = 1.15$, $\Gamma_{Q_2} = .24$.

Whether or not this interpretation ultimately proves viable for this particular system, this example provides an excellent illustration of the kind of effects one must consider when vertex corrections are taken into account. Our analysis also offers an explanation for the behavior of the relative phase $\phi(\rho K) - \phi(K^*\pi)$ below the ρK threshold. Thus, as a consequence of including the subenergy dependence (and having a low mass Q_2 state), $\Delta\phi$ starts off near -180° at 1.04 GeV, rather than at 0°. Such behavior is quite anomalous from the standpoint of the standard isobar model, and hence the $\Delta\phi$ data below 1.2 GeV has invariably been discarded. In our treatment, however, it emerges in a very natural way.

(9) As an interesting byproduct of our development, we have been forced to discard the physical picture in which the meson decays into three particles, two of which subsequently interact to form an isobar. This view of the dissociation process is not compatible with the mass-dependence of the diffractive cross section, as we have shown. Instead, one must postulate a (sequential) decay of the meson into an isobar-spectator, with the isobar subsequently decaying into the observed two-particle state. This would appear to give increased credibility to the quark model viewpoint of the isobar as an elementary particle. Finally, in the course of verifying this result numerically, we were forced to construct a model three-body decay amplitude with some useful properties for future applications.

In conclusion, we consider a simple recipe for incorporating the primary effects we have investigated in a model suitable for data analysis. Thus, we wish to include both subenergy effects and the possibility of suppression at high mass due to vertex corrections. To take into account the latter, we make use of the approximate factorization found in our calculations, and thus set $\tau_\alpha^p = B_\alpha^p(M_{\beta\gamma})\,F(M_3)$. The function $F(M_3)$ then appears as an overall multiplicative factor for the entire amplitude, $T = \sum_\alpha \tau_\alpha$. In particular, one might take $F(M_3) = \exp(-M_3^2/M_o^2)$, and vary M_o as a free parameter. To handle the subenergy dependence, we rely on our observation that $B_\alpha^p(M_{\beta\gamma})$ is readily approximated by a low-order polynomial in $M_{\beta\gamma}$. We thus expand $B_\alpha^p(M_\beta) = \Sigma_k b_k^{\alpha;p}\,\phi_k^\alpha(M_{\beta\gamma}, \bar{M}_3)$, where $\bar{M}_3$ is fixed at the upper end of the relevant three-body mass interval (e.g., $\bar{M}_3 \simeq 1.6$ GeV for the $K\pi\pi$ problem). Instead of committing ourselves to a model for the production mechanism, we consider the $b_k^{\alpha;p}$ as free parameters (for the diffraction process they can be taken as pure imaginary). Similarly, we take the coefficients C_k^α, defined previously by Eq. (6.41), as free real-valued parameters. Employing Eqs. (6.35), (6.39) and (6.40), we then obtain

$$
\begin{aligned}
b_k^\alpha(M_3) &= b_k^{\alpha;p} + C_k^\alpha \sum_\beta X_{\alpha\beta}(M_3)\, I_\beta(M_3) \;, \\
I_\alpha(M_3) &= \Sigma_{k,k'}\, I_{kk'}^\alpha(M_3)\, C_k^\alpha\, b_{k'}^{\alpha;p} \;, \\
\rho_\alpha(M_3) &= \Sigma_{k,k'}\, I_{kk'}^\alpha(M_3)\, C_k^\alpha\, C_{k'}^\alpha \;, \\
I_{kk'}^\alpha(M_3) &\equiv \int_0^{K_\alpha} \frac{d k_\alpha\, k_\alpha^2}{\epsilon_\alpha}\; \frac{\kappa_\alpha^{2\ell_\alpha}\, k_\alpha^{2\lambda_\alpha}\, \phi_k^\alpha(M_{\beta\gamma}, \bar{M}_3)\, \phi_{k'}^\alpha(M_{\beta\gamma}, \bar{M}_3)}{g_\alpha^2(M_{\beta\gamma}^2)\, D_\alpha(M_{\beta\gamma}^2)} \;.
\end{aligned}
\qquad (8.1)
$$

Thus, by computing and storing $I_{kk'}^\alpha(M_3)$ (for the necessary discrete set of M_3), one can rapidly compute ρ_α and I_α for a given set $b_k^{\alpha;p}$, C_k^α. The array $X_{\alpha\beta}(M_3)$ is then computed via the (resonance) parametrization $\lambda_{\alpha\beta}(M_3)$ from Eq. (6.31), and $b_k^\alpha(M_3)$ is calculated from Eq. (8.1). Finally, one forms $\tau_\alpha = B_\alpha(M_{\beta\gamma})\,F(M_3)$, where B_α is computed using the coefficients $b_k^\alpha(M_3)$.

Alternatively, for fitting to the isobar "cross sections" $\sigma(\alpha)$, one can simply work with Eq. (7.2), using $b_k^\alpha \rightarrow b_k^\alpha (M_3)\, F(M_3)$.

The technique so defined appears extremely easy to work with, and allows one to freely vary the subenergy dependence of the production ($b_k^{\alpha;p}$) and resonance (C_k^α) terms in the fit. Furthermore, since the C_k^α's and b_k^α's have a fixed phase, one can go beyond the isobar model (to the level of linear $M_{\beta\gamma}$ dependence) at the expense of two real parameters per channel. In fact, since it is the difference in subenergy behavior which is most crucial, one might pick either C_2^α or $b_2^{\alpha;p} = 0$ and work with a single real parameter. Applications of this technique are now being studied.

In conclusion, it should be noted that a general computer code for implementing the approach discussed in this article has been developed, and is available to anyone who might be interested in employing these techniques. The author also wishes to express his appreciation to Dr. Thomas A. Lasinski for many helpful and stimulating discussions during the course of this work.

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11. This could well be the case with regard to the A_1 result claimed by Longacre and Aaron; see Ref. 1.
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17. In practice the condition $G_p^{\ell m}$ small is rather likely to be satisfied; this is clearly true if the off-shell variations are individually weak.

18. For a particularly elegant version of the unitarized isobar model see J. L. Basdevant and E. L. Berger, Phys. Rev. Lett. 37, 977 (1976); also Ref. 1.

19. Experimental values have been obtained from the review of diffractive processes by D. W. G. S. Leith, SLAC-PUB-1526 (January, 1975).

20. Our conventions regarding Clebsch-Gordan coefficients, spherical harmonics, etc. are taken from M. E. Rose, Elementary Theory of Angular Momentum (Wiley, New York, 1957).

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23. Modifications for general ℓ_α, isospin, etc. are messy but straightforward. For the particular case of $\ell_\alpha=1$, one may proceed along the lines discussed by Aaron, Amado and Young in Ref. 10.

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25. The procedure is identical with that developed in Ref. 12.

26. See, for example, the curves of Basdevant and Berger in Refs. 1 and 18.

27. G. W. Brandenburg, et al., Phys. Rev. Lett. 36, 703 (1976).

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29. Since the experimental results correspond to an average over a finite interval Δt, the slope parameter b in $\sigma(t)/\sigma(0)=\exp(bt)$ was determined for the model, and a similar averaging was performed.

30. D. D. Brayshaw, Phys. Rev. Lett. 39, 371 (1977).

31. A peak would occur, however, in coupled channels; e.g., $X \rightarrow A_1 \rightarrow \rho\pi$. This effect can be seen in $\rho K \rightarrow Q_2 \rightarrow K^* \pi$ in our fits by introducing some coupling of the ρK channel to the low mass state.

32. The work of Ascoli et al., in Ref. 15 (for the 3π system) also suggests this possibility, in a comparison of exact model results vs. a computer analysis of Monte Carlo generated "data".

33. G. Alexander et al., DESY preprint 77/78, December 1977 (submitted to Phys. Lett.); J. A. Jaros et al., SLAC-PUB-2084 (February 1978).

TABLE 1.

Elastic Scattering Parameters

System	$\alpha (GeV/c)^{-2}$	$\beta (GeV/c)^{-3}$	σ_{tot}^{∞} (mb)	γ (mb $[GeV/c]^{\frac{1}{2}}$)
K^+p	2.38	0.437	17.4	0.0
K^-p	7.60	0.000	17.1	17.1
π^+p	6.24	0.141	21.3	11.2
π^-p	7.52	0.030	21.3	17.6

TABLE 2.

$K\pi\pi$ Fit Parameters (1^+0^+)

Type	$N_{\rho K}/4\pi$	$N_{K^*\pi}/4\pi$	s_{11} $(GeV/c)^2$	s_{22} $(GeV/c)^2$	$\lambda_{11}^{(o)}$ $(GeV/c)^2$	β_2 $(GeV/c)^2$	$\lambda_{22}^{(o)}$ $(GeV/c)^{-2}$
$n_f=2$	2.92	2.36	1.93	1.46	7.22	0.154	9.05
Gaussian	4.82	1.82	1.93	1.70	7.16	0.190	6.42

FIGURE CAPTIONS

1. Series of diagrams defining T_m for (a) an isolated m-body system, and (b) in the presence of n-m non-interacting particles.

2. (a) The Feynman diagram for one-particle exchange. (b) and (c) The equivalent RST diagrams. (d) Scattering process related to the RST vertex $a+e \rightarrow c$.

3. (a) Production diagram leading to the two-particle final state (k_1, k_2). (b) Production followed by rescattering. The latter is characterized by the off-shell amplitude t_{12}.

4. (a) "Sequential" production model; the isobar k_R decays into the observed final state (k_β, k_γ). (b) Production of a true three-body state $(k'_\alpha, k'_\beta, k'_\gamma)$ followed by the interaction (t_α) of β', γ'. The latter precedes the interaction (t_2) of α' with particle i. (c) Production of the three-body system without rescattering. (d) Production with t_2 preceding t_α. Diagrams (c) and (d) together constitute the "simple" production model.

5. The RST diagram which, together with Fig. 4a, reproduces the Deck amplitude.

6. Isospin conventions for the production model. In general, i_j is the particle isospin, and μ_j the third-component of isospin; I_α is the isobar isospin.

7. (a) Diagram corresponding to the three-particle vertex (simple model). (b) The two-particle scattering process associated by crossing.

8. (a) Diagram defining the sequential three-body vertex. (b) The associated crossed diagram, representing scattering via the isobar of mass $\mathcal{M}_\alpha$.

9. (a) The "fixed" coordinate system defined by the vectors $\vec{k}_i, \vec{p}_i, \vec{p}_f$ in the three-body c.m. (b) The "α" coordinate system defined by the vectors $\vec{k}_\alpha, \vec{k}_\beta, \vec{k}_\gamma$ in the three-body $(\alpha\beta\gamma)$ c.m. The vector notation corresponds to Fig. 4a.

10. Subenergy dependence of the isobar amplitudes $\tilde{\tau}_p^{\alpha}$ defined in Eq. (5.4) for M_3=1.5 GeV. The 1^+0^+ amplitudes associated with ρK and $K^*\pi$ are shown in (a) and (b), respectively. The solid, dashed, and dashed-dot curves correspond to the respective choices n_g=0, 1, -1; the curves are arbitrarily normalized to unity at the isobar mass.

11. The differential cross sections corresponding to (a) ρK and (b) $K^*\pi$ for the parametrized production model. The curves correspond to n_g=0, 1, -1 according to the conventions of Fig. 10. The ρK cross sections have been arbitrarily normalized to the value 0.5 mb/GeV^3 at M_3=1.28 GeV; the $K^*\pi$ cross sections are normalized to 1.0 mb/GeV^3 at M_3=1.34 GeV.

12. Subenergy dependence of the isobar amplitudes for M_3=1.5 GeV. The 1^+0^+ ρK and $K^*\pi$ amplitudes are shown in (a) and (b), respectively. The solid, dashed, and dashed-dot curves correspond to n_f=1, 2, 0; they are normalized to unity at the isobar mass.

13. The differential cross sections for (a) ρK and (b) $K^*\pi$ for n_f=0, 1, 2 according to the conventions of Fig. 12. The normalizations have been adjusted as in Fig. 11.

14. (a) Schematic representation of production leading to a two-particle interaction (or isobar) t_α. (b) Production plus three-particle rescattering, culminating in interaction t_α. (c) Integral equation for the three-body operator T_3.

15. Real (dashed curves) and imaginary (solid curves) parts of $\rho_\alpha(M_3)$, as defined in Eq. (6.35). Using the parametrization of Eq. (6.36), the results shown in (a) and (b) correspond to ρK and $K^*\pi$, respectively.

16. Fits to the 1^+0^+ state of $K^+\pi^+\pi^-$. The solid curves correspond to the production model $n_g=0$, $n_f=2$ and the resonance parametrization of Eqs. (7.1), (7.3), (7.4); the dashed curves correspond to $n_g=0$ and the Gaussian vertex function of Eq. (7.5). Results are shown for (a) the $K^*\pi$ intensity, (b) the ρK intensity, and (c) the relative phase $\phi(\rho K)-\phi(K^*\pi)$; experimental points are taken from Ref. 27.

17. The relative phase corresponding to the Gaussian fit of Fig. 16, calculated with four terms in the subenergy expansion (solid curve), and with just a single term (dashed curve).

18. Comparison of the Gaussian fit to the ρK intensity (solid curve), versus a calculation (dashed curve) using the same fit parameters with the subenergy fixed at the ρ mass ($M_{\beta\gamma}=M_\rho$).

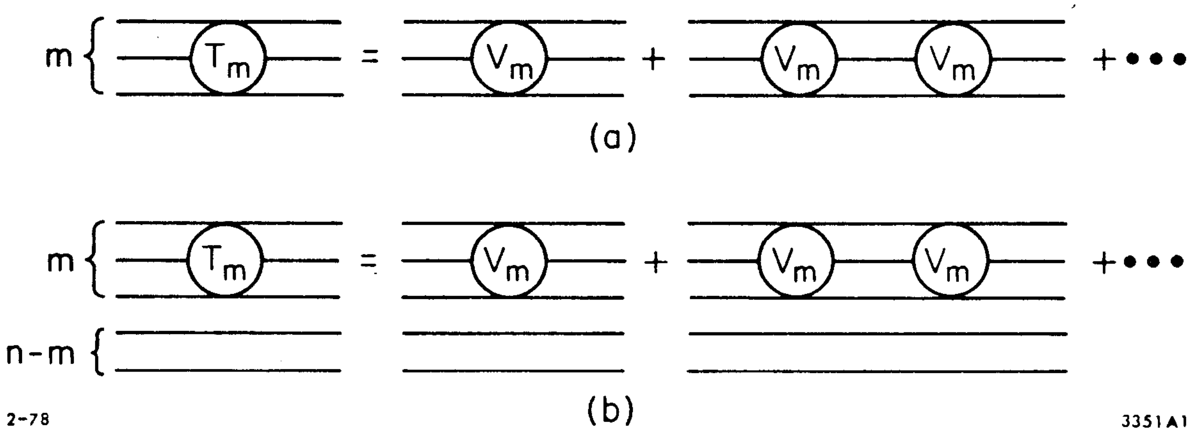


Fig. 1

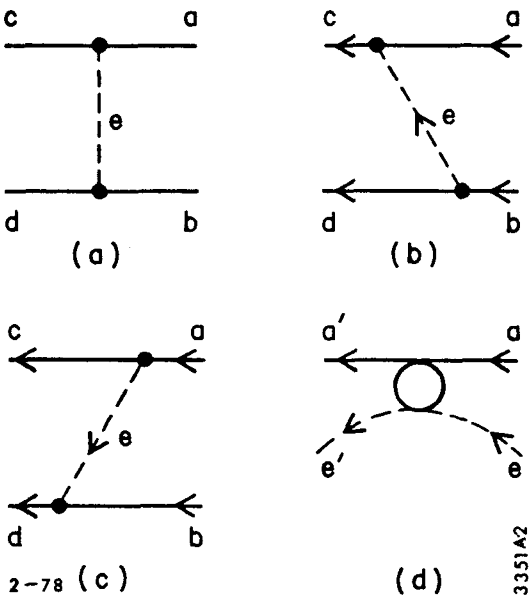


Fig. 2

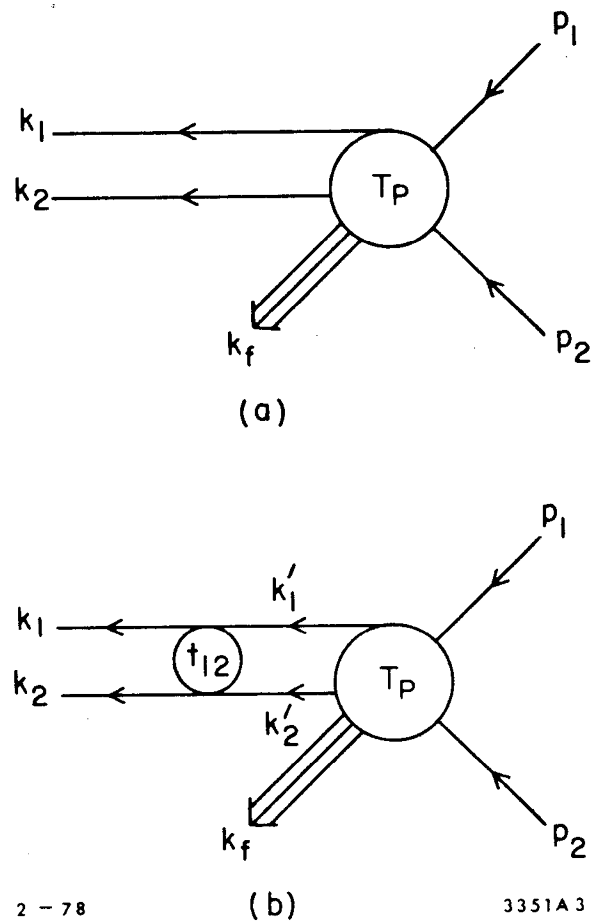
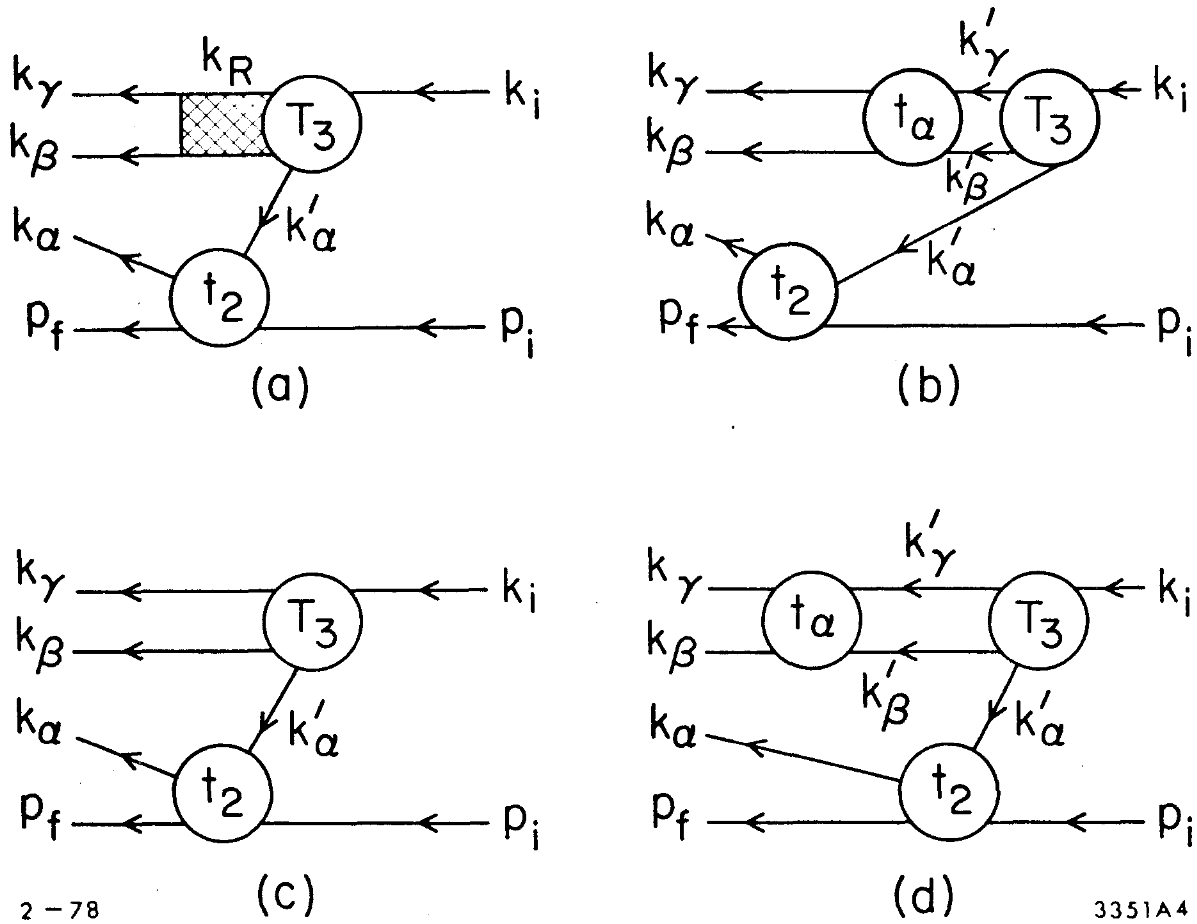


Fig. 3



k_γ
k_R
k_i
k_β
T_3
k_α
k'_α
t_2
p_f
p_i
(a)
k'_γ
t_α
k'_β
(b)
(c)
(d)
2 – 78
3351A4

Fig. 4

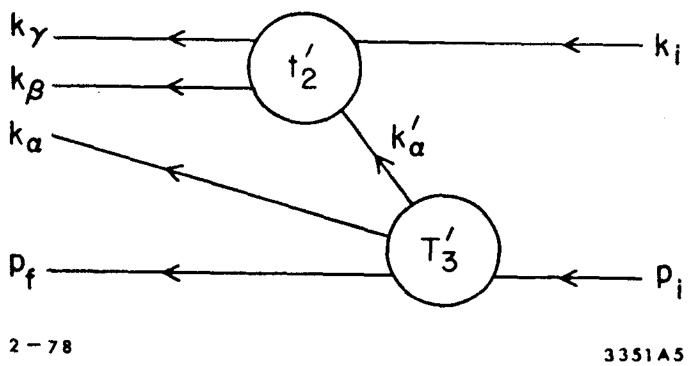


Fig. 5

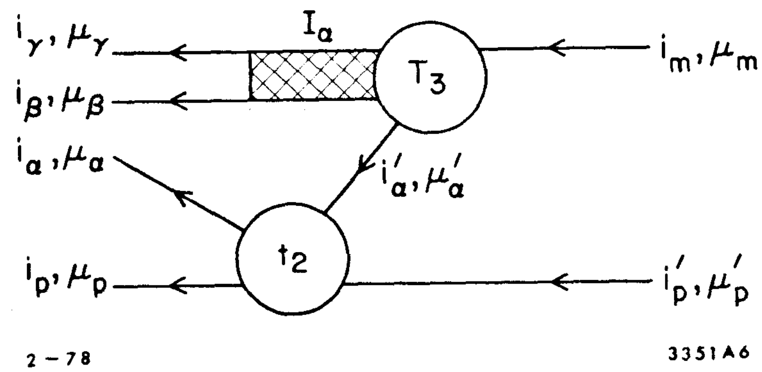

i_γ, μ_γ
I_α
T_3
i_m, μ_m
i_β, μ_β
i_α, μ_α
i'_α, μ'_α
t_2
i_p, μ_p
i'_p, μ'_p
2 − 78
3351A6

Fig. 6

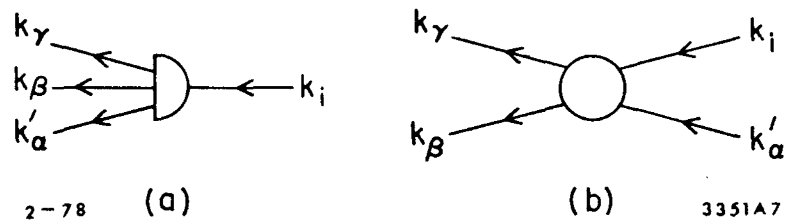
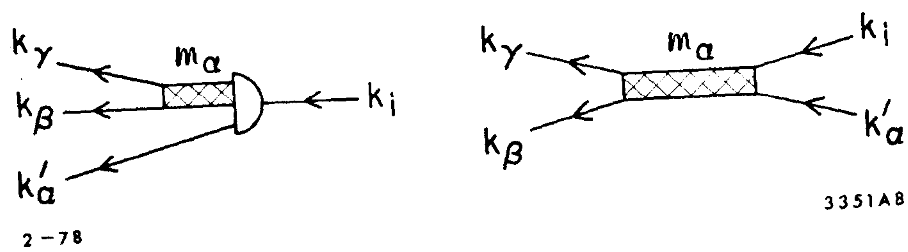
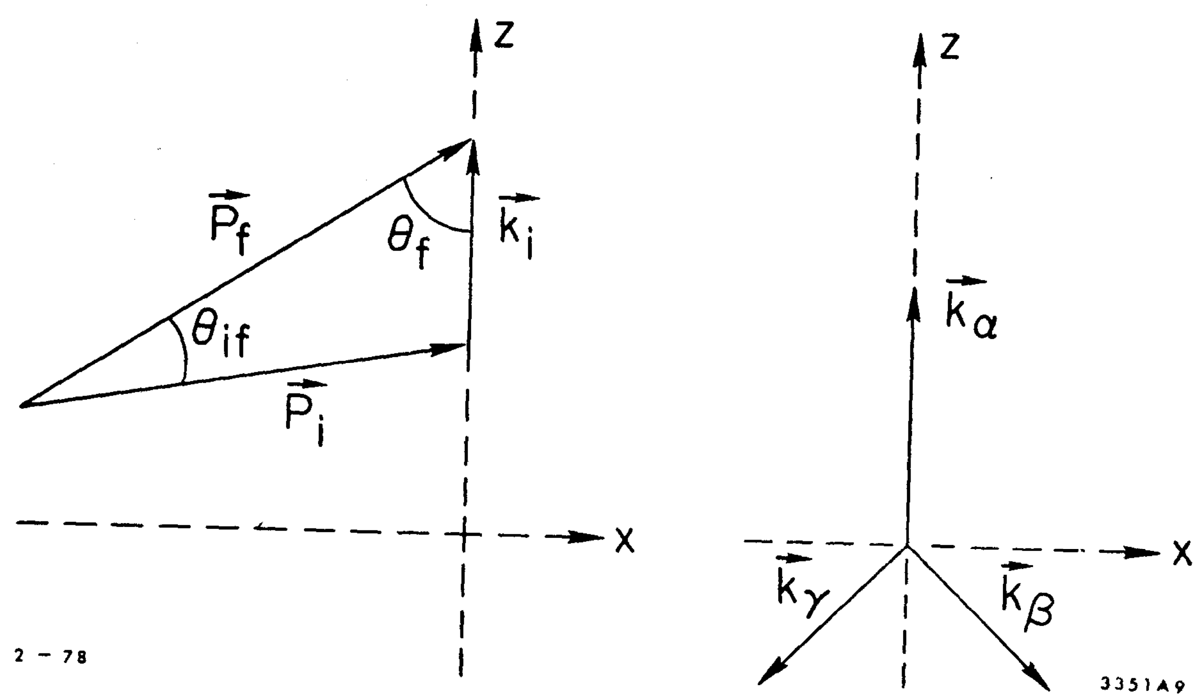


Fig. 7



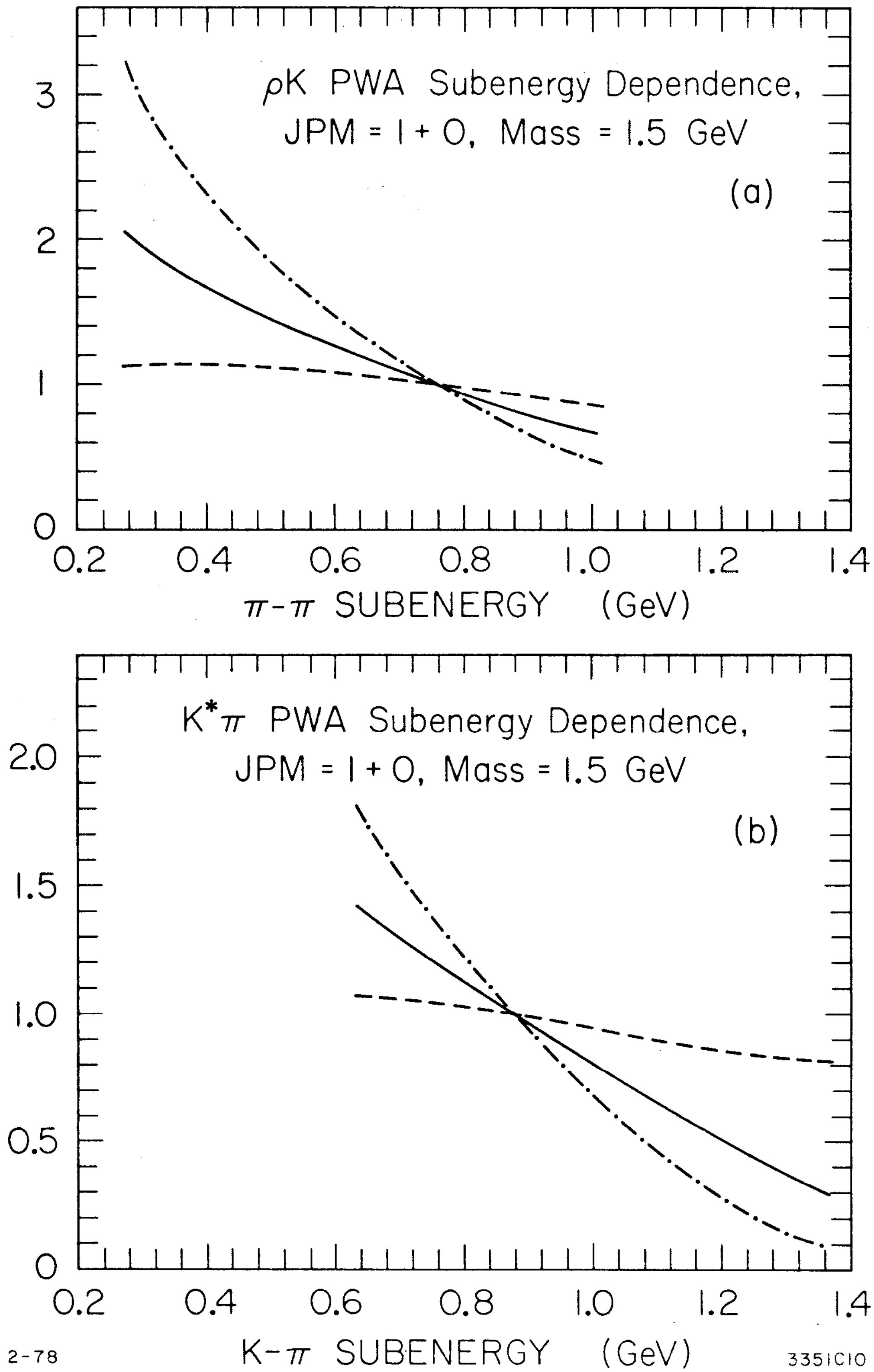
k_γ
m_α
k_β
k_i
k'_α
2-78
k_γ
m_α
k_i
k_β
k'_α
3351A8

Fig. 8



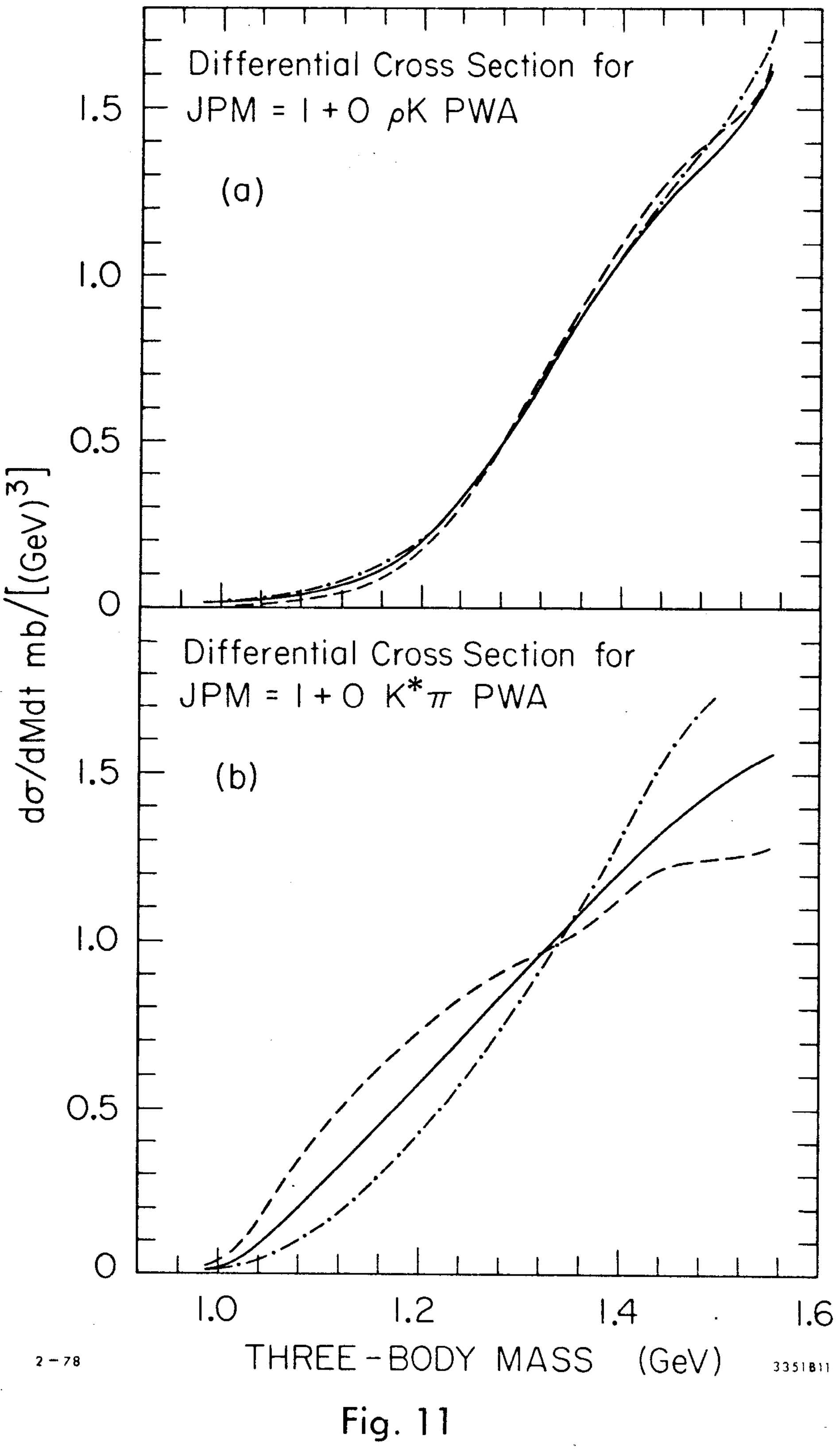
z
$\vec{P}_f$
θ_f
$\vec{k}_i$
θ_{if}
$\vec{P}_i$
x
2 – 78
z
$\vec{k}_\alpha$
x
$\vec{k}_\gamma$
$\vec{k}_\beta$
3351A9

Fig. 9



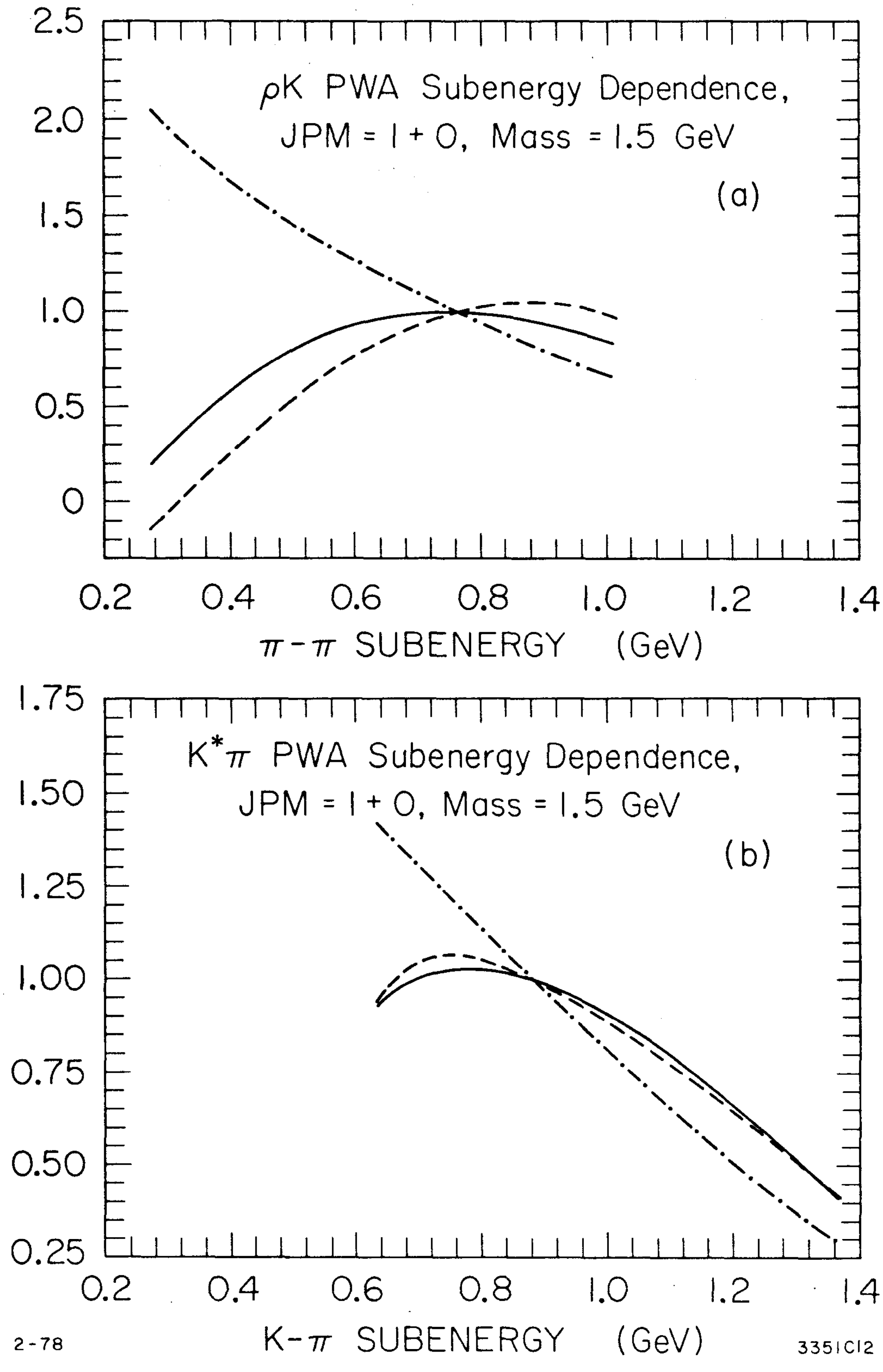
ρK PWA Subenergy Dependence,
JPM = 1 + 0, Mass = 1.5 GeV
(a)
0
1
2
3
0.2
0.4
0.6
0.8
1.0
1.2
1.4
π-π SUBENERGY (GeV)
K*π PWA Subenergy Dependence,
JPM = 1 + 0, Mass = 1.5 GeV
(b)
0
0.5
1.0
1.5
2.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
K-π SUBENERGY (GeV)
2-78
3351C10

Fig. 10



Differential Cross Section for
JPM = 1 + 0 ρK PWA
(a)
Differential Cross Section for
JPM = 1 + 0 K*π PWA
(b)
dσ/dMdt mb/[(GeV)³]
0
0.5
1.0
1.5
1.0
1.2
1.4
1.6
THREE-BODY MASS (GeV)
2-78
3351B11

Fig. 11



2-78

3351C12

Fig. 12

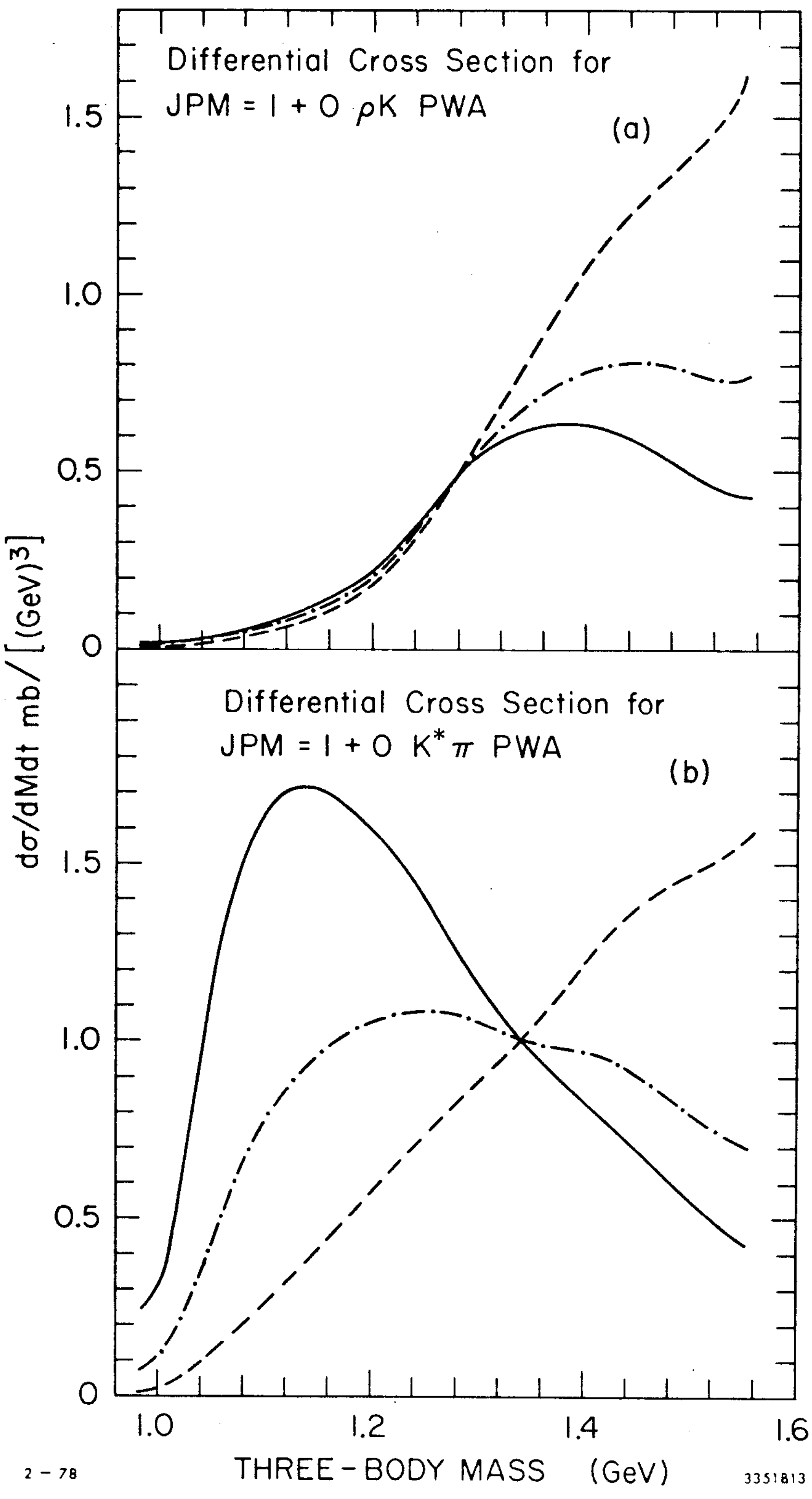
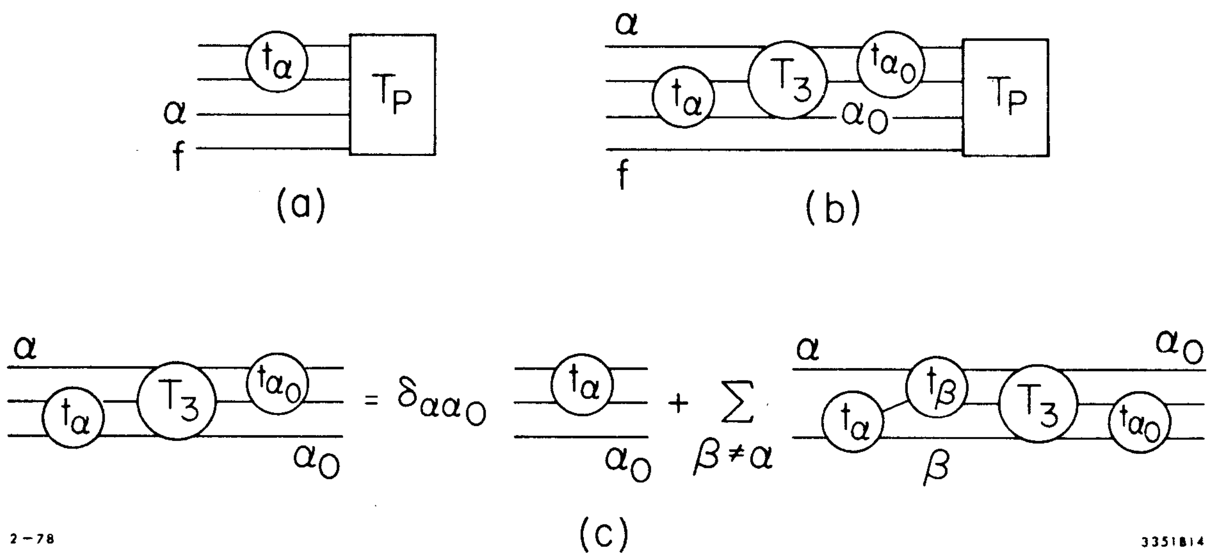
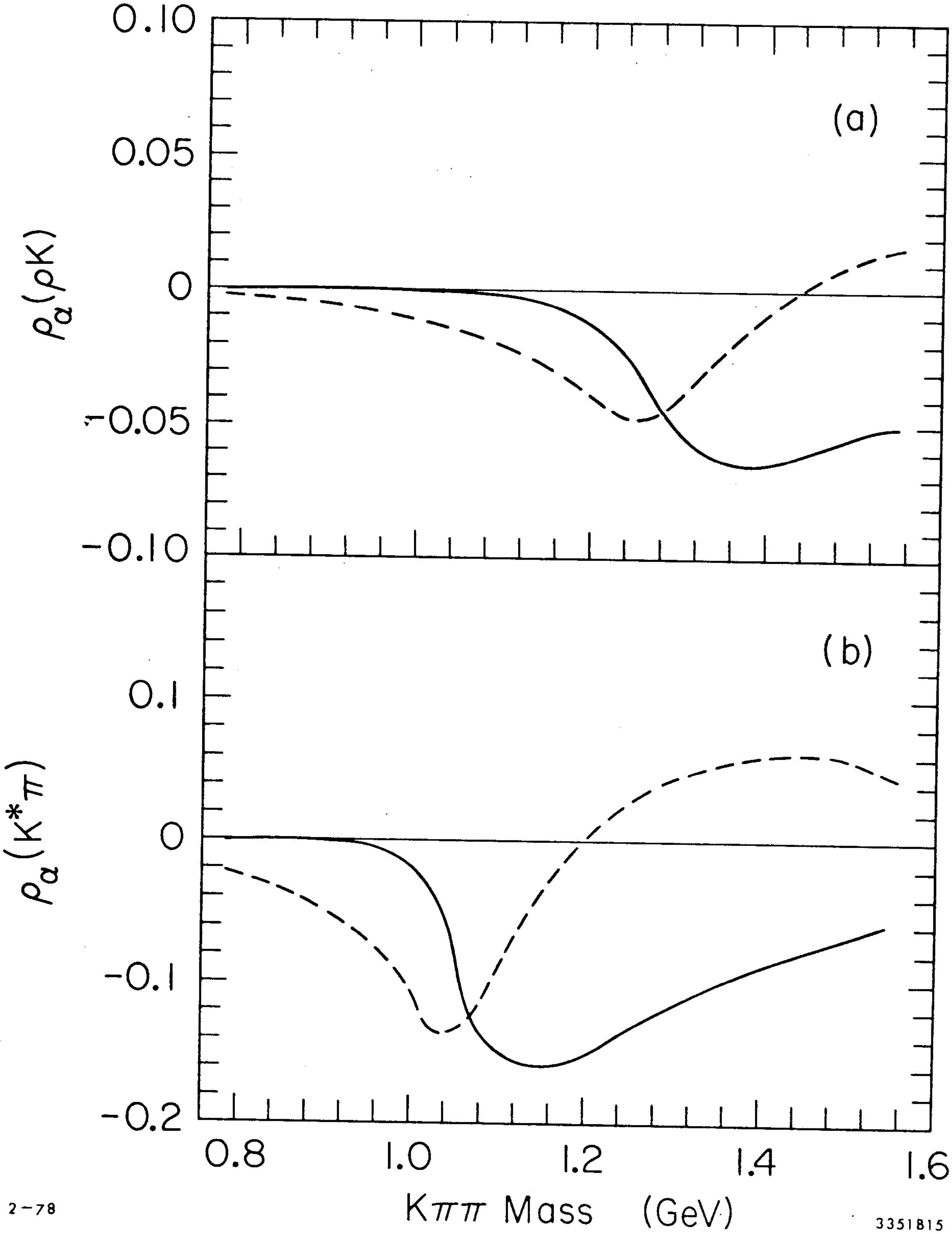

Differential Cross Section for
JPM = 1 + 0 ρK PWA
(a)
Differential Cross Section for
JPM = 1 + 0 K*π PWA
(b)
dσ/dMdt mb/[(GeV)³]
0
0.5
1.0
1.5
1.0
1.2
1.4
1.6
THREE-BODY MASS (GeV)
2-78
3351B13

Fig. 13



t_α
T_P
α
f
(a)
α
t_α
T_3
t_{α_0}
α_0
T_P
f
(b)
α
t_α
T_3
t_{α_0}
α_0
$= \delta_{\alpha\alpha_0}$
t_α
α_0
$+ \sum_{\beta \neq \alpha}$
α
α_0
t_α
t_β
T_3
t_{α_0}
β
2-78
(c)
3351B14

Fig. 14



0.10
0.05
0
-0.05
-0.10
$\rho_\alpha(\rho K)$
(a)
0.1
0
-0.1
-0.2
$\rho_\alpha(K^*\pi)$
(b)
0.8
1.0
1.2
1.4
1.6
$K\pi\pi$ Mass (GeV)
2-78
3351B15

Fig. 15

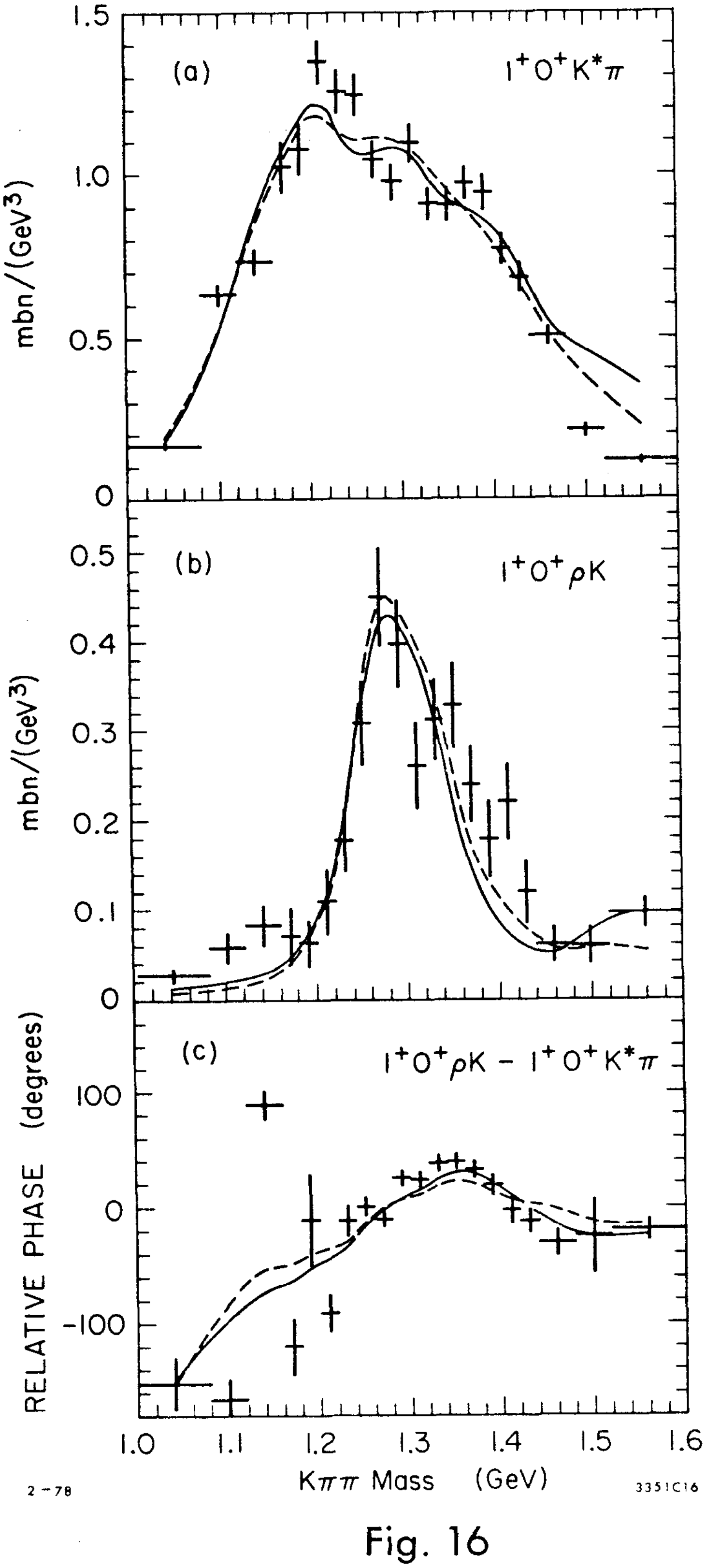
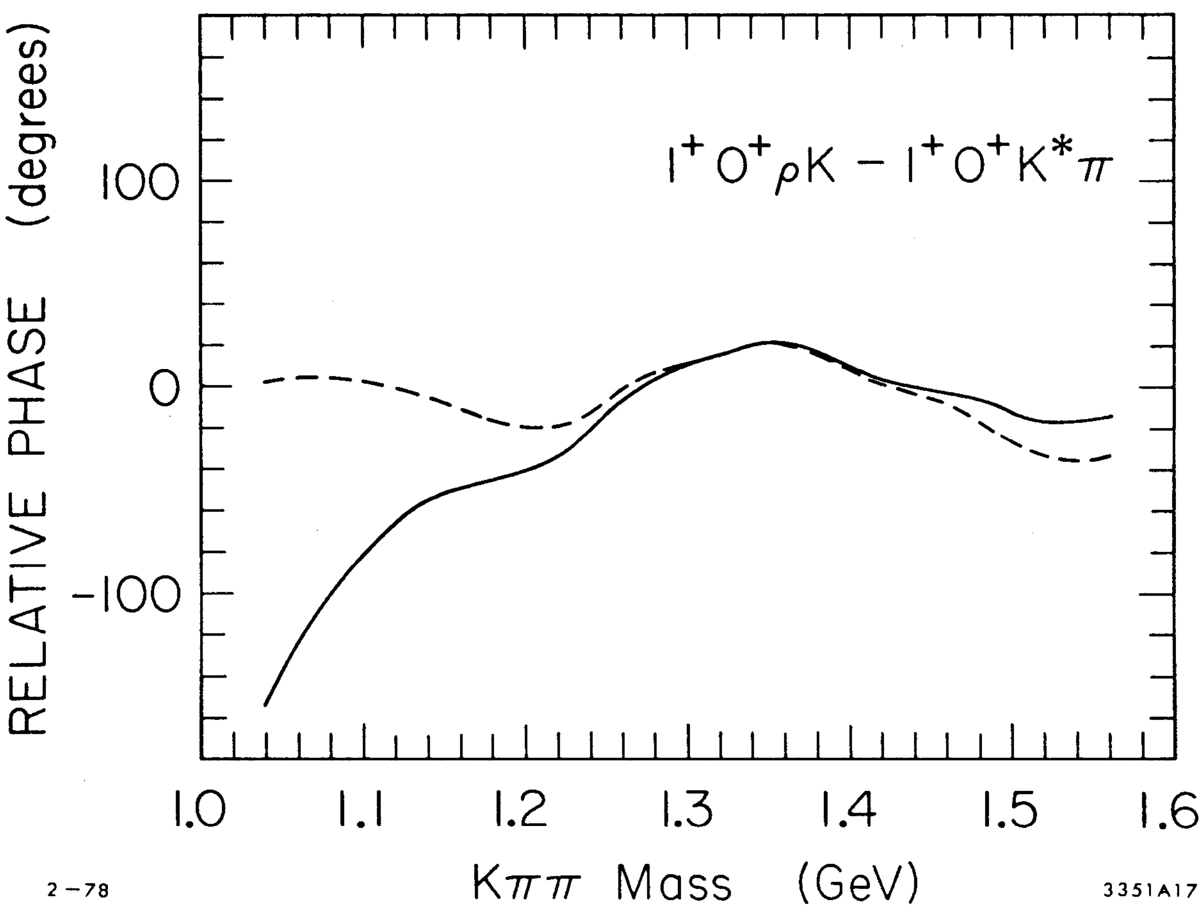

(a)
1⁺0⁺K*π
(b)
1⁺0⁺ρK
(c)
1⁺0⁺ρK − 1⁺0⁺K*π
mbn/(GeV³)
mbn/(GeV³)
RELATIVE PHASE (degrees)
Kππ Mass (GeV)
2-78
3351C16

Fig. 16



RELATIVE PHASE (degrees)
100
0
-100
$1^+0^+\rho K - 1^+0^+K^*\pi$
1.0
1.1
1.2
1.3
1.4
1.5
1.6
2-78
$K\pi\pi$ Mass (GeV)
3351A17

Fig. 17

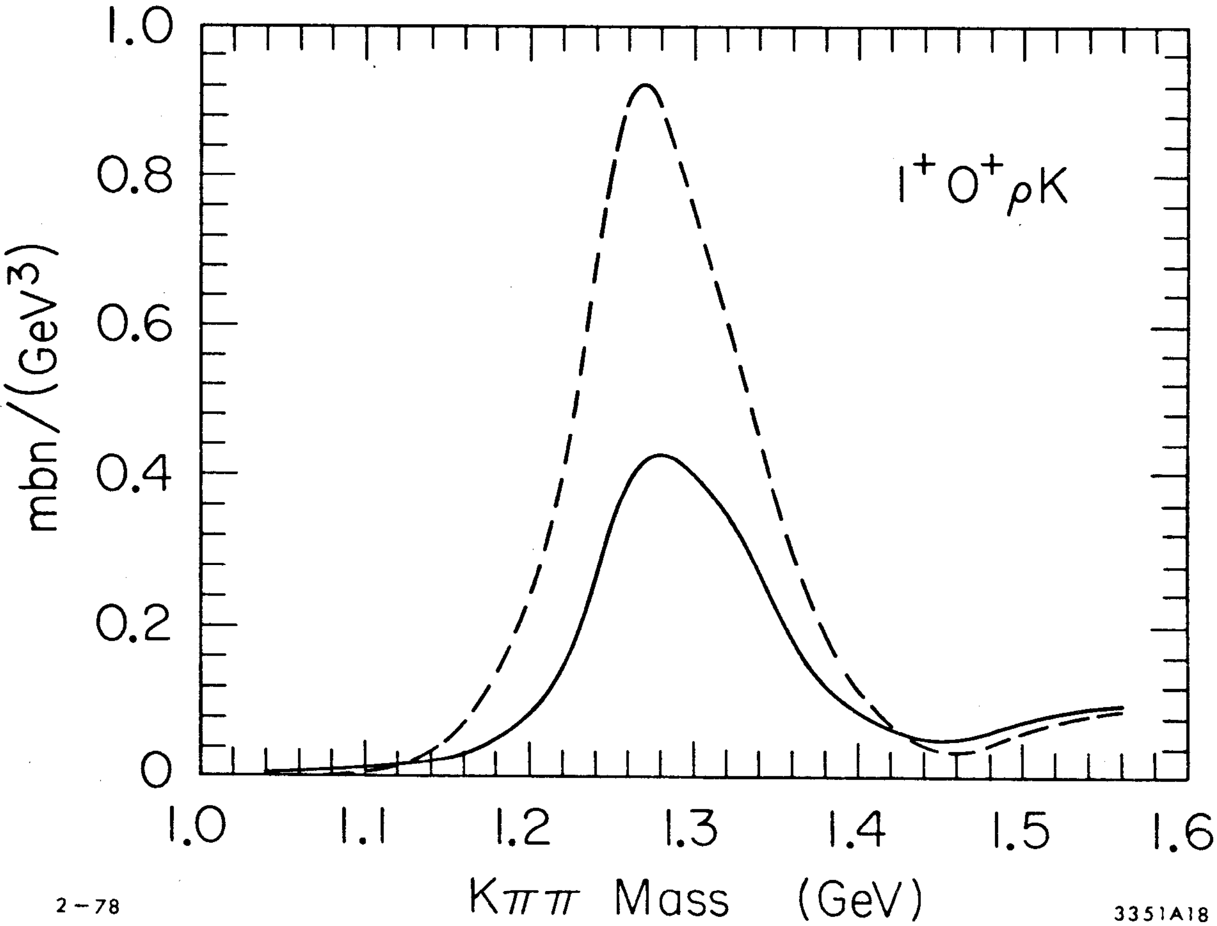

1.0
0.8
0.6
0.4
0.2
0
mbn/(GeV^3)
$1^+0^+\rho K$
1.0
1.1
1.2
1.3
1.4
1.5
1.6
$K\pi\pi$ Mass (GeV)
2-78
3351A18

Fig. 18