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REMARKS ON THE TOPOLOGY OF GAUGE FIELDS*

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In contrast to the short distance behavior of quantum chromodynamics (QCD), which is within the scope of perturbation theory and thus can be subjected to quantitative tests, the large distance or strong coupling regime of QCD is not well understood yet even qualitatively. An overriding problem of interest in quark confinement, and various theoretical schemes have been put forward to show that the quarks can indeed be confined. These schemes, though varying from one to another in detail, rely on the idea that the growing coupling constant at large distances plays a key role. It is not clear yet, however, whether confinement is a natural consequence of QCD alone, or it requires some independent and extraneous assumptions.

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Another important aspect of gauge fields is the existence of topologically nontrivial configurations such as vortices, monopoles and instantons. They may be regarded as natural consequences of gauge physics. Most likely these configurations also are crucial to the understanding of confinement as is claimed by a number of recent papers. For this reason I have planned this session to be organized mainly around the topological problems.

My own remarks on these questions will be very brief:

1. Action versus Free Action

There is a formal analogy between statistical mechanics and the Feynman formulation of quantum field theory, which seems to become especially relevant in gauge theories. This analogy was emphasized by the Princeton group[1] who made use of the concept of entropy. Let us write down the Feynman integral (in the Euclidean form) for a gauge field

$$Z = \int \exp\left[-\frac{1}{4g^2}\int F_{\mu\nu} F_{\mu\nu} d^4x\right] D(A_\mu) \tag{1}$$

where the coupling constant does not enter the definition of $F_{\mu\nu}$. The functional measure is not clearly defined, but we intend to integrate over the A_μ without imposing special gauge conditions. For a given value of the action, there is a corresponding phase space volume which we write symbolically as exp S :

$$\begin{aligned} Z &= \int \exp\left[-\frac{1}{g^2} I + S(I)\right] dI \\ &= \exp\left[-\Theta/g^2\right], \end{aligned} \tag{2}$$

$$\Theta \approx (I - g^2 S)_{min.}$$

Let us call Θ free action in contrast to the action I. In principle, we should minimize Θ rather than I, and this difference could become important for large g^2, which plays the role of temperature.

This statement takes on a real significance when the topology of gauge fields is considered. Suppose we start from the lowest value of I, i.e. the pure gauges: $F_{\mu\nu} = 0$, $I = 0$. This part of phase space can be parametrized by a unitary matrix field u,

$$A_\mu = iu^+ \partial_\mu u \ , \quad u^+ u = 1 \tag{2a}$$

The field u, however, may be topologically non-trivial. For example, choose

$$u = \tau_\mu x_\mu / |x| \ , \quad \tau_\mu = (\vec{\tau}, i) \tag{3}$$

in the case of SU(2) theory. Then u becomes singular at the origin, and $F_{\mu\nu}$ and I cannot be identically zero since the Pontrjagin index

$$\frac{1}{16\pi^2} \mathrm{Tr} \int F_{\mu\nu} \tilde{F}_{\mu\nu} d^4 \chi = \int Q_\mu d S_\mu \ ,$$

$$Q_\mu = -\frac{1}{3} \mathrm{T_r}\, u^+ \partial_\nu u \partial_\lambda u^+ \partial_\rho u \epsilon^{\mu\nu\lambda\rho} \tag{4}$$

is computed to be nonzero. Thus we might say that pure gauges are not a well defined concept, or the entropy cannot be defined for given I, at least not for $I = 0$. There seems to be a sort of uncertainty principle between I and S due to topology. Clearly this is related to the Gribov problem[2] and Singer's observation[3] of its generality. At any rate we are forced to enlarge pure gauges to their neighborhood which I will call almost pure gauges. It is conceivable that for large enough g^2 the gain in S by including various topologies can outweigh the cost of larger I, thus actually owering the free action. Then the topology-averaged field configurations which minimize the free action may deviate substantially from the naive classical configurations which minimize only I.

2. A class of almost pure gauges.[4]

It is both natural and convenient to consider the following class of almost pure gauges

$$A_\mu = ifu^+\partial u \qquad (5)$$

where f is a scalar function which vanishes at the singularities of u. The topological characterization of a singularity will not be altered by this if $f \to 1$ in a region surrounding it. From Eq. (5) we obtain

$$F_{\mu\nu} = f(1-f)\partial_\mu u^+ \partial_\nu u + \partial_\mu f u^+ \partial_\nu u - (\mu \leftrightarrow \nu)\,. \qquad (6)$$

It is not surprising that this class of configurations cover various known examples of nontrivial topology such as instanton, meron, monopole, and string.

A more interesting point is that we can generalize Eq. (5) further as

$$A_\mu = i\sum_i f_i u_i^+ \partial_\mu u_i\,. \qquad (7)$$

The many-instanton solution of 't Hooft (in the singular gauge) is indeed of this form. Furthermore, Eq. (7) can be written as a London relation[5]

$$A_\mu = \lambda J_\mu = \lambda \sum_{i=1}^{N} J_{i\mu}\,,$$

$$J_{i\mu} = i h_i u_i^+ D_\mu u_i$$

where

$$D_\mu u = \partial_\mu u + i u A_\mu\,, \qquad (8)$$

f_i and h_i are related by

$$f_i = \lambda h_i / (1 + \lambda \Sigma h_j) . \tag{9}$$

Then a simple superposition principle holds for the currents $J_{i\mu}$ to generate many instantons, and the total current is conserved: $D_\mu J_\mu = 0$. It might also be instructive to observe that Eq. (8) can be further simplified if J_μ is defined in terms of a $2N\times 2$ rectangular matrix U:

$$U = \Sigma \oplus h_i^{\frac{1}{2}} u_i \quad (\text{assuming } h_i \geq 0), \quad U^+ U = (\Sigma h_i) 1 ,$$

$$J_\mu = \frac{i}{2} [U^+ (D_\mu U) - (D_\mu U^+) U] . \tag{10}$$

U has a guaged SU(2) symmetry acting from the right, and a global $SU(2)\times S_N$ symmetry acting from the left which may be generalized to SU(2N). These two are independent, like color and flavor.

3. A remark concerning the Wilson criteria.

The Wilson criterion is widely used to test the confinement property of a theory. In the context of functional integration, one evaluates

$$\langle W \rangle = \Sigma \; W_i \; \rho_i \; ,$$

$$W = \mathrm{Tr} \exp [i \oint A_\mu \, d x_\mu] \tag{11}$$

where ρ_i is the weight of a configuration i. In general Eq. (11) is expected to yield an asymptotic form

$$\langle W \rangle \sim a \exp [- \lambda L] + b \exp [- \mu L^n] + \cdots , \quad n > 1 \tag{12}$$

where L is the linear dimension of the Wilson loop. What I would like to emphasize is that to prove confinement, one must first show the absence of configurations contributing to the first term rather than the presence of configurations contributing to the second term. I suppose that the former task is more demanding and difficult than the latter.

REFERENCES

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4. See also Y. Nambu, Cal Tech preprint CALT 68-634, 1977.

5. Y. Nambu, Annals of the New York Academy of Sciences 294 (Five Decades of Weak Interactions) p. 74 (1977).