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A RELATIVISTIC DESCRIPTION OF THE ELECTROMAGNETIC FORM FACTOR OF THE DEUTERON*

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ABSTRACT

A covariant model of the electromagnetic form factor of the deuteron in the impulse approximation is presented. The treatment includes spin and allows for a complete determination of the two elastic structure functions. Our results are in good agreement with experimental data.

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I. INTRODUCTION

The study of electromagnetic form factors of bound states at large momentum transfer constitutes a powerful tool in determining its degree of compositeness and underlying dynamics. This fact has been known for some time, and the dimensional counting prediction for n elementary constituents $F_n \sim (q^2)^{1-n}$ appears to be consistent with experiment for a range of bound states starting with pions (n = 2) up to light nuclei such as He^4 (n = 12).[1] For the nuclear case an alternative yet complementary explanation for this behavior has been given in terms of the short distance characteristics of the nucleon-nucleon force.

Schmidt and Blankenbecler in Ref. 2 developed a model with scalar nucleons, which was fully relativistic and which incorporated in a very simple form the short distance behavior of the nucleon force. In particular it was applied to the deuteron case, and predictions were given for the He^3 and He^4 form factors.

In the present work we have extended the previous ideas on the deuteron form factor to include spin. This enables us to describe the two elastic structure functions corresponding to charge-quadrupole and magnetic scattering. The paper is organized as follows: In Section II we present our model and give all necessary formulae. Section III is devoted to a numerical analysis and comparison to experimental data and finally our main conclusions are exposed in Section IV. An appendix contains those detailed results of calculations which we do not include in Section II.

II. DEUTERON FORM FACTOR

We shall write the covariant decomposition of the elastic form factor of the deuteron as:[3]

$$G^{\mu}(q^2) = \frac{1}{\sqrt{2D^{o}\,2D'^{o}}} \langle D'| j^{\mu} |D\rangle$$

$$= -e\left\{ G_1(q^2)(\xi'^{*}\cdot\xi)d^{\mu} + G_2(q^2)\left[\xi^{\mu}(\xi'^{\mu}\cdot q) - \xi'^{\mu}(\xi\cdot q)\right] \right. \tag{2.1}$$

$$\left. - G_3(q^2)\,\frac{\xi\cdot q\,\xi^{*\prime}\cdot q}{2M^2}\,d^{\mu}\right\}$$

where ξ and ξ' are the polarization vectors for the incoming and outgoing deuterons of momenta D and D$'$ (see Fig. 1) satisfying the conditions

$$\xi\cdot D = \xi'\cdot D' = 0 \quad , \quad \xi^2 = \xi'^2 = -1$$

and

$$d^{\mu} = D'^{\mu} + D^{\mu} \quad , \quad q^{\mu} = D'^{\mu} - D^{\mu} \tag{2.2}$$

M stands for the deuteron mass and e the electric charge. G_1, G_2, and G_3 are Lorentz scalar functions of the invariant momentum transfer q^2 of the problem.

IIa. IMPULSE APPROXIMATION

In this approximation the deuteron is coupled via strong interactions only to the two nucleon channel, so that the electromagnetic elastic vertex will be described by the diagram depicted in Fig. 1. Conventional Feynman rules give for this diagram[4]

$$G^{\mu}(q^2) = \int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left\{ \bar{\Lambda}^{\beta}(D-k, k+q)\,\frac{1}{\not k + \not q - m}\, f^{\mu}(k, k+q) \right.$$

$$\left. \frac{1}{\not k - m}\,\Lambda^{\alpha}(k, D-k)\,\frac{1}{\not k - \not D - m}\right\} \xi_{\alpha}\xi^{*\prime}_{\beta} \tag{2.3}$$

where $\Lambda^{\alpha}(p,p')$ is the Bethe-Salpeter (B-S) vertex that describes the covariant coupling of the deuteron to two nucleons arbitrarily off mass shell.

In the previous equation $f^{\mu}(p,p')$ is the elastic electromagnetic vertex of the nucleon and m is its mass. We will carry out one of the four integrals in eq. (2.3) using a very convenient method developed by M. Schmidt.[5]

First of all let us introduce Brodsky's parametrization of the vectors of the problem

$$
\begin{aligned}
D &= \left(D + \frac{M^2}{4D}\,,\ \vec{0}_\perp\,,\ D - \frac{M^2}{4D}\right)\\
q &= \left(q_\perp^2/4D\,,\ \vec{q}_\perp\,,\ -q_\perp^2/4D\right) \qquad (2.4)\\
k &= \left(xD + \frac{k^2+k_\perp^2}{4xD}\,,\ \vec{k}_\perp\,,\ xD - \frac{k^2+k_\perp^2}{4xD}\right).
\end{aligned}
$$

With the momenta written this way one gets

$$
d^4k = \frac{d^2k_\perp\, dk^2 dx}{2|x|} \qquad (2.5)
$$

where k^2 and x run from $-\infty$ to $+\infty$.

Now the integral over k^2 can be carried out directly since only the singularity coming from the pole in the propagator of the non-interacting nucleon contributes to the integral. This occurs because the poles from the other two propagators are always in the lower half k^2 plane. For x values between 0 and 1 however, the pole is in the upper half plane and one finds that

$$
\begin{aligned}
G^{\mu}(q^2) = \frac{i}{(2\pi)^3}\int_{-\infty}^{+\infty}\int_0^1 \frac{d^2k_\perp\, dx}{2x}\ \frac{1}{(k+q)^2-m^2}\ \frac{1}{k^2-m^2}\\
\mathrm{Tr}\Big\{\bar{\Gamma}^{\beta}(D-k)\left[\not{k}-\not{q}+m\right] f^{\mu}(k,\,k+q)\left[\not{k}+m\right]\Gamma^{\alpha}(k) \qquad (2.6)\\
\left[\not{k}-\not{p}+m\right]\Big\}\,\xi_\alpha\,\xi_\beta^{*\prime}
\end{aligned}
$$

with the additional condition

$$k^2 = \frac{x}{1-x}\left[M^2(1-x) - m^2 - \frac{k_\perp^2}{x}\right]$$

The function $\Gamma^\alpha(p)$ was introduced by Blankenbecler and Cook[6] and results from $\Lambda^\alpha(p, p')$ when $p'^2 = m^2$, i.e. it is the B-S vertex (deuteron $\rightarrow$ two nucleons) with one leg on shell, and describes completely the deuteron structure in the approximation we are working in. The most general form for these functions is given by

$$\Gamma^\alpha(p) = F(p^2)\,\gamma^\alpha - \frac{G(p^2)}{M}\,p^\alpha - \frac{M - \not{p}}{M}\left[H(p^2)\,\gamma^\alpha - \frac{I(p^2)}{M}\,p^\alpha\right] \qquad (2.7)$$

IIb. THE MODEL

Having presented our basic mathematical framework we next turn to the actual construction of the model. As a first step we will restrict ourselves to the use of the functions F and G in the B-S vertex. This we justify by noting that in the on mass shell limit the functions H and I do not contribute. However if we proceed strictly in this way and try to compute the form factor in Eq. (2.6), we obtain a non-gauge invariant result, being the gauge-invariance violating terms of the order of the binding energy. Since the H and I contributions to the amplitude are of this same order, it is clear that one can choose the functions F, G, H and I related in such a way that gauge invariance automatically follows.

An alternative and equivalent way to obtain a gauge invariant amplitude is to use only functions F and G and the substitution rule

$$k \rightarrow Dx - \frac{q}{2}(1-x) \qquad (2.8)$$

under the integral sign in Eq. (2.6).

This rule is also implicit in the gauge invariant scalar form factor of the deuteron. In fact, in this latter case[5]

$$F(q^2)(2D+q)^\mu = \int \frac{d^2k_\perp dx}{1-x} \psi^* \psi (2k+q)^\mu \tag{2.9}$$

and by a trivial use of (2.4) we obtain

$$F(q^2) = \int \frac{d^2k_\perp dx}{1-x} \psi^* \psi x \tag{2.10}$$

so that the rule (2.8) is obviously fulfilled. We do not explicitly have to choose the H and I invariant functions using this method.

In the Appendix we collect all formulae derived from the computation of the trace in (2.6) and the use of (2.8).

Having established how to deal with gauge invariance we now make a choice for the functions F and G in the B-S vertex. It is known that the nonrelativistic limits of F and G are[4]

$$\begin{aligned} \frac{F}{k^2 - m^2} &\longrightarrow u_0 - \frac{w_2}{\sqrt{2}} \\ \frac{G}{k^2 - m^2} &\longrightarrow \frac{3M^2}{\vec{k}^2} \frac{w_2}{\sqrt{2}} \end{aligned} \tag{2.11}$$

where u_0 and w_2 are the usual phenomenological S and D wavefunctions of the deuteron in momentum space.

We are now prepared to make a definite relativistic ansatz for F and G. For u_0 this has been done by Blankenbecler and Schmidt.[2] Their result is:

$$u_0 \longrightarrow \frac{N_0(x)(1-x)^{I+1}}{(k_\perp^2 + M^2(x))(k_\perp^2 + M^2(x) + \delta_0^2)^I} \tag{2.12}$$

where $M^2(x) = m^2 - x(1-x)M^2$.

In the same philosophy we generalize the w_2 wavefunction to be:

$$w_2 \longrightarrow \frac{N_2(x)(1-x)^{I+2}(k_\perp^2 + M^2(x) - m\epsilon)}{(k_\perp^2 + M^2(x))(k_\perp^2 + M^2(x) + \delta_2^2)^{I+1}} \tag{2.13}$$

This we achieved by generalizing the simple effective D-wave

$$w_2 \sim \frac{\vec{k}^2}{(m\epsilon + k_\perp^2)^3}$$

where ϵ is the binding energy of the deuteron. The choice of the power I in the wavefunctions is dictated by the underlying dynamical interactions one assumes. In the scalar case[2] this underlying dynamics was taken to be the exchange of vector mesons with monopole form factors at each vertex. We assume the same mechanism to take place. We therefore choose the power $I = 4$. This is also the value dictated by the familiar counting rules in quark constituent models. In this way we reproduce the large q^2 behavior of the deuteron form factor predicted by those models.[1]

III. NUMERICAL RESULTS

To make contact to the experimental data we first define the three "physical" form factors in terms of the G_1, G_2 and G_3 form factors given in the text.

$$G_c = G_1 - \frac{q^2}{6M^2} G_Q$$

$$G_Q = G_1 - G_2 + \left(1 - \frac{q^2}{4M^2}\right) G_3 \tag{3.1}$$

$$G_M = G_2$$

Their experimental values at $q^2 = 0$ are: $G_c(0) = 1$, $G_M(0) = 1.71$ (in units of $e/2M$) and $G_Q(0) = 25.84$ (in units of M^{-2}). Hence the values at the origin for G_1, G_2 and G_3 are:

$$G_1(0) = 1 \quad , \quad G_2(0) = 1.71 \quad \text{and} \quad G_3(0) = 26.55.$$

The elastic structure functions A and B in the Rosenbluth formula

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}\bigg|_{\text{Mott}} \left\{ A+B+\tan^2 \frac{\theta_e}{2} \right\}$$

are given by

$$A = G_c^2 + \frac{q^4 G_Q^2}{18M^4} - \frac{q^2}{6M^2}\left(1 - \frac{q^2}{4M^2}\right) G_M^2$$

$$B = -\frac{q^2}{3M^2}\left(1 - \frac{q^2}{4M^2}\right) G_M^2 \tag{3.2}$$

and these are the quantites we are going to compare with our model.

In our actual numerical analysis we will restrict ourselves to the most economical set of parameters in order not to introduce too many adjustable variables.

First of all, we know that the proportion of D-wave in the deuteron is small. We therefore neglect in F the D-wave content compared to the leading S-wave contribution. Since, however, the G function is pure D-wave we will keep it in our analysis.

Our parameter set is K_0, ρ, δ_0 and δ_2, where K_0 is an overall normalization factor and ρ measures the relative proportion of F and G in Eq. (2.7).

The resulting values of our fit are

$$\delta_0^2 = .72$$

$$\delta_2^2 = .20 \tag{3.4}$$

$$\rho = .16$$

and K_0 is such that G_1 is normalized to unity at $q^2 = 0$. In Fig.'s 2, 3 and 4 we give the curves for A and B compared to experimental data.[8, 9]

The set of parameters (3.4) renders the following values of the form factors at $q^2 = 0$:

$$G_1(0)=1$$
$$G_2(0)=1.71$$
$$G_3(0)=29.16$$

As one can see from these figures, the theoretical curves are remarkably good especially for the B function where the agreement extends over the whole measured q^2 range (although Fig. 4 does not show the very low q^2 data points). For the A function our results are only excellent for very low q^2 and beyond $-q^2 \geq 2$ GeV^2. This is not surprising since the model is best suited for the high q^2 tail. However, a more careful choice of the F functions, i.e. the inclusion of the small D-wave admixture in it, would likely render a better matching in the low q^2 regime. As it stands now, we basically get for A the same behavior and shape as in a previous naive scalar calculation.[2]

IV. CONCLUSIONS

In this paper we have presented a fully covariant description of the electromagnetic form factor of the deuteron. In this model the triangular diagram of the impulse approximation has been calculated using Bethe-Salpeter wavefunctions with one leg on shell, which are simple relativistic generalizations of the phenomenological S and D Hulthen wavefunctions whose falloff has been dictated by the asymptotic counting rules. This extension to include spin has permitted a prediction for both A and B elastic structure functions in the Rosenbluth formula.

We have compared our model with existing experimental data. We obtain good agreement for A in the high q^2 region ($q^2 \geq 2$ GeV^2), and excellent agreement over the whole momentum range for the structure function B.

Since in our actual numerical analysis we always made the simplest choices, it is obvious that taking into account all the potential richness of the model, would probably render even better results, especially for the low q^2 region in the elastic structure function A. As a final remark, being equipped with a B.—S. vertex for the deuteron with spin, one can obviously use it to predict the inelastic structure functions of deuterium.

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APPENDIX

Here we present all expressions for the three form factors. These formulae are obtained by making explicit use of the substitution rule Eq. (2.8). We should stress once more that this is a quite definite and natural (since it is directly inspired from the scalar form factor derivation) recipe to obtain uniquely gauge invariant results.

For i = 1, 2, 3 we get

$$G_i = \int \frac{d^2 k_\perp dx}{x} \Big[F F^* \left\{ f_1^N E_i(1) + f_2^N E_i(2) \right\} + G G^* \left\{ f_1^N E_i(3) + f_2^N E_i(4) \right\} -$$

$$- F^* G \left\{ f_1^N E_i(5) + f_2^N E_i(6) - G^* F \left\{ f_1^N E_i(7) + f_2^N E_i(8) \right\} \right\} \Big]$$

where

$$f^N_{1,2} \equiv f^p_{1,2} + f^n_{1,2}$$

$f^{p,n}_{1,2}$ are usual dipole form factors of proton and neutron.

F and G are functions of x and $k_\perp$.

F* and G* are displaced functions of x and $k_\perp + (1-x) q_\perp$.

The expressions for the E_i are:

$$E_1(1) = 2 \left[(1+x)(m^2 - k^2) - D \cdot q\,(1-x) + 2xD \cdot k \right]$$

$$E_1(2) = q^2$$

$$E_1(3) = E_1(4) = E_1(5) = E_1(6) = E_1(7) = E_1(8) = 0$$

$$E_2(1) = 2 \left[(3-x)(m^2 - k^2) - D \cdot q\,(1-x) + 2D \cdot k \right]$$

$$E_2(2) = 2 \left[(m^2 - k^2) + 2D \cdot k + \frac{q^2}{2}(1-x) \right]$$

$$E_2(3) = E_2(4) = 0$$

$$E_2(5) = E_2(7) = \frac{m}{M}(1-x)(D\cdot q + k^2 - m^2)$$

$$E_2(6) = E_2(8) = \frac{m}{2M} q^2 (1-x)$$

$$E_3(1) = 4M^2 x(1-x)^2$$

$$E_3(2) = 4M^2 x$$

$$E_3 = (1-x)^2 \left[k^2 (1+x) - 2x\, D\cdot k + (1-x) D\cdot q + m^2 (3x-1) \right]$$

$$E_3(4) = \frac{q^2}{2}(1-x)^2 (2x-1)$$

$$E_3(5) = E_3(7) = 2mM(2x^3 - 4x^2 + 3x - 1)$$

$$E_3(6) = E_3(8) = -mM(1-x)^2 - \frac{M}{m}\left[k^2(1-x^2) - 2D\cdot k(1-x) - \frac{q^2}{2}\, x\,(1-x)^2 \right]$$

FIGURE CAPTIONS

1. Feynman diagram for the electromagnetic form factor of the deuteron in the impulse approximation.

2. The elastic structure function A compared with experiment

$$(0 \leq |q^2| \leq 6\,\mathrm{GeV}^2).$$

3. The elastic structure function A compared with experiment

$$(2\,\mathrm{GeV}^2 \leq |q^2| \leq 12\,\mathrm{GeV}^2).$$

4. The elastic structure function B compared with experiment.

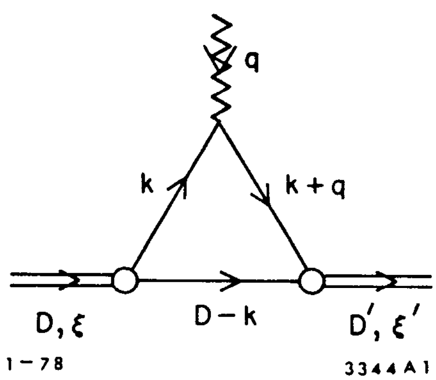
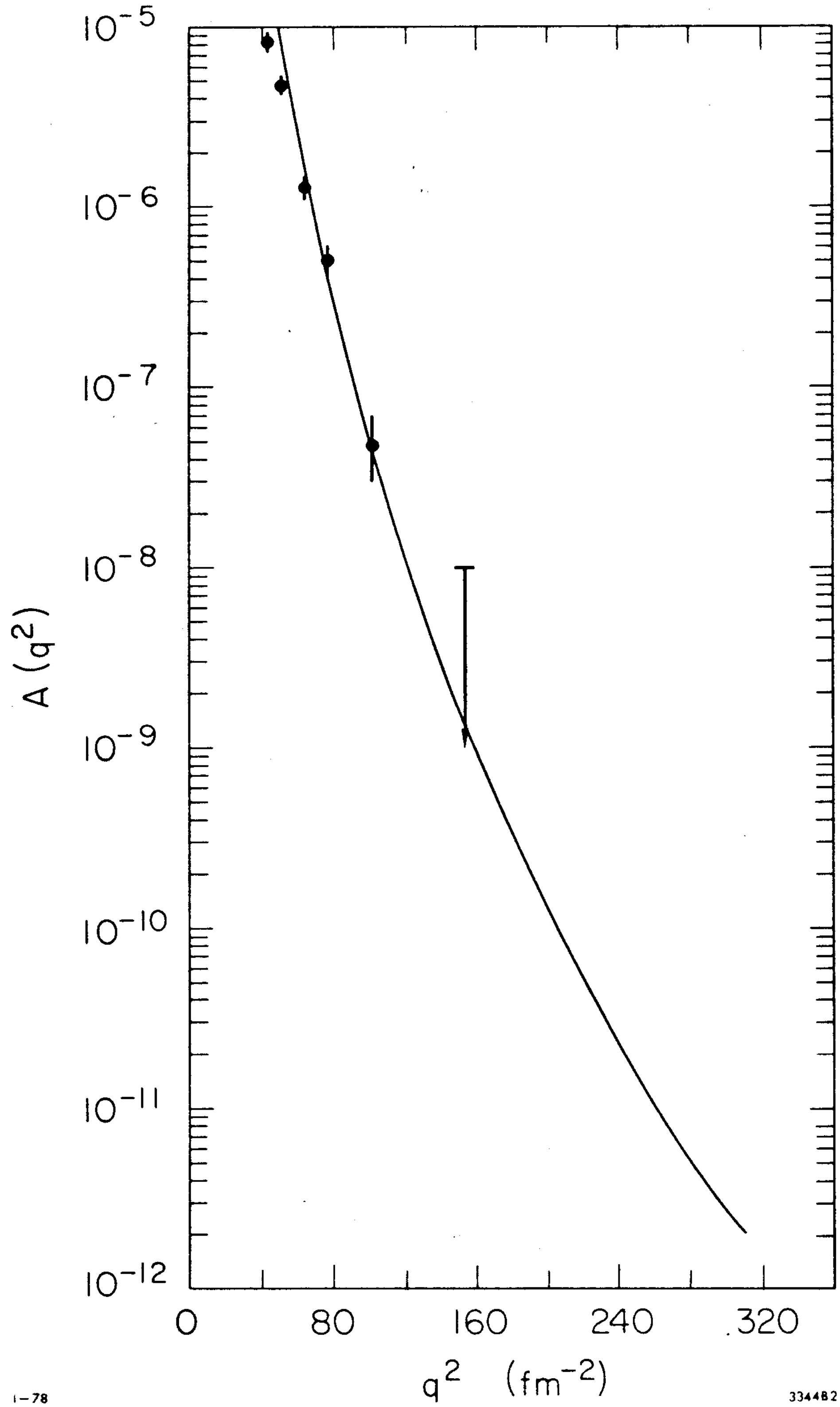


Fig. 1



10^-5
10^-6
10^-7
10^-8
10^-9
10^-10
10^-11
10^-12
A (q^2)
0
80
160
240
320
q^2 (fm^-2)
1-78
334482

Fig. 2

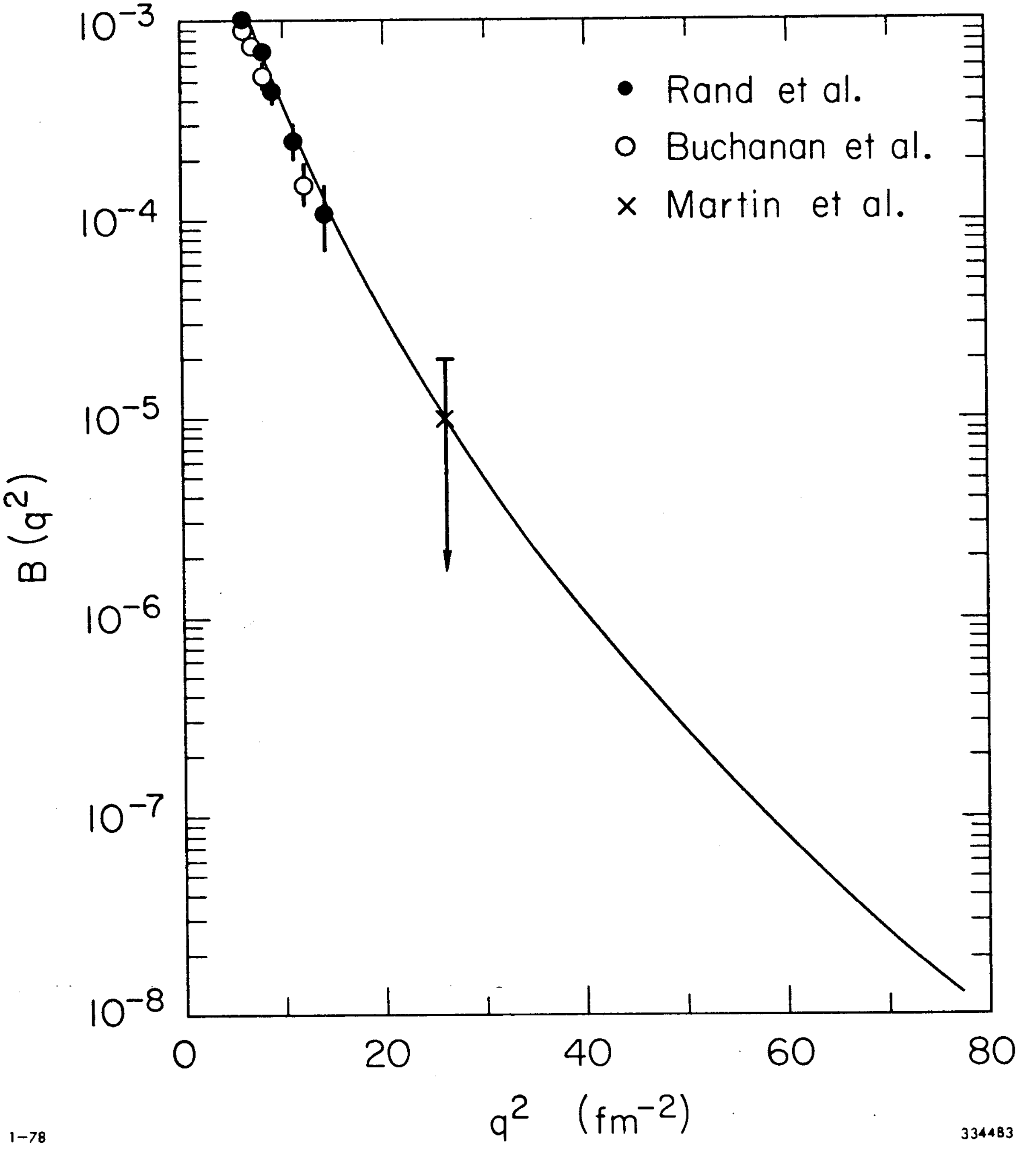


Fig. 3

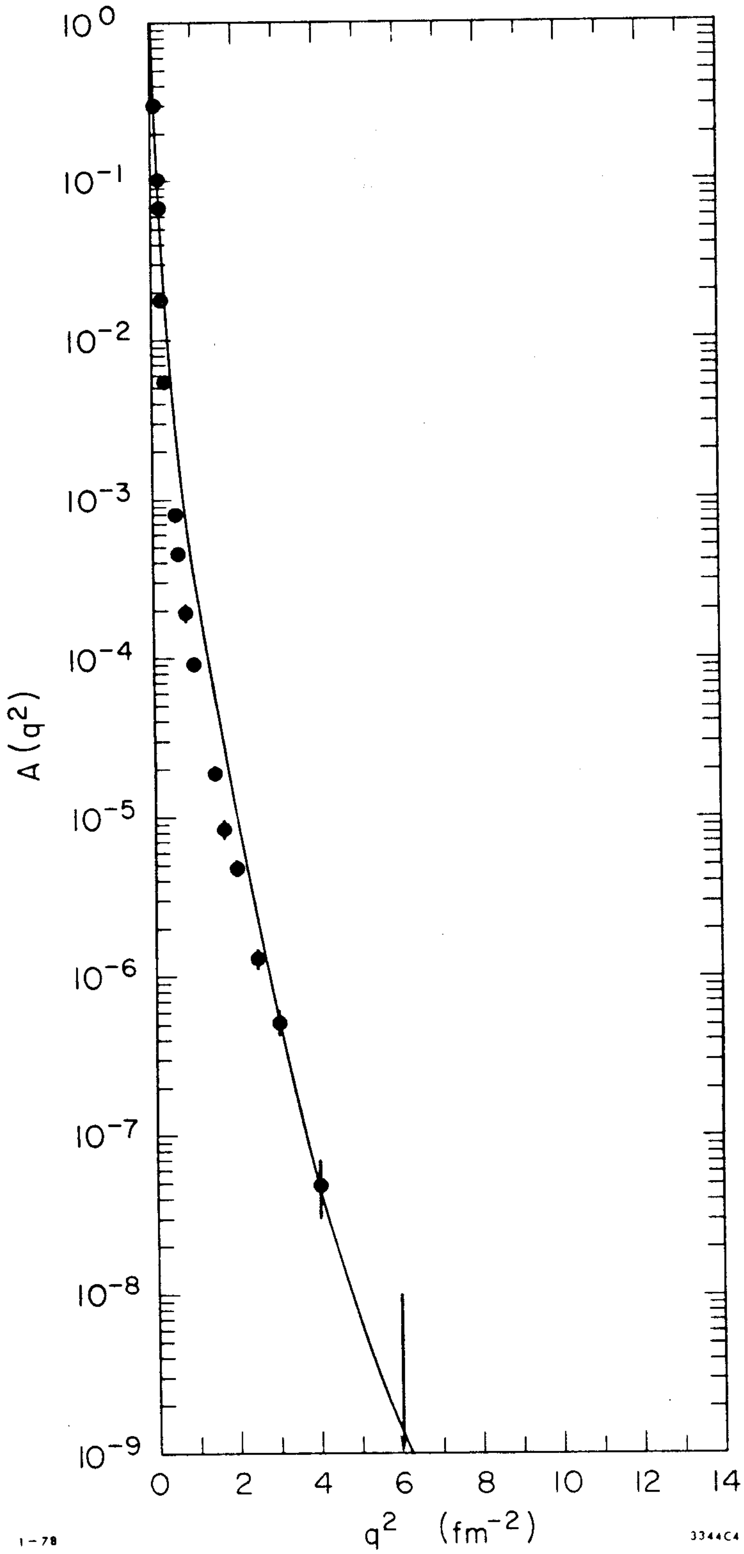

10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
$A(q^2)$
0
2
4
6
8
10
12
14
q^2 (fm^{-2})
1-78
3344C4

Fig. 4