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ASYMPTOTICALLY FLAT SELF-DUAL SOLUTIONS TO EUCLIDEAN GRAVITY *

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ABSTRACT

In an attempt to find gravitational analogs of Yang-Mills pseudoparticles, we obtain two classes of self-dual solutions to the Euclidean Einstein equations. These metrics are free from singularities and approach a flat metric at infinity.

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The discovery of pseudoparticle solutions to the Euclidean SU(2) Yang-Mills theory[1] has suggested the possibility that analogous solutions might occur in Einstein's theory of gravitation. The existence of such solutions would have a profound effect on the quantum theory of gravitation.[2,3] Since the Yang-Mills pseudoparticles possess self-dual field strengths, one likely possibility is that gravitational pseudoparticles are characterized by self-dual curvature.

In fact it has been pointed out by Hawking[3] that the Taub-NUT metric[4], when appropriately continued to Euclidean space-time, produces a self-dual curvature and hence is a possible candidate for a gravitational pseudoparticle. He has also given a generalized multi-Taub-NUT metric. However, these metrics do not approach a flat metric at infinity.[5] To see this, let us write the Euclidean Taub-NUT solution as

$$(ds)^2 = \frac{R + m}{R - m} dR^2 + 4 (R^2 - m^2) \{\sigma_x{}^2 + \sigma_y{}^2 + (\frac{2m}{R + m})^2 \sigma_z{}^2\} \quad (1)$$

where σ_x, σ_y, σ_z form a standard Cartan basis,

$$\begin{aligned} \sigma_x &= \frac{1}{2} (- \cos \psi \, d\theta - \sin \theta \sin \psi \, d\phi) \\ \sigma_y &= \frac{1}{2} (\sin \psi \, d\theta - \sin \theta \cos \psi \, d\phi) \qquad (2) \\ \sigma_z &= \frac{1}{2} (-d\psi - \cos \theta \, d\phi) \end{aligned}$$

obeying the structure equations of the exterior algebra,[6]

$$d\sigma_x = 2\sigma_y \wedge \sigma_z, \text{ etc.} \tag{3}$$

Here θ, ψ and ϕ are Euler angles on S^3 with ranges $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, $0 \leq \psi \leq 4\pi$. Then it is easy to see that the above metric describes a distorted 3-dimensional hypersphere S^3 for any fixed value of $R > m$.

Since a Yang-Mills pseudoparticle approaches a pure gauge at infinity and is interpreted as inducing transitions between topologically inequivalent vacua, one might require that gravitational analogs have a similar asymptotic behavior. In this letter we explore the possibility of gravitational pseudoparticles which possess a self-dual curvature and approach a flat metric at infinity. In the following we present two classes of such solutions. They are both singularity-free in the entire spacetime and their manifolds have a simple topological structure.

In deriving these solutions we exploit a particularly useful choice of gauge (local Lorentz frame). First we define a local orthonormal frame using the vierbeins $e^a{}_\mu$, and take

$$e^a = e^a{}_\mu dx^\mu . \tag{4}$$

In terms of the e^a, the metric is expressed as $ds^2 = (e^0)^2 + (e^1)^2 + (e^2)^2 + (e^3)^2$. Then the connection one-form $\omega^a{}_b$ is defined by

$$de^a = - \omega^a{}_b \wedge e^b, \; \omega^a{}_b = - \omega^b{}_a . \tag{5}$$

Latin indices are raised and lowered by a flat metric. Then we define the curvature two-form by

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \ . \tag{6}$$

Now we note that if $\omega^a{}_b$ is self-dual,

$$\omega^0{}_1 = -\,\omega^2{}_3\ ,\ \text{etc.}, \tag{7}$$

then $R^a{}_b$ is self-dual. This follows directly from the definition (6) of $R^a{}_b$. Since any self-dual curvature gives a vanishing Ricci tensor, any metric yielding a self-dual connection is a solution to the Einstein equation. On the other hand, it is easy to show that any self-dual curvature can be obtained, by a suitable change of gauge, from a metric yielding a self-dual connection.* In this "self-dual gauge", the problem of finding a self-dual solution to the Einstein equation[7] is therefore reduced to one of finding self-dual connections and hence solving first-order differential equations generated by Eq. (5). This is quite analogous to the Yang-Mills case.[1]

In the following we consider two types of metrics having axial symmetry as in the Taub-NUT case:**

$$\text{I:}\quad (ds)^2 = f^2(r)dr^2 + r^2g^2(r)(\sigma_x{}^2 + \sigma_y{}^2) + r^2\sigma_z{}^2 \tag{8}$$

$$\text{II:}\quad (ds)^2 = f^2(r)dr^2 + r^2(\sigma_x{}^2 + \sigma_y{}^2) + r^2g^2(r)\sigma_z{}^2\ . \tag{9}$$

Here we consider these metrics directly in the Euclidean space and do not regard them as a result of some continuation from the Minkowski regime. Asymptotic flatness requires that

$$\lim_{r\to\infty} f(r) = \lim_{r\to\infty} g(r) = 1\ . \tag{10}$$

Taking as our orthonormal frames

$$\mathrm{I:}\quad e^a = (f(r)dr,\ rg(r)\sigma_x,\ rg(r)\sigma_y,\ r\sigma_z) \tag{11}$$

$$\mathrm{II:}\quad e^a = (f(r)dr,\ r\sigma_x,\ r\sigma_y,\ rg(r)\sigma_z)\ , \tag{12}$$

we find after some simple algebra that the self-duality of the connection implies

$$\mathrm{I:}\quad g^2 = f(2g^2 - 1),\ f = g(g + rg') \tag{13}$$

$$\mathrm{II:}\quad fg = 1,\ f(2 - g^2) = g + rg'\ . \tag{14}$$

Asymptotically flat solutions are given, respectively, by

$$\mathrm{I:}\quad f(r) = \frac{1}{2}\left(1 + [1 - (a/r)^4]^{-\frac{1}{2}}\right) \tag{15}$$

$$g(r) = \left\{\frac{1}{2}\ (1 + [1 - (a/r)^4]^{\frac{1}{2}})\right\}^{\frac{1}{2}} \tag{16}$$

$$\mathrm{II:}\quad g(r) = f^{-1}(r) = [1 - (a/r)^4]^{\frac{1}{2}}\ , \tag{17}$$

where $\underline{a}$ is an integration constant. The curvature components of case II are given by

$$\begin{aligned}
R^0{}_1 &= - R^2{}_3 = - \frac{2a^4}{r^6}\ (e^0 \wedge e^1 - e^2 \wedge e^3) \\
R^0{}_2 &= - R^3{}_1 = - \frac{2a^4}{r^6}\ (e^0 \wedge e^2 - e^3 \wedge e^1) \\
R^0{}_3 &= - R^1{}_2 = + \frac{4a^4}{r^6}\ (e^0 \wedge e^3 - e^1 \wedge e^2)\ .
\end{aligned} \tag{18}$$

The curvatures for case I have the same algebraic form with the replacement

$$\frac{2a^4}{r^6} \to - \frac{a^4}{2r^6g^6} \quad . \tag{19}$$

Hence in both cases the curvatures are regular everywhere for $r \geq a$ and fall off like $1/r^6$ at infinity. For comparison, we note that the Taub-NUT curvature produced by Eq. (1) is obtained by the replacement

$$\frac{2a^4}{r^6} \to \frac{m}{(R + m)^3} \tag{20}$$

and thus goes like $1/R^3$ at infinity.

The manifolds described by the above metrics have the topology $R \times S^3$. Although the metrics have an apparent singularity at $r = a$, it can be eliminated by a change of variable,

$$u^2 = r^2(1 - (a/r)^4) \ . \tag{21}$$

For instance, the solution II now takes the form

$$(ds)^2 = du^2/(1 + (a/r)^4)^2 + u^2\sigma_z{}^2 + r^2(\sigma_x{}^2 + \sigma_y{}^2) \ . \tag{22}$$

Our next task is to compute topological invariants of the manifold. Here, as in the Taub-NUT case[8], we have to be careful about possible contributions from the boundary of the manifold.

Â-genus (axial anomaly).

The Atiyah-Patodi-Singer theorem[9] gives the Â-genus of the manifold $[r_1,r_2] \times S^3$ as

$$\hat{A}(r_1,r_2) = \hat{A}_{vol} - \left(\hat{A}_{surf} + \frac{1}{2}(h_D + \eta_D)\right)\Big|_{r_1}^{r_2} . \tag{23}$$

$\hat{A}_{vol}$ is the volume integral of the Riemann curvature tensor contracted with its dual and $\hat{A}_{surf}$ gives the contribution due to the deviation of the metric from a product metric on the boundary.[10] h_D is the number of harmonic spinors of the Dirac operator restricted to the boundary and η_D gives its spectral asymmetry.[9,11] Using the formulas in references 8 and 11 we obtain

$$\hat{A}(r_1 = a,\ r_2 = \infty) = \frac{1}{4} - 0 + \left(- \frac{1}{6} - \frac{1}{12}\right) = 0 \tag{24}$$

for both solutions I and II. Thus these solutions by themselves will not induce chiral symmetry breakdown, just as in the Taub-NUT case.[8]

Euler-Poincaré characteristic (trace anomaly).

The Euler-Poincaré characteristic χ is related to the thermal effects of gravitational pseudoparticles.[3,12] To calculate χ, we apply the Chern-Gauss-Bonnet theorem,[13]

$$\chi = \chi_{vol} - \chi_{surf}\Big|_{r_1}^{r_2} \tag{25}$$

where χ_{vol} and χ_{surf} are the analogs of $\hat{A}_{vol}$ and $\hat{A}_{surf}$ in in Eq. (23). Using the known formulas, we find for both solutions I and II the Euler characteristic***

$$\chi(r_1 = a, r_2 = \infty) = 3 - (-1) + (-4) = 0 . \tag{26}$$

This of course agrees with the combinatorial calculation for $R \times S^3$.

We observe that at large r, our curvatures fall like $1/r^6$; in contrast, the Euclidean Taub-NUT and Schwarzschild solutions fall like $1/r^3$. This suggests that our metrics describe gravitational "dipoles" while Taub-NUT and Schwarzschild describe monopoles. This is probably a sign that our Euclidean solutions will not have a meaningful continuation to Minkowski space, as is the case for the Yang-Mills pseudoparticle.

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Footnotes

* The proof involves decomposing any given spin connection $\omega^a{}_b$ into self-dual and anti-self-dual parts. If $R^a{}_b$ is self-dual, the anti-self-dual part of $\omega^a{}_b$ is a pure $O(4)$ gauge transformation, $\Lambda^a{}_c(d\Lambda^{-1})^c{}_b$, and can be gauged away.

**The spherically symmetric ansatz, $ds^2 = f^2dr^2 + r^2g^2(\sigma_x{}^2 + \sigma_y{}^2 + \sigma_z{}^2)$, leads to a trivially flat metric when we impose self-duality.

***It appears that the manifold of solution II can be compactified by adding an S^2 at $r = a$. In this case (see Eq. (22)) the manifold acquires the local topology of $D^2 \times S^2$; since as $r \to a$, the D^2 shrinks to a point, the manifold is homotopic to S^2. If we then omit the $r = a$ boundary term in Eq. (26), we obtain $\chi = 4$. However, we know $\chi = 2$ for a manifold homotopic to S^2. Hence the Chern-Gauss-Bonnet theorem requires a "corner" correction in this case. A similar situation occurs if one puts a metric on a cone and tries to compute the Euler characteristic using the Gauss-Bonnet theorem without correcting for the apex. For solution I, analogous arguments indicate that the manifold compactified at $r = a$ is homotopic to the manifold of $SO(3)$. Then the apparent Euler characteristic is 4, while the true value is $\chi = 0$. The compactified manifolds admit a spin structure because the second Stiefel-Whitney classes vanish[14]. However, in practice the "corners" may make it difficult to treat the Dirac operator on the whole manifold. If such an operator can be defined, the $\hat{A}$ genus (axial anomaly) would also require "corner" corrections. This problem is under study.

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