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LATTICE FIELD THEORIES: NON-PERTURBATIVE METHODS OF ANALYSIS*

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Introduction

In the past few days we have heard several beautiful lectures describing the way in which people hope to extract interesting physical information from quantum field theories by studying their semi-classical versions. Being in the mountains it seems appropriate to describe these attempts as an attack on the semi-classical face of quantum field theory. Since all mountains have more than one face, I would like to describe in my next few lectures attempts which have been made to launch a direct attack on the quantum face (Fig. 1); hence these lectures are in a sense complementary to the preceding ones.

To be precise I will first show how one can, from the very beginning, consider the problem of solving for the spectrum of states of any given continuum quantum field theory as a giant Schroedinger problem and then explain some non-perturbative methods for diagonalizing the Hamiltonian of the theory without recourse to semi-classical approximations. Along the way the notion of a lattice will appear as an artifice to handle the problems associated with the familiar infrared and ultra-violet divergencies of continuum q.f.t. and in fact, for all but gauge theories, I will show you how to go back and forth between specific lattice theories and continuum quantum field theories formulated with spatial and momentum cutoffs. This is an important thing to be able to do in principle, since it is by no means a priori clear that the situation is as shown in Fig. 1. It may be that the situations shown in Fig. 1 where two groups are attacking different faces of the same mountain is a trick of perspective and a more Olympian view of the situation would reveal that, contrary to our prejudices, the situation is more like Fig. 2, where we see we are in fact scaling different mountains. As this dreadful possibility could be the case in reality we must from the outset define the rules of the game and list our eventual goals so that you will understand where we are going and how we hope to get there.

First, let us address the question of goals. Here is where we get to list all the good stuff everyone has in his shopping list. We would like to understand on the basis of Lagrangian field theory --

(1) Why -- as we have seen in the lecture of F. Gilman and G. Feldman--the naive quark model gives such a remarkably nice qualitative picture of hadron phenomenology. (Especially things which can be reduced to counting on our fingers kinds of questions.)

(2) If the successes of the naive quark model point to the existence of bound quarks as elementary constituents of matter, where are they? (i.e. why haven't they been seen in final states to date?)

(3) If color gauge theories are in fact the right place to look for a theory of hadrons--is there any truth to the folklore that asymptotic freedom and confinement are two sides of the same coin.

Clearly, to be able to answer these questions in a really convincing way within the framework of conventional field theory it is absolutely necessary to develop techniques which are powerful enough to allow us to

(1) Find the hadrons as bound states of the fundamental degrees of freedom.

(2) Calculate the ratio of the energy of a widely separated quark-antiquark pair to a typical bound state (e.g. the proton) in order to see if confinement does (or heresy--does not) occur in color gauge theories.

Even without specifying how we hope to develop such techniques, we can see from the fact that we are asking for the answers to questions which clearly go beyond the scope of perturbation theory that there will be certain steps which must be taken as we proceed along the way to our eventual goals. First, since we will not be able to rely on the conventional tools of Feynman graph perturbation theory we will have to find a way of formulating field theory so as to be able to discuss the problem without having to confront either infinite volume or short

distance divergences. That is not to say that we will attempt to remove all divergencies from the continuum theory at the outset, but rather--the idea is to first impose sufficient cutoffs to render the theory finite. Then, in principle solve it exactly and then let all cutoffs go to infinity at the same time taking the bare parameters to the appropriate values so as to achieve a non-trivial relativistic theory with a finite spectrum of states. Obviously, if the lessons of perturbation theory hold any water for the full theory, then the fact that the multiplicative renormalization scheme can be carried out tells us that the scheme we have described must be feasible--in principle.

Hence, I will first describe how to impose sufficient cutoffs on a given continuum theory to render all computations finite. Next I will show how to recast the resulting cutoff continuum theory in terms of a unitarily equivalent lattice theory which will enable us to better understand the quantum mechanical nature of the problem facing us. Finally, I will formulate a non-perturbative technique we propose to use to diagonalize any given lattice Hamiltonian. Clearly, from this point of view the problem of analyzing quantum field theories breaks into two distinct parts -- the first being the development of a formalism which allows us to recast any cutoff continuum field theory in the form of an equivalent lattice theory; the second being the development of techniques for solving any given lattice theory independently of how it was obtained. Except for today's lecture--whose purpose is to exhibit techniques for going back and forth between lattice theories and corresponding continuum field theories--I will focus attention on the second problem. Moreover, since the problems are not truly connected, I will study our proposed non-perturbative techniques as applied to model lattice field theories for which some exact results are known. The reason for focusing on these special models--as you will see--has nothing to do with the fact that our

method is especially suited to the analysis of these 1-space-1-time dimensional models; but rather, we focus attention on these models in order to show that our methods are not producing incorrect results.

Before jumping into our technical discussion let me spend a few moments talking about the general way our approach fits into the framework of the other non-perturbative attacks on the problems of quantum field theory currently under way. Since I have been on the road so long in coming here I have to be forgiven for choosing to summarize the picture by the following road map (Fig. 3). This map--as all good maps--is essentially self-explanatory and so I will limit myself to a few brief remarks. In the upper lefthand corner we see the figure representing what I have labeled the lattice path-integral formalism. This, of course, stands for the program pioneered by Wilson[1] and collaborators and I haven't much to say about it. I would note that the initial notion that it would provide a super-highway which led to the mysterious black box containing the secrets of quark-confinement and the explanation of the quark model ran into a brick wall. Since then this program has followed a more torturous path occasionally bogging down in muck and mire and at present it is obscured in a cloud of computer computation. On the upper righthand side of the map you see the figure standing for the analysis of Euclidean path integral in terms of stationary points or semi-classical states. We have certainly heard much about this scheme in the lovely lectures by G.'t Hooft and R. Jackiw[2]--but I think it is fair to say that while the concepts one encounters are fascinating and have provided some insight into the U(1)-problem the hope that they would provide a super-highway to an understanding of quark confinement, etc., has also run into a brick wall. At present the interest in instantons, merons[3] (the darling of the Princeton group working on this problem), and other exotic beasties is based upon the hope that they will lead to an understanding of the physics of color-gauge theories; but I have not yet seen any

compelling reason to think this has yet happened. I have chosen--with no malice intended--to signify the current state of affairs by saying that the current program is traversing the instanton roundabout but it is not yet out of the woods.

At the bottom of the map you see the figures representing attacks upon the problem based upon Hamiltonian techniques. All such methods have in one way or another made use of lattice techniques. The lefthand super-highway labeled Padé from strong coupling stands for the study of lattice theories by the methods introduced by Kogut and Susskind[4]--who by the way deserve tremendous credit for initiating the program of converting the Wilson program to a Hamiltonian formalism. This program has received much attention in recent years, but the question of whether or not the method of continuing a strong coupling expansion of a lattice theory to weak coupling by Padé approximants will prove adequate to study the questions of interest is now shrouded in the fog of massive computer calculations. One can only await the results of these studies to judge their applicability to our world. Finally, I come to the much less well traveled path to which I will devote the next three lectures. This path, labeled variational renormalization group approach, signifies the program initiated at SLAC.[5] Our approach has been to proceed much less rapidly and study a series of simpler theories in order to achieve insight into the way our methods work, and--more importantly--know well they work. As with all other approaches we feel ours to be very promising and exciting but honesty forces me to say that we too are still lost in a haze of computation. If for some reason you notice that this path seems closer than others to the rainbow marking the "pot of gold" or in this case "black box" let me hasten to add this is probably a trick of personal perspective and as with all other theories it is the roads which are still under construction which will provide the true test of all the ideas put forward to date. As to what is in the black box, if folklore is right, presumably it is

the secret of confined quarks. However, one should not forget there is always the possibility that Fairbanks could be right and (Fig. 4) the box really holds a free, hungry, colored quark.

This completes the general remarks I will make and I would now like to present the plan of the discussion to follow, and then dive into the discussion of point 1.

Plan of Lectures

1. Introduction of Basic Concepts

   (a) Bosons
   (b) Fermions
   } Lattice versions

2. Introduce General Non-Perturbative Method of Calculation

3. Method for Solving Any Lattice Theory for Ground State and Spectrum of Low Lying States

   (a) Ising model

   (b) Thirring model

   (c) U(1)-Goldstone model

4. Discussion of a Simple Gauge Theory

   (a) Formulation of lattice gauge-field theory

   (b) Higgs Model in 1 + 1 dimensions

Basically the plan of these lectures is as it is for obvious reasons. I wish to say a few words about why I have chosen to spend only the remainder of this first lecture introducing the notion of a lattice theory which is unitarily equivalent to a given cutoff continuum field theory, and then devoting the bulk of our time to specific lattice models. I feel this needs discussion because while I choose to focus upon the non-perturbative variational scheme we propose for studying any lattice theory, I do not wish to leave the impression that I

consider the full development of the methods for relating continuum and lattice theories, as well as variations upon these methods, as unimportant. Far from it. In fact, it is my belief that when one really wishes to turn the tools I will describe upon the problem of analyzing the behavior of a specific continuum theory, or when one wishes to know the relationship between Feynman graph perturbation theory and calculations based upon the methods to be described, or when one wishes to ask which continuum theory a given lattice theory corresponds to in the limit in which one removes all cutoffs, the fullest exploitation of the tricks I will describe only briefly in the context of free field theory will be as important as the variational techniques I will talk about for diagonalizing any given lattice Hamiltonian. I am choosing to give these questions short shrift only because of the time constraints imposed upon me by the format of this lecture series and because I wish to limit discussion to those aspects of the general problem which have been most fully explored. I hope, however, that you will bear in mind that we have only begun to scratch the surface of what can be done by means of these techniques and will be encouraged to try your own hand at pushing them much further than we have done to date.

One further remark is in order, and that is that I probably will run out of time before I get to gauge theories and so probably I will only be able to make a few general remarks about the state of the art as of now--and refer you to a forthcoming series of papers on the subject.[6]

By way of giving credit where credit is due, I wish to state that the work to follow has been done in collaboration with S. Drell, S. Yankielowicz, Ben Svetitsky and H. Quinn.

LECTURE 1. FREE SCALAR FIELD

Let us begin our discussion of the way in which one can introduce cutoffs into a free scalar field theory and then transform it to an equivalent lattice theory. Our starting point is the usual Lagrangian

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \phi(x)\right)^2 - \frac{\mu^2}{2}\,\phi(x)^2 \tag{1.1}$$

and for the sake of notational simplicity alone let us specialize to $\mu = 0,1$ (i.e. a theory is 1 space + 1 time dimension). From this one forms the Hamiltonian by defining

$$\pi(x) \equiv \frac{\delta \mathscr{L}}{\delta(\partial_0 \phi(x))} = \partial_0 \phi(x) \tag{1.2}$$

and, assuming the theory is defined in a volume L, we obtain

$$H = \int_{\frac{-L}{2}}^{\frac{+L}{2}} dx \left(\frac{\pi(x)^2}{2} + \frac{\left(\partial_1 \phi(x)\right)^2}{2} + \frac{\mu^2}{2}\,\phi(x)^2\right) \tag{1.3}$$

At this point we define the quantum version of the classical theory specified in (1.1)-(1.3) by defining the equal time commutator of $\pi(x)$ and $\phi(y)$. It is here that we will choose to introduce a fundamental length in the theory and so cut off all short distance divergences. We do this by defining the modified commutator

$$[\pi(x),\phi(y)] = -i\,\delta_\Lambda(x - y) \tag{1.4}$$

where

$$\delta_\Lambda(x - y) \equiv \Lambda\,\frac{\sin\left(\pi\Lambda(x - y)\right)}{(2N+1)\sin\left(\frac{\pi\Lambda(x-y)}{2N+1}\right)} \tag{1.5}$$

and where Λ is a small parameter (e.g. 10^{10} GeV) and L is defined to be

$$L = (2N + 1)/\Lambda \tag{1.6}$$

Clearly, relation (1.4) implies that the fields $\pi(x)$ and $\phi(x)$ are overcomplete, and that there must exist a subset of the operators satisfying canonical commutation relations such that $\pi(x)$ and $\phi(x)$ can be written as functions of the smaller set of operators. There are many ways to see that this must be true; one way is to Fourier transform (1.4) and study the fields π_k and ϕ_k'. However, the simplest way to discover the relevant independent set of variables is to observe that

$$\delta_\Lambda(x - y) = \Lambda\delta_{j_1,j_2} \tag{1.7}$$

when $x = j_1/\Lambda$ and $y = j_2/\Lambda$ for arbitrary integers j_1, j_2.

If we define

$$\phi(j) \equiv \phi(x = j/\Lambda) \tag{1.8}$$

it is then easy to show that

$$\phi(x) = \sum_{p=-N}^{+N} \frac{e^{ik_p x}}{\sqrt{2N+1}} \phi_{k_p} \tag{1.9}$$

where

$$k_p \equiv \left(\frac{2\pi\Lambda}{2N+1}\right) p \tag{1.10}$$

and where

$$\phi_{k_p} \equiv \sum_j \frac{e^{-ik_p j/\Lambda}}{\sqrt{2N+1}} \phi(j) \tag{1.11}$$

Using these formulae the Hamiltonian (1.3) can be rewritten in terms of the independent degrees of freedom as

$$H = \Lambda\left[\sum_j \left\{\frac{\tilde{\pi}(j)^2}{2} + \frac{\tilde{\mu}^2}{2}\,\phi(j)^2\right\} + \sum_{j_1,j_2}\left\{\frac{1}{2}\,D(j_1 - j_2)\,\phi(j_1)\,\phi(j_2)\right\}\right] \tag{1.12}$$

where we have defined dimensionless fields, mass parameters, etc., by

$$\tilde{\pi}(j) = \pi(j)/\Lambda\ , \tag{1.13}$$

$$\tilde{\mu}^2 = \mu^2/\Lambda^2\ , \tag{1.14}$$

and

$$D(j) \equiv \frac{1}{\Lambda^2}\sum_{p=-N}^{+N} \frac{(k_p)^2}{(2N+1)}\, e^{ik_p j/\Lambda} \tag{1.15}$$

N.B. the function D(j) has a particularly simple form in the limit $N \to \infty$, namely:

$$D(j) = \begin{cases} \pi^2/3 & \text{if } j = 0 \\ (-1)^j/j^2 & \text{if } j \neq 0 \end{cases} \tag{1.16}$$

Obviously, since we are dealing with a quadratic Hamiltonian (1.13) and since $D(j_1-j_2)$ is a function of only the difference of j_1 and j_2 this Hamiltonian can be diagonalized by going to k-space. If we do this we find

$$H \equiv \Lambda \sum_{p=-N}^{N}\left[\frac{\tilde{\pi}_{-\tilde{k}_p}\tilde{\pi}_{\tilde{k}_p}}{2} + \frac{(\tilde{k}_p^2 + \tilde{\mu}^2)}{2}\,\phi_{-\tilde{k}_p}\,\phi_{\tilde{k}_p}\right] \tag{1.17}$$

where

$$\tilde{k}_p \equiv \left(\frac{2\pi}{2N+1}\right) p \tag{1.18}$$

If we now introduce creation and annihilation operators in the usual way, we see that

$$H = \Lambda \sum_{p=-N}^{N} \left(a^{+}_{\tilde{k}_p} a_{\tilde{k}_p} + 1/2 \right) \sqrt{\tilde{k}_p^{\,2} + \tilde{\mu}^2} \tag{1.19}$$

This form of H is quite instructive since we see that by introducing a fundamental length as in (1.4)-(1.5) we have done nothing more or less than defining a maximum momentum cutoff on the free field theory. Our spectrum as a function of k is totally relativistic except that it terminates at $k_{max} = 2\pi N/(2N + 1)$. It is worth comparing this result to the more common way of latticizing the free scalar field where one defines a lattice Hamiltonian

$$H_{nn} = \Lambda \sum_{j} \left[\tilde{\pi}_j^{\,2} + \frac{\tilde{\mu}^2}{2} \phi_j^2 + \frac{1}{2} (\phi_{j+1} - \phi_j)^2 \right] \tag{1.20}$$

where $\phi_{N+1} \equiv \phi_{-N}$.

The k-space form of this Hamiltonian gives

$$H_{nn} = \Lambda \sum_{p=-N}^{N} \left[\frac{\tilde{\pi}_{-\tilde{k}_p} \tilde{\pi}_{\tilde{k}_p}}{2} + \frac{1}{2} \left(\tilde{\mu}^2 + 4 \sin^2(\tilde{k}_p/2) \right) \phi_{-\tilde{k}_p} \phi_{\tilde{k}_p} \right]$$

$$= \Lambda \sum_{p=-N}^{N} (a^{+}_{\tilde{k}_p} a_{\tilde{k}_p} + 1/2) \sqrt{\tilde{\mu}^2 + 4 \sin^2(\tilde{k}_p/2)}$$

which approximates a relativistic spectrum only for $\tilde{k} << 1$. A comparison of the two spectra is shown in Fig. 5 and one sees that for momenta much smaller than the lattice mass there is no important difference between the two approaches insofar as the low momentum spectrum is concerned. There is however a huge difference in point of view since our Hamiltonian is related in a definite way to a given continuum theory and so we know how to go back and forth between the two languages at will. The first important difference between the two approaches at the level of the low energy spectrum will occur when we next study the free fermion theory.

Free Fermion Field Theory

We begin our discussion of the free fermion theory with the continuum Hamiltonian

$$H = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \left\{\psi^+(x)\left(\frac{\alpha\partial_1}{i} + \beta M\right)\psi(x)\right\} \tag{1.22}$$

and once again modify the equal time commutation relations to read

$$\left\{\psi^+(x),\psi(y)\right\} \equiv \delta_\Lambda(x - y) \tag{1.23}$$

As before, we note that this implies that the fields $\psi^+(x)$ and $\psi(y)$ are over-complete and we go to the independent fields

$$\tilde{\psi}(j) = \left(\frac{1}{\Lambda}\right)^{1/2} \psi(x = j/\Lambda) \quad , \tag{1.24}$$

in terms of which one can write

$$\psi(x) = \Lambda^{1/2} \sum_{p=-N}^{N} e^{i\tilde{k}_p \Lambda x}\, \tilde{\psi}_{\tilde{k}_p} \tag{1.25}$$

where

$$\tilde{\psi}_{\tilde{k}_p} = \sum_{j=-N}^{N} e^{-i\tilde{k}_p j}\, \tilde{\psi}(j) \quad . \tag{1.26}$$

Using (1.24)-(1.26) we rewrite H as

$$H = \Lambda\left[\sum_{j_1,j_2}\left(\tilde{\psi}^+(j_1)\,\frac{\alpha}{i}\,\delta'(j_1 - j_2)\tilde{\psi}(j_2) + \sum_j\left(\tilde{\psi}^+(j)\beta\tilde{\psi}(j)\right)\tilde{M}\right]\right. \tag{1.27}$$

where $\tilde{M} = M/\Lambda$ and

$$\delta'_\Lambda(j_1 - j_2) = \frac{\Lambda}{\sqrt{2N+1}} \sum_p i\tilde{k}_p\, e^{i\tilde{k}_p(j_1 - j_2)} \tag{1.28}$$

which, in the limit $N \to \infty$, becomes

$$\delta'_{\Lambda}(j) = \begin{cases} 0 & \text{if } j = 0 \\ (-)^j/j & \text{if } j \neq 0 \end{cases} \tag{1.29}$$

If we rewrite this in k-space we find

$$H = \Lambda \sum_{p=-N}^{N} \tilde{\psi}^{+}_{\tilde{k}_p} \left(\alpha \tilde{k}_p + \beta \tilde{M}\right) \tilde{\psi}_{\tilde{k}_p} \tag{1.30}$$

which can be diagonalized to yield a theory of free fermions with an energy momentum dispersion formula given by

$$E^2(\tilde{k}_p) = \Lambda^2(\tilde{k}_p^2 + \tilde{M}^2) \tag{1.31}$$

which is clearly relativistic except that it cuts off at $k_{max} = 2\pi\Lambda/(2N + 1)$. If we compare this to the definition of $\nabla\tilde{\psi}(j)$ given by the usual nearest neighbor formula

$$\nabla\tilde{\psi}(j) \equiv \tilde{\psi}(j + 1) - \tilde{\psi}(j) \tag{1.32}$$

with

$$H = \Lambda \sum_{j} \tilde{\psi}^{+}(j) \left(\frac{\alpha\nabla}{i} + \beta\tilde{M}\right)\tilde{\psi}(j) \tag{1.33}$$

we find that the k-space form of (1.33) is

$$H = \Lambda \sum_{p=-N}^{N} \tilde{\psi}^{+}_{\tilde{R}_p} \left(\alpha \sin(\tilde{R}_p) + \beta\tilde{M}\right) \psi_{\tilde{k}_p} \tag{1.34}$$

which, when diagonalized, yields an energy momentum dispersion formula

$$E^2_{nn}(\tilde{k}_p) \equiv \Lambda\left(\sin^2(\tilde{k}_p) + \tilde{M}^2\right) \tag{1.35}$$

Since $|\tilde{k}_p| \leq \pi$ we see, as shown in Fig. 6, that this formulation of lattice fermions introduces a serious problem in that it leads to a doubling at the number of fermionic states having any given energy. Since this is disastrous for the free field case this is not what we wish to have happen at all; moreover, in

higher dimensions one gets $(2)^d$ times as many states at a given energy as the continuum theory would predict. Several methods other than the one we have described have been introduced by Wilson and Kogut and Susskind in order to avoid this problem; however, they all suffer from the undesirable feature that they destroy continuous chiral symmetry of massless fermion theory so long as the cutoff (of lattice spacing $a = 1/\Lambda$) is held finite. The method we have proposed has the virtue of being simply related to the continuum theory, is as relativistic as possible and yields a chirally symmetric free massless fermion theory.

Summary of Lecture 1

Let us close this lecture with a summary of the points we have covered. First, I have shown you one way in which one can transcribe a cutoff quantum field theory as a specific lattice theory. This had the advantage that one knows precisely, for any given transcription procedure, how to smear the fields $\phi(j)$ or $\psi(j)$ so as to obtain continuum fields satisfying δ-function commutation relations in the limit $\Lambda \to \infty$. Clearly, the same goes for all operators made out of polynomials in these fields and so one could, by exploiting this information, study the $\Lambda \to \infty$ limit of operator product expansions, equations of motion, etc. Secondly, we have shown that this transcription procedure handles fermions and bosons in the same way, breaks no symmetries (e.g. chiral invariance) and maintains approximate Lorentz invariance of the free field theory. Moreover, although we have not discussed this point, one could construct the operators $\Theta_{\mu\nu_\Lambda}(x)$ and Lorentz generators and study in an operator way how good Lorentz invariance is for low-lying states. Finally, I wish to remind you that although we only carried out the transcription procedure for the free field case, there is no difficulty in carrying out the same procedure for interacting theories.

With this discussion behind us we will now forget about the question of where we got any particular lattice theory and concentrate on the way in which one deals with any given theory without asking whether it was obtained from a continuum theory by our procedure or one of its myriad variations. We do this because, even though the quantum mechanical aspects of our problem are more easily appreciated for a lattice theory than a continuum theory, there is no reason to believe that solving a very large degree of freedom Schroedinger problem will be any more tractable than summing infinite numbers of Feynman graphs. The purpose of the next three lectures is to try and convince you that reliable non-perturbative methods for rendering the problem manageable do exist.

LECTURE 2. ISING MODEL IN A TRANSVERSE MAGNETIC FIELD

In this lecture and the lectures to follow I will focus on the question of how to do a variational calculation for the ground state and first few low-lying states of any given lattice field theory. Although the method of analysis is quite generally applicable I will choose to develop it within the framework of specific examples for the obvious pedagogic reasons. I will begin with an analysis of a model which can be called a one-dimensional Ising model in the presence of a transverse magnetic field. Before diving into calculations let us spend a few moments discussing the motivation for studying this model.

Why Study a Lattice Ising Model?

The lattice Ising model is particularly interesting for several reasons. First, it is an example of a theory which undergoes a phase transition for a critical value of coupling constant. In this case when I use the word phase transition I am not talking about temperature dependence of a system, but about a change in the symmetry properties of the ground state of the system. Thus, for the Ising model we will discuss, we will see that at a critical value of transverse field the groundstate of the theory changes from being two-fold degenerate to being unique. Moreover, we will see that this corresponds to a certain discrete symmetry of the theory going from being a spontaneously broken symmetry to a normal symmetry as the coupling increases through its critical value. It is this ability to see a theory go from a spontaneously broken phase to a normal symmetry phase which makes this model so interesting, since if our present understanding of approximate hadronic symmetries has any validity we believe that the successes of the PCAC hypothesis (e.g. Adler self-consistency conditions, Adler-Weisberger relations, pion low energy theorems, $K_{\ell 3}$-predictions, etc.) point to the fact that the hadronic theory possesses a spontaneously broken

or Goldstone symmetry. Since the reason why this sort of symmetry breaking takes place is a mystery, it becomes important to have a method for calculating in a theory in order to see if this sort of effect exists for any range of couplings. Having established the necessity of being able to find phase transitions when they occur as a requisite for a good non-perturbative calculational scheme, we turn to the next question, i.e., why study an artificial model like the Ising model in a transverse field rather than the manifestly more interesting σ-model. The answer to this is that the Ising model can be solved exactly and so we can know precisely how well our variational methods are doing. Unfortunately, the same cannot be said for the σ-model and so it would be hard to know whether or not our analysis of this theory for strong or intermediate coupling held water. Thus, we conclude that the Ising model to be studied is interesting in that it is an example of an exactly soluble model with a phase transition and hence it can serve as a benchmark against which to test our methods. Hopefully, since our methods are based on techniques which easily generalize to other theories in higher dimensions, and make no use of the special features of the theory which make it exactly soluble, the exercise is one step along the road of developing familiarity with and confidence in the variational scheme to be described. So much for motivation. Let's now go on to analyze the model in question.

The Ising Model in a Transverse Magnetic Field

The model we are interested in analyzing is defined by the Hamiltonian

$$H = \sum_j \left[\frac{\epsilon_0}{2} \sigma_z(j) - \Delta_0 \sigma_x(j) \sigma_x(j+1) \right] \tag{2.1}$$

where σ_z and σ_x are two Pauli matrices,

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; ; \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and ε_0, Δ_0 are two free parameters. Clearly, up to an overall scale factor the Hamiltonian really only depends upon the ratio $y_0 = \varepsilon_0/\Delta_0$.

Exact Results

Since this model is exactly soluble it behooves us to spend a few moments summarizing those important features of the theory we will try to reproduce. First, for all values of y_0 the theory is invariant under the discrete symmetry which takes $\sigma_z(j) \to \sigma_z(j)$ and $\sigma_x(j) \to -\sigma_x(j)$ and which is generated by

$$U = e^{i\pi\sum_j \sigma_z(j)} \tag{2.2}$$

as can be seen by noting that

$$U^+ \sigma_z(j) U = \sigma_z(j)\ U^+U = \sigma_z(j) \tag{2.3}$$

whereas

$$\begin{aligned} U^+\sigma_x(j)\,U &= e^{-i\pi\sigma_z(j)}\sigma_x(j)\ e^{+i\pi\sigma_z(j)} \\ &= \sum_n \frac{(-i\pi)^n}{n!}\left[\sigma_z(j),\left[\ldots\ldots[\sigma_z(j),\sigma_x(j)]\right]\ldots\right] \\ &= \cos(\pi)\sigma_x(j) + i\sin(\pi)\sigma_y(j) \\ &= -\sigma_x(j) \end{aligned} \tag{2.4}$$

The operator U is the discrete symmetry transformation which is spontaneously broken, and since the operator $\sigma_x(j)$ has non-trivial transformation properties under U it is the fact that its vacuum expectation can become non-zero which implies the existence of a doubly degenerate ground state. To be precise, it is known that for $y_0 \leq 2$

(1) the model has a doubly degenerat ground state, or in other words there exist two states which we shall denote by $|+>$ and $|->$ s.t. $U|+> = |->$ and $U|+> = |->$ for which

$$(2) \quad < \pm \, |\sigma_x(j)| \, \pm > \quad \pm \left(1 - \frac{y_0^2}{4}\right)^{1/8} \tag{2.5}$$

and moreover,

(3) it is known that the theory has an excitation spectrum whose energy momentum dispersion is given by

$$E_{\tilde{k}_p} \cong \sqrt{(\varepsilon_0 - 2\Delta_0)^2 + \tilde{k}_p^2} \tag{2.6}$$

Now for $y_0 > 2$ it is known that the ground state, $| \psi_0 >$, of the theory becomes unique and is an eigenstate of U such that $U | \psi_0 > = | \psi_0 >$. The mass gap and energy-momentum dispersion relation are the same as for $y_0 \leq 2$ and obviously

$$< \psi_0 | \sigma_x | \psi_0 > = < \psi_0 | U^+ \sigma_x U | \psi_0 > = - < \psi_0 | \sigma_x | \psi_0 > \tag{2.7}$$

implies $< \psi_0 | \sigma_x | \psi_0 > = 0$.

One can also calculate the ground state energy density as a function of y and the second derivative, $-\partial^2 E_g / \partial y_0^2$, will be of interest to us at a later point.

Some Trivial Considerations

Although I will make no attempt to explain the machinations one must go through to prove these exact results--since they involve doing a Jordan-Wigner transformation, identifying a conservation law of the theory and diagonalizing the resulting quadratic fermion Hamiltonians--I do want to spend a few moments discussing the limits $\varepsilon_0 = 0$, Δ_0 arbitrary and $\Delta_0 = 0$, ε_0 arbitrary so that the basic notions of spontaneous symmetry breaking for this model will be clear. Let us begin with the case $\varepsilon_0 = 0$, i.e.

$$H_{\varepsilon_0 = 0} = \sum_{j=-N}^{N} [-\Delta_0 \sigma_x(j) \sigma_x(j+1)] \tag{2.8}$$

Clearly, since all the operators $\sigma_x(j)$ commute with $H_{\varepsilon_0=0}$ diagonalizing $H_{\varepsilon_0=0}$ amounts to writing down all the eigenstates of $\sigma_x(j)$. If we denote the eigenstate of $\sigma_x(j)$ of eigenvalue +1 (-1) by an arrow to the right (left) as shown in Fig.(7a), then it follows from (2.8) that the two states $|\psi_{+1}>$ and $|\psi_{-1}>$, shown in Fig. (7a) are degenerate. Note that since U as defined in (2.2) takes σ_x to $-\sigma_x$ it maps the state $|\rightarrow>_j$ into $|\leftarrow>_j$ and vice versa, hence $U|\psi_{\pm 1}> = |\psi_{\mp 1}>$ and so the ground state of the $\varepsilon_0 = 0$ system is twofold degenerate as promised. The states shown in Fig. (7b) are two of the lowest-lying excitations of the system.

It is essentially a matter of choice which of the two ground states of the system, or linear combination of the ground states, one chooses to base a theory on; and a definite prescription can be arrived at only from other considerations. The way one usually decides this issue is to add a small external field in the σ_x direction; i.e. one adds a term

$$V(J) = -J \sum_{j=-N}^{N} \sigma_x(j) \tag{2.9}$$

to $H_{\varepsilon_0=0}$. One then studies the ground state of the system as $J \to 0$. Obviously, the energy of $|\psi_{+1}>$ is given by $E_0 - J(2N)$ and $|\psi_{-1}>$ by $E_0 + J(2N)$; hence for all $J > 0$ $|\psi +1>$ is the groundstate of the system. Other arguments are based upon the desire to have cluster decomposition, but we will not go into that now.

Let us now consider the limit $\Delta_0 = 0$, ε_0 arbitrary. In this case the Hamiltonian is

$$H_{\Delta_0=0} = \sum_{j=-N}^{N} \frac{\varepsilon_0}{2} \sigma_z(j) \tag{2.10}$$

As in the previous case, since $\sigma_z(j)$ commutes with $H_{\Delta_0=0}$ we can label all eigenstates of $H_{\Delta_0=0}$ by giving the eigenvalues of $\sigma_z(j)$. If we let $|\uparrow>_j$ and

$|\downarrow\rangle_j$ denote the states such that $\sigma_z |\uparrow\rangle_j = |\uparrow\rangle_j$ and $\sigma_z|\downarrow\rangle_j = -|\downarrow\rangle_j$, then the groundstate of the theory is the unique state shown in Fig. (8a) and a typical lowest-lying excitation is shown in Fig. (8b).

Approximate Solution by Recursive Methods

Having set the stage let us now introduce the general method by which we hope to analyze this and all other lattice field theories. As we have noted, the method we wish to use should be non-perturbative and should not rely upon any special features of the 1-dimensional problem. The method we have turned to is the Rayleigh-Ritz variational procedure, and our innovation is to devise a scheme for constructing a trial wave function for the groundstate, since guessing the correct form of an infinite parameter wave function is beyond our mortal powers.

Our constructive technique is an iterative one based upon a procedure of thinning degrees of freedom. To be precise, it is based upon the observation that if one has an orthonormal set of states $|\psi_j\rangle$ then the problem of minimizing the expectation value

$$\langle \psi(\alpha) |H| \psi(\alpha) \rangle / \langle \psi(\alpha) | \psi(\alpha) \rangle \tag{2.11}$$

where

$$|\psi(\alpha)\rangle \equiv \sum_j \alpha_j |\psi_j\rangle \tag{2.12}$$

is equivalent to diagonalizing the "truncated" Hamiltonian

$$H_{ij}^{TR} = \langle \psi_i |H| \psi_j \rangle$$

Our procedure will be to begin with a complete set of orthonormal states and thin out this set of states by throwing some away. Thus, we reduce the problem to that of finding a good variational wave function over states generated by this smaller set of independent states. This, however, can be shown to be equivalent to diagonalizing a new Hamiltonian of the same form but having different coefficients. We carry out this procedure of thinning out the remaining set of

states and generating a new effective Hamiltonian until our new Hamiltonian takes a form which can be solved. At each stage we base our decision of which states to keep and which to discard on a simple physically intuitive algorithm.

In order to make the ideas more clear let us abandon generalities and dive into our analysis. I will begin with a discussion of the thinning procedure based upon a very simple algorithm and then discuss the results of a slightly more sophisticated analysis.

Let us begin by introducing the notation $j = 2p + r$; $r = 0,1$ and rewriting H as

$$\begin{aligned} H &= \sum_j \left[\frac{\varepsilon_0}{2}\sigma_z(j) - \Delta_0\sigma_x(j)\sigma_x(j+1)\right] \\ &= \sum_p \left[\frac{\varepsilon_0}{2}\left(\sigma_z(2p) + \sigma_z(2p+1)\right) - \Delta_0\sigma_x(2p)\sigma_x(2p+1)\right] \\ &\quad - \Delta_0\sum_p \sigma_x(2p+1)\sigma_x\left(2(p+1)\right) \end{aligned} \tag{2.13}$$

By this device we divide the lattice into blocks labeled by the integer 'p' containing two sites each, and at the same time we divide it into two terms--the first containing operators referring to a single block and the second containing products of operators in neighboring blocks.

Having done this, we now turn for inspiration to a study of the Hamiltonian describing any single block 'p'; i.e. we study any one

$$h_p \equiv \frac{\varepsilon_0}{2}\left(\sigma_z(2p) + \sigma_z(2p+1)\right) - \Delta_0\sigma_x(2p)\sigma_x(2p+1) \tag{2.14}$$

If we label the states which correspond to the different possible values of $\sigma_z(2p)$ and $\sigma_z(2p+1)$ as $|\downarrow\downarrow>$, $|\downarrow\uparrow>$, $|\uparrow\downarrow>$ and $|\uparrow\uparrow>$ respectively then we see that

$$
\begin{aligned}
h_p|\downarrow\downarrow> &= -\varepsilon_0|\downarrow\downarrow> - \Delta_0|\uparrow\uparrow> \quad , \\
h_p|\downarrow\uparrow> &= -\Delta_0|\uparrow\downarrow> \quad , \\
h_p|\uparrow\downarrow> &= -\Delta_0|\downarrow\uparrow> \quad , \\
h_p|\uparrow\uparrow> &= \varepsilon_0|\uparrow\uparrow> - \Delta_0|\downarrow\downarrow> \quad .
\end{aligned}
\tag{2.15}
$$

The four eigenstates of h_p are shown in Table I, where we have also given their eigenvalues and the difference in energy between the lowest state in each block and the 3 excited states within a block.

Our thinning procedure will be to define the two states per block $|\downarrow>_p$ and $|\uparrow>_p$ where $|\downarrow>_p \equiv (|\downarrow\downarrow> + a_0|\uparrow\uparrow>)/\sqrt{1 + a_0^2}$,

$$
|\uparrow>_p = (|\uparrow\downarrow> + |\downarrow\uparrow>)/\sqrt{2} \tag{2.16}
$$

and then observe that the orthonormal set formed by taking all possible products of these two states over all blocks 'p'. That these states should be able to provide a reasonably good approximation for the groundstate of the theory is intuitively obvious, since the states we have thrown away have higher energy.

Having decided upon which two states (out of the possible four states per block) to keep, our next step is to compute the truncated Hamiltonian. This is easily done. Note that

$$
H = \sum_p \left[h_p - \Delta_0 \sigma_x(2p + 1)\sigma_x\big(2(p + 2)\big) \right] \tag{2.17}
$$

and since the way h_p acts on a given product state is given in Table I, we need only see how the terms $-\Delta_0\sigma_x(2p+1)\sigma_x\ 2(p + 1)$ act.

The way $\sigma_x(2p)$ acts upon such a state $|\downarrow>_p$ or $|\uparrow>_p$ is given by

$$
\sigma_x(2p)|\downarrow>_p = (|\uparrow\downarrow> + a_0|\downarrow\uparrow>)/\sqrt{1 + a_0^2} \tag{2.18}
$$

and so

$$ {}_p\langle\downarrow|\sigma_x(2p)|\downarrow\rangle_p = {}_p\langle\uparrow|\sigma_x(2p)|\uparrow\rangle_p = 0 \tag{2.19} $$

and

$$ {}_p\langle\uparrow|\sigma_x(2p)|\downarrow\rangle_p = {}_p\langle\downarrow|\sigma_x(2p)|\uparrow\rangle_p = \frac{(1 + a_0)}{\sqrt{2(1 + a_0^2}} \tag{2.20} $$

Similarly, $\sigma_x(2p + 1)$ has the matrix elements

$$ {}_p\langle\uparrow|\sigma_x(2p + 1)|\uparrow\rangle_p = {}_p\langle\downarrow|\sigma_x(2p + 1)|\downarrow\rangle_p = 0 $$

and

$$ {}_p\langle\uparrow|\sigma_x(2p + 1)|\downarrow\rangle_p = {}_p\langle\downarrow|\sigma_x(2p + 1)|\uparrow\rangle_p = \frac{1 + a_0}{\sqrt{2(1 + a_0^2}} \tag{2.21} $$

Hence, combining (2.17)-(2.21) we find that we can write H^{TR} in terms of 2 x 2 matrices referring to each block 'p', i.e.

$$ H^{TR}_{(1)} = \sum_p \left[C_1 \mathbb{1}_p + \frac{\varepsilon_1}{2} \sigma_z(p) - \Delta_1 \sigma_x(p)\sigma_x(p + 1) \right] \tag{2.22} $$

where

$$ \begin{aligned} C_1 &= -\frac{1}{2}\left(\Delta_0 + \sqrt{\varepsilon_0^2 + \Delta_0^2}\right) \\ \varepsilon_1 &= \sqrt{\varepsilon^2 + \Delta_0^2} - \Delta_0 \\ \Delta_1 &= \frac{\Delta_0(1 + a_0)^2}{2(1 + a_0^2)} \end{aligned} \tag{2.23} $$

This effective Hamiltonian embodies all the information contained in our choice of a family of trial wave functions and, by construction, it follows that diagonalizing $H^{TR}_{(1)}$ will provide an upper bound upon the true groundstate energy. If either $\varepsilon_1 = 0$ or $\Delta_1 = 0$ we could diagonalize $H^{TR}_{(1)}$ exactly. If $\varepsilon_1 << \Delta_1$ or $\Delta_1 << \varepsilon_1$ we could use perturbation theory to study the structure of the theory. However, in general neither case will apply and our only recourse will be to apply the same

thinning procedure to the theory defined by $H^{TR}_{(1)}$. In this way we generate a sequence $H^{TR}_{(2)}$, $H^{TR}_{(3)}$, etc. and exactly diagonalizing any one of them will yield an upper bound of the groundstate energy. The process is carried out until $H^{TR}_{(n)}$ takes a simple diagonalizable form or until $H^{TR}_{(n+1)} = H^{TR}_{(n)}$ at which point further iteration will avail us little.

To be specific, we could follow the general procedure just outlined and generate from

$$H^{TR}_{(n)} = \sum_p \left[d_n 1_p + \frac{\varepsilon_n}{2} \sigma_z(p) - \Delta_n \sigma_x(p)\sigma_x(p+1) \right] \tag{2.24}$$

a new Hamiltonian

$$H^{TR}_{(n+1)} = \sum_p \left[d_{n+1}\, 1_p + \frac{\varepsilon_{n+1}}{2} \sigma_z(p) - \Delta_{n+1} \sigma_x(p)\sigma_x(p+1) \right] \tag{2.25}$$

where

$$\begin{aligned}
\varepsilon_{n+1} &= \left(\varepsilon_n (1 - a_n^2) - \Delta_n (1 - a_n)^2 \right) / (1 + a_n^2) \\
\Delta_{n+1} &= \frac{\Delta_n}{2} \frac{(1 + a_n)^2}{(1 + a_n^2)} \quad , \\
C_{n+1} &= -\left[\frac{\varepsilon_n (1 - a_n^2) + \Delta_n (1 + a_n)^2}{2(1 + a_n^2)} \right] \quad , \\
d_{n+1} &= C_{n+1} + 2dn \quad , \\
a_n &= \left(\sqrt{\varepsilon_n^2 + \Delta_n^2} - \varepsilon_n \right) / \Delta_n \quad .
\end{aligned} \tag{2.26}$$

Clearly, this recursion relation for the coefficients d_{n+1}, C_{n+1}, Δ_{n+1} and ε_{n+1} can be easily studied numerically and I will now summarize the results of such a study. Actually, for this simple recursion relation it is very helpful to observe that the ratio $\varepsilon_{n+1}/\Delta_{n+1}$ is given in terms of the ratio ε_n/Δ_n alone. If we let

$y_n = \varepsilon_n/\Delta_n$ we can learn almost everything about the way in which (2.26) will iterate if we plot the function

$$R(y_n) = y_{n+1} = y_n = \frac{2(1 - a_n^2)(1 + a_n)}{(1 + a_n^2)^2}\,[y_n(1 + a_n) + a_n - 1] - y_n \qquad (2.27)$$

shown in Fig. 9, which tells us how the ratio ε_n/Δ_n changes with each iteration. Clearly, for any starting value of 'y' such that $R(y) < 0$ we have--after one iteration--a value of $\varepsilon_{n+1}/\Delta_{n+1}$ which is smaller than it was; similarly, for $R(y) > 0$ we are driven to a still larger value of y. Thus Fig. 9 tells us that for $y < y_c$ successive iterations drive us to a limiting form of the H_n^{TR} for which $\varepsilon_n \to \varepsilon_\infty > 0$ and $\Delta_n \to 0$ as $n \to \infty$. So, the theory for $y < y_c$ is a theory with a degenerate groundstate. For $y > y_c$ we are driven to $\Delta_n \to \Delta_\infty > 0$ and $\varepsilon_n \to 0$. Hence for $y > y_c$ the theory has a unique groundstate. The special point $y = y_c$ is a critical point at which the symmetry properties of the theory change. Since $R(y_c) = 0$ we see that the ratio of ε_n/Δ_n is unchanged with successive iterations and more complete study (2.26) shows that $H_{n+1}^{TR} = \rho_c H_n^{TR}$ where $\rho_c < 1$. This tells us that the Hamiltonian reproduces itself up to a scale factor, and thus at the critical point the physics of all length scales is the same--as the folklore would have it.

Location of the value $y_c = 2.55...$ for which $R(y_c) = 0$ is easily accomplished and recalling that the exact value of y_c is 2 we see that this primitive algorithm doesn't do too badly.

A More Sophisticated Algorithm

Going back to (2.26) we see that the algorithm we have adopted depended upon two distinct choices. First, we assumed that we would keep only two states per box at each iteration. Second we chose $a_n(y_n)$ to be given by the naive algorithm that we should diagonalize the 2-site Hamiltonian at each stage. A more sophisticated algorithm is to let $a_n(y_n)$ be an undetermined function of y and then carry

out that iteration for 80 or so steps for a class of functions parametrized by one or more variables and then vary over these parameters so as to minimize the groundstate energy density $\varepsilon(y_0) = \lim_{n\to\infty}(d_n/2^n)$. In this way, except for specifying its general form, the recursion relation itself is undetermined and one varies over a system of possible "renormalization group transformations" to obtain the best possible upper bound on the groundstate energy density. We will now discuss the results of such a calculation for the 1-parameter family of functions

$$a_\rho(y) = \tan^{-1}\left[\frac{\pi}{4}\left(1 - \tanh(\rho y)\right)\right] \quad . \tag{2.28}$$

Figure 10 shows a plot which compares the values of $\varepsilon_0(y_0)$ obtained from the exact solution to the problem (solid line) with the result of our one-parameter variational calculation (dashed line) over the only part of the range of y for which the difference is at all significant. Figure (11) plots the exact form of $\langle\sigma_x\rangle$ as a function of y_0 (eq. (2.5)) against that of our approximate calculation. As you can see the value of the critical point becomes slightly worse, $y_c^{var} \cong 2.7$, but the dashed curve provides a better than one percent fit to $(1 - (y/y_c)^2)^{-.19}$ from $0 \leq y \leq \left(y_c^{var} - 1 \times 10^{-5}\right)$. This again is not bad for such a crude approximation. Finally, Fig. (12) shows that this crude calculation is capable of revealing a structure surprisingly similar to the logarithmic singularity in $-\partial^2\varepsilon_0/\partial y^2$ possessed by the exact solution for $\varepsilon_0(y)$. This is quite a subtle non-perturbative property of the theory and the fact that one is reproducing this phenomenon is quite striking.

Summary

To conclude this lecture let me summarize what we have seen. First we have shown that even a very naive calculation exhibits the fact that a phase transition exists. Second, we have seen that even the slight change of going to a variational calculation allows us to do a remarkably accurate calculation of the groundstate energy over the entire range of ε_0 and Δ_0. Third, we see that the general behavior

of the order parameter $\langle\sigma_x\rangle(y)$ is done reasonably well even with a crude calculation. Finally, although I have not discussed it, one can show that one also does a similar job in predicting the spectrum of excited states for all values of ε_0/Δ_0.

LECTURE 3. LATTICE THIRRING MODEL

In today's lecture I want to present an analysis of what I shall call a lattice version of the Thirring model. The method of analysis will be exactly the same as that used to analyze the Ising model in a transverse magnetic field except now we will be dealing with a theory of fermions. Before presenting the analysis let us first discuss the reasons I have chosen to discuss this model as a second example of the variational renormalization group method presented in the last lecture.

Why?

There are several reasons why this model is a very attractive one to study. To begin with it is the first serious model with fermions, and there is something more physical about fermions than about lattice spins. The next good reason for studying this model is that, as we saw in the first lecture, fermionic theories are the first one to require the use of the non-nearest (or long-range) gradient operator. Since there has been great skepticism about whether or not it is possible to carry out calculations with this form of the gradient, it is worth demonstrating that it is easily worked with. Third, this is a model whose continuum version is solvable and exhibits a peculiar behavior for the fermion wave-function renormalization, $Z_z(g)$, in that it vanishes for a finite value of g. We will see that this feature also occurs in the lattice theory and that because we know how to go back and forth between the would-be continuum operators of the theory and lattice fields we know how to establish the fact that it is the same phenomenon. Fourth, in our list of reasons, is that this is an example of a theory with explicit chiral invariance and our analysis will show how the iteration procedure works in such a setting. Finally, the lattice model of the theory exhibits an interesting non-perturbative phenomenon in that for values of g for which $Z_z(g) = 0$ the theory exhibits a kind of confinement.

The Thirring Model

The Hamiltonian we will study is

$$H = \sum_{j_1,j_2} \left[i\delta'_\Lambda(j_1 - j_2)\ \psi^+_{j_1}\ \alpha\ \psi_{j_2} \right] - g_0 \sum_j (\psi^+_j\ \beta\ \psi_j)^2 \tag{3.1}$$

where

$$\delta'_\Lambda(j) = \begin{cases} 0 & \text{for } j = 0 \\ (-1)^j/j & \text{for } j \neq 0 \end{cases} \tag{3.2}$$

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ ; \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.3}$$

and ψ_j is a two component fermi field satisfying anticommutation relations

$$\left\{ \psi^+_{j_1}, \psi_{j_2} \right\} = \delta_{j_1 j_2}\ \mathbb{1} \tag{3.4}$$

Our method of analysis is the same as before. First we will dissect the lattice into blocks of three sites. We will then solve the three site problem exactly and truncate the space of states to the subspace generated by products of the lowest $Q = 0$, $Q = \pm 1$ states per block.

In order to simplify the discussion let us begin by showing how the approach will work in the special case $g = 0$, i.e. the free massless fermion field in 1 + 1 dimensions. This special case is interesting because it is exactly soluble by going to momentum space and so one might expect this naive truncation procedure to be at its worst. As we will see it does surprisingly well.

Free Field: g = 0 Limit

If we denote the separate components of the two component field by

$$\psi(j) = \begin{pmatrix} b_j \\ d^+_j \end{pmatrix} \tag{3.5}$$

(3.4) becomes

$$\left\{b_{j_1}, b^+_{j_2}\right\} = \left\{d_{j_1}, d^+_{j_2}\right\} = \delta_{j_1, j_2} \tag{3.6}$$

all other commutators being zero. Substituting this in (3.1), we rewrite H as:

$$H = \sum_{j_1, j_2} \left[i\delta'(j_1 - j_2)(b^+_{j_1} b_{j_2} - d^+_{j_1} d_{j_2}) \right] \tag{3.7}$$

Before breaking the problem up into 3-site blocks let us list some useful symmetries of the Hamiltonian. First, inspection of H shows that one can only absorb a b(d) at one point and create a $b^+(d^+)$ at some other point, hence H never changes the total number of b's or d's in a state. If we define

$$n_b(j) = b^+_j b_j \quad ; \quad n_d(j) = d^+_j d_j \tag{3.8}$$

we can define two conserved operators, the electric charge

$$Q = \sum_j \left(n_b(j) - n_d(j) \right) = \sum_j q(j) \tag{3.9}$$

and the chiral or axial charge

$$Q_5 = \sum_j \left(n_b(j) + n_d(j) - 1 \right) = \sum_j q_5(j) \quad .$$

This notation is chosen so that for a single site 'j' we can introduce the state $|0_j\rangle$ s.t.

$$b_j |0_j\rangle = d_j |0_j\rangle = 0 \tag{3.10}$$

and then define the other three possible states for a site 'j' to have the quantum numbers given in Table II.

Useful discrete symmetries are C, P and the anti-linear operator defined by

$$
\begin{aligned}
C\, b_j\, C^{-1} &= b_j^+ \quad , \\
C\, d_j\, C^{-1} &= d_j^+ \quad , \\
C\, Q\, C^{-1} &= -Q \quad , \\
C\, Q_5\, C^{-1} &= -Q_5 \quad ,
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
P\, b_j\, P^{-1} &= d_{-j}^+ \quad , \\
P\, d_j\, P^{-1} &= b_{-j}^+ \quad , \\
P\, Q\, P^{-1} &= Q \quad , \\
P\, Q_5\, P^{-1} &= -Q_5 \quad ,
\end{aligned}
\tag{3.12}
$$

and

$$
\begin{aligned}
\textcircled{H}\, b_j\, \textcircled{H}^{-1} &= d_j \quad , \\
\textcircled{H}\, d_j\, \textcircled{H}^{-1} &= b_j \quad , \\
\textcircled{H}\, Q\, \textcircled{H}^{-1} &= -Q \quad , \\
\textcircled{H}\, Q_5\, \textcircled{H}^{-1} &= Q_5 \quad ,
\end{aligned}
\tag{3.13}
$$

with the phase convention

$$C|0_j\rangle = i|\pm j\rangle$$

$$P|0_j\rangle = |\pm j\rangle$$

and

$$\textcircled{H}|0_j\rangle = |0_j\rangle \quad . \tag{3.14}$$

With these preliminaries behind us let us now define $j = 3p + r$, $r = -1$, 0, +1 and rewrite H as

$$H = \sum_p \left[\sum_{r \neq r'} i\delta'(r - r') \left(b^+_{3p+r} b_{3p+r'} - d^+_{3p+r} d_{3p+r'} \right) \right]$$
$$+ \sum_{p \neq p'} \left[\sum_{r,r'} i\delta' \left(3(p - p') + r - r' \right) \left(b^+_{3p+r} b_{3p'+r'} - d^+_{3p+r} d_{3p'+r'} \right) \right] \tag{3.15}$$

The first set of terms, $p = p'$, constitute the single block Hamiltonian and the $p \neq p'$ terms give the block-block recoupling terms. Hence, let us restrict attention to a single block 'p' and diagonalize

$$h_p = \sum_{r \neq r'} i\delta'(r - r') \left(b^+_{3p+r} b_{3p+r'} - d^+_{3p+r} d_{3p+r'} \right) \quad . \tag{3.16}$$

Since there are four states per site, i.e. $|0\rangle$, $b^+|0\rangle$, $d^+|0\rangle$ and $b^+d^+|0\rangle$, we see that there are $4^3 = 64$ states per block, and so to diagonalize h_p we must at first blush diagonalize a 64 x 64 matrix. It's not so bad, though. If we look at sectors of definite Q, Q_5, C or P or CP the problem vastly simplifies. For example, in Tables 3 and 4 we see the states of $Q = 0$ and $Q = +1$ divided according to their Q_5 eigenvalues. Since the states of $Q_5 = \pm 3$ are the only $Q = 0$ states of this quantum number they are eigenstates of h_p. Since C maps a $Q = 0$, $Q_5 = +1$ state into a state of $Q = 0$, $Q_5 = -1$ one need only diagonalize the 9 x 9 submatrix corresponding to the $Q = 0$, $Q_5 = -1$ sector in order to find the lowest $Q = 0$ eigenstates. There are, of course, two degenerate states at each energy since there is one of $Q_5 = +1$ obtained by applying C to the $Q_5 = -1$ eigenstates. Next we observe that iCP transforms the nine states of $Q = 0$, $Q_5 = -1$ among themselves and so one can reduce the problem to studying h_p restricted to states of definite iCP. This simplifies the problem to diagonalizing a 6 x 6 and 3 x 3 matrix. Actually, having reduced the problem this far we can now straightforwardly diagonalize the 6 x 6 and 3 x 3 problems. The same arguments can be used to simplify the

$Q = +1$ problem. Although it is not really necessary for me to do so, in order to prove it can be done, let me exhibit for you the exact form of the lowest energy state of $Q = 0$, $Q_5 = -1$ and $Q = +1$, $Q_5 = 0$; i.e.

$$|0_p> = \frac{1}{18}(3 - 4i)|+0-> + \frac{1}{18}(3+4i)|-0+>$$
$$-\frac{1}{9}(3+i)(|+-0> + |0+->) - \frac{1}{9}(3-i)(|0-+> + |-+0>)$$
$$+\frac{4}{9}i\left(|0\pm 0> + \frac{5}{8}(|\pm 0\,0> + |0\,0\,\pm>)\right) \qquad (3.17)$$

and

$$|+>_p = \frac{1}{18}(4 - 3i)|\pm+0> - \frac{1}{18}(4+3i)|0+\pm>$$
$$+\frac{1}{9}(1-3i)\left(|+0\pm> + |0\pm+>\right) - \frac{1}{9}(1+3i)\left(|\pm 0+> + |+\pm 0>\right)$$
$$+\frac{4}{9}\left(|+-+> + \frac{5}{8}(|++-> + |-++>)\right) \qquad (3.18)$$

Having found explicit forms for $|0_p>$, $|+_p>$, we can define $|-_p> = C|+_p>$ and $|\pm_p> = -iC|0_p>$, and go on to computing the form H takes when truncated to the system of states spanned by products of these 4 states per box over all boxes. Since the eigenvalues of h_p corresponding to these 4 states are $E_0 = -3$, h_p can be replaced by -3 times the unit operator. Hence, the problem of computing the truncated form of H reduces to computing the truncated forms of operators like b^+_{3p+r}, d^+_{3p+r}, etc.

To compute $(b^+_{3p+r})^{TR}$, etc., it helps to observe that when the b^+ operates on a state it raises the charge of that state by one unit and raises the Q_5 of a state by one unit also. On the other hand d^+ raises the Q_5 by a unit but lowers Q by one unit. Hence, the only possible non-zero matrix elements of b^+_{3p+r} and d^+_{3p+r} between the states $|0_p>$, $|+_p>$, $|-_p>$, and $|\pm_p>$ are

$$
\begin{aligned}
&\langle +_p | b^+_{3p+r} | 0_p \rangle = u_r \quad , \\
&\langle \pm_p | b^+_{3p+r} | -_p \rangle = t_r \quad , \\
&\langle -_p | d^+_{3p+r} | 0_p \rangle = v_r \quad , \\
&\langle \pm_p | d^+_{3p+r} | +_p \rangle = w_r
\end{aligned}
\tag{3.19}
$$

and it follows from the symmetries C, P and (H) that

$$
u_r = t^*_r \; ; \quad v_r = w^*_r \quad \text{and} \quad u_r = v_{-r} \quad . \tag{3.20}
$$

This information can be all summarized in operator form by introducing anti-commuting operators B^+_p, B_p, D^+_p and D_p defined by

$$
\begin{aligned}
&\langle +_p | B^+_p | 0_p \rangle = 1 \\
&\langle -_p | D^+_p | 0_p \rangle = 1, \quad \text{etc.}
\end{aligned}
\tag{3.21}
$$

and writing

$$
\begin{aligned}
(b^+_{3p+r})^{TR} &= B^+_p (u_r \, Q^2_{5p} + u^*_r \, Q^2_p) \quad , \\
(d^+_{3p+r})^{TR} &= D^+_p (u^*_r \, Q^2_{5p} + u_r \, Q^2_p) \quad ,
\end{aligned}
\tag{3.22}
$$

with

$$
Q_{5p} = (B^+_p \, B_p + D^+_p \, D_p - 1)
$$

and

$$
Q_p = (B^+_p \, B_p - D^+_p \, D_p) \quad . \tag{3.23}
$$

Note that if u_r and v_r are real, then since $Q^2_p + Q^2_{5p}$ is the identity operator, (3.22) becomes

$$(b^{+}_{3p+r})^{TR} = u_r \, B^{+}_p \quad ,$$
$$(d^{+}_{3p+r})^{TR} = u_r \, D^{+}_p \quad . \tag{3.24}$$

Substituting the general expression (3.21) into (3.15), we arrive at the form of $H^{TR}_{(1)}$, and then this process can be carried out iteratively. Actually, for the free field Hamiltonian it is easy to show that u_r is real and so $H^{TR}_{(1)}$ becomes, using (3.24),

$$H^{TR}_{(1)} = -3\left(\frac{2N+1}{3}\right)\mathbb{1} + \sum_{p\neq q}\left[i\left(\sum_{r,r'=-1}^{1} \delta'(3(p-q)+r-r')\, u_r\, u_{r'}\right)^{*} \right.$$
$$\left. (B^{+}_p B_q - D^{+}_p D_q)\right] \tag{3.25}$$

which has exactly the same form as the original Hamiltonian if we define a new δ' function to be

$$\delta'_{(1)}(p-q) \equiv \sum_{r,r\leq -1}^{1} \delta'\left(3(p-q)+r-r'\right) u_r\, u_{r'} \quad . \tag{3.26}$$

If one carries out this iteration in detail we get an upper bound on the groundstate energy of a free massless fermion theory which is (in units of the cutoff Λ^2)

$$E_0^{approx} = (-1.217\ldots)L \tag{3.27}$$

which is to be compared to the exact answer which is

$$E_0^{exact} = -\frac{\pi}{2} L \tag{3.28}$$

The agreement here is to 20 percent which is not bad considering the naive truncation procedure we have adopted. It is simple to show that one can do considerably better by making only slight changes in the procedure, but we will not bother discussing that here.

This discussion sets up the notation of the general calculation. Let us now turn to a discussion of the $g \neq 0$ case.

Full Thirring Model

Returning to the full Hamiltonian, (3.1), let us now summarize what happens if one carries out exactly the same procedure except for $g_0 \neq 0$. In this case u_r is no longer real and one must use the more complicated form (3.22) to compute $H^{TR}_{(n)}$. After the first iteration the Hamiltonian takes the following general form which is then reproduced in each succeeding iteration, i.e.:

$$\begin{aligned} H^{TR}_{(n)} &= \sum_{p_1,p_2} iX_n(p_1 - p_2)(B^+_{p_1} B_{p_2} - D^+_{p_1} D_{p_2}) \\ &- g_n \sum_p (B^+_p B_p + D^+_p D_p - 1)^2 + E_n 1_p \\ &+ i \sum_{p_1,p_2} \Big[Z_n(p_1 - p_2)(B^+_{p_1} Q^2_{p_1} Q^2_{p_2} B_{p_2} - D^+_{p_1} Q^2_{5p_1} Q^2_{5p_2} D_{p_2}) \\ &+ Z^*_n(p_1 - p_2)(B^+_{p_1} Q^2_{5p_1} Q^2_{5p_2} B_{p_2} - D^+_{p_1} Q^2_{p_1} Q^2_{p_2} D_{p_2}) \Big] \end{aligned} \tag{3.28}$$

and Tables 5, 6 and 7 show how the various parameters g_n, $X_n(p)$ and $Z_n(p)$ change with succeeding iterations for different values of g_0.

Before discussing the tables in detail let us note that we should expect something very peculiar to happen for $g_0 \gg 1$. This is the case because for $g_0 \gg 1$ we expect the single site part of H (3.1) to dominate and so we can first study the individual term h_j, where

$$h_j = -g_0(\psi^+_j \beta \psi_j)^2 = -g_0\Big(n_b(j) + n_d(j) - 1\Big)^2 \tag{3.29}$$

It is obvious from (3.29) that the neutral states $|0_j\rangle$ and $|\pm_j\rangle$ are degenerate and have energy, -g, whereas the charged states have energy, 0. Hence, the cost

of creating two separated charges from a distribution of tightly bound pairs is 2g and so we would expect that the excitation spectrum of the theory (in the limit Λ large) not tohave separated fermions but to have massless bound state excitations. In an earlier paper, Drell, Yankielowicz and I[7] showed that for $g_0 >> 0$ this problem could be converted to almost a Heisenberg anti-ferromagnetic and that our conclusions about massless bound states could be proven to be correct if the interactions in this system were of nearest neighbor form. In this case the Heisenberg anti-ferromagnet problem is exactly soluble, the solution having been given by Bethe in 1931, and it is indeed massless. I mention this because, as can be seen from Tables 5-7, for g_0 smaller than $g_c \approx 1$, the iterative calculation converges to a theory with $\lim_{n\to\infty} g_n = 0$ and $X_{(n)}(p)$ going over, up to a scale factor, to a function of 'p' which is quite similar to $\delta'(p)$. Hence, for $g < g_c$ we conclude that we are dealing with a theory whose behavior is similar to a theory with massless fermions and both scalar and fermion excitations of zero mass exist. On the other hand, for $g > g_c$ we see that $\lim_{n\to\infty} g_n = g_\infty > 0$ and $X_n(p)$ goes over to a nearest neighbor form. This says that after a finite number of iterations we are in the situation of studying a theory which is equivalent to the one studied in Ref. (7), and so we know that it describes massless bound configurations, and essentially non-propagating fermion excitations of mass g_∞ (in units of the inverse lattice spacing Λ).

This basically completes the general description of the way in which the iterative variational calculation proceeds. While there is much which can be said about other aspects of the theory, I would like to conclude our discussion of this model by asking if the existence of a g_c has any counterpart in the

solvable continuum theory, and if we can see how our g_c will correspond to the continuum behavior. I would also like to make a few simple remarks about how one could go beyond what I will describe now and study approximations to continuum two point functions, the Schwinger term, etc.

To address the first point, it is true that the continuum theory does exhibit a strange behavior; namely, depending upon the point splitting procedure used to define the composite operators of the theory, one finds that there exists a finite coupling $g_0 = \bar{g}_c$ past which the continuum theory fails to exist. This occurs because for $g_0 > \bar{g}_c$ the continuum Hamiltonian written in terms of currents alone develops a negative coefficient--and since the solution is based upon the assumption that the currents are Bose operators, the theory fails to have a groundstate. Clearly, this cannot happen in a fermion theory, since the sign of the Hamiltonian is irrelevant in a fermion theory--changing the sign simply changes the way in which we choose to define the filled sea of negative energy states. Since on the lattice we are dealing with a fermion theory, the currents are not true bosons in that they remember there is an exclusion principle and so for all values of g_0 the groundstate of the theory must exist. Question: Is our lattice g_c related to the continuum $\bar{g}_c$, and one can understand what it is about the lattice theory which makes the passage to a continuum model as $\Lambda \to \infty$ for $g > \bar{g}_c$ impossible? The answer to this question would seem to be that there is indeed a tight correspondence between the change in behavior of the lattice theory and the strange behavior of the continuum theory. To see this we study the combination of lattice fields which go over to the continuum operator $\psi(x)$ as $\Lambda \to \infty$; i.e., we study

$$\psi_\Lambda(x) = \lim_{N\to\infty} \sum_{j=-N}^{N} \left[\frac{\sin\left(\pi(\Lambda x - j)\right)}{(2N+1)\sin\left(\frac{\pi(\Lambda x - j)}{2N+1}\right)} \begin{pmatrix} b_j \\ d_j^+ \end{pmatrix} \right]$$

and compute the matrix element

$$\langle 0 | \int dx\, \psi_\Lambda(x) | + \rangle \equiv Z_2(g_0)^{1/2} \quad .$$

by means of the obvious iteration procedure defined by (3.22). To normalize this calculation to the free field limit ($g_0 = 0$) calculated the same way, we directly compute $\sqrt{Z_2(g_0)/Z_2(0)}$ and the result is shown in Fig. (13). As you can see for $g_0 \leq g_c$, $\sqrt{Z_2(g_0)/Z_2(0)}$ is finite, whereas for $g_0 \geq g_c$ the ratio is zero. If one goes back to the continuum theory, this is exactly what happens for $g = \bar{g}_c$.

At this point, I leave you to draw your own conclusions as to how tight the relation between the behavior of the lattice model and continuum theory really is. To really nail it down, we should work out things like $\langle 0 | \psi_\Lambda^+(x)\psi_\Lambda(0) | 0 \rangle$, the Schwinger term, and operator equations of motion. We have not studied this in any detail, although we have computed the Schwinger term and obtained entirely reasonable results. I refer you to Ref. (7) since time does not permit further discussion of this point. I would like to conclude with a brief explanation of what is happening for $g > g_c$ and why one cannot conventionally define the continuum Thirring model beyond this point by taking the limit $\Lambda \to \infty$ and multiplicatively renormalizing H. In a sense I have just made the relevant point: what is breaking down for $g > g_c$ is the usual program of multiplicative renormalization. This is so because the usual multiplicative renormalization scheme requires that one rescale the coupling constant and $\psi(x)$ in constant in such a way as to render the spectrum

finite and at the same time so as to make $\langle 0|\int dx\psi(x)|+\rangle$ be unity. As we have seen, for $g_0 > g_c$ this second condition is impossible even for finite value of Λ. This happens because for $g_0 > g_c$ the fermions suddenly acquire a mass proportional to the cutoff and so leave the set of physical states. While states of zero mass do exist for $g_0 > g_c$, they correspond to tightly bound states. Thus, while a finite relativistic theory exists for $g_0 > g_c$, it cannot be obtained by multiplicatively renormalizing H in the manner prescribed by perturbation theory--although it is true that such a scheme can be carried out if H is rewritten in terms of composite "magnon" creating operators of the Heisenberg anti-ferromagnet. In some sense, this shows that this lattice model exhibits a version of confinement in that, although the theory is simply described by fermionic operators, there are no physical states of the theory created by these operators.

Summary

To close, I would like to highlight the important messages I would like you to get from this analysis. First, computing with our definition of the gradient operation is easy. Second, this variational analysis makes complete sense and does surprisingly well for the free field theory and seems to make sense for the $g_0 \neq 0$ version of the Thirring model. Third, the existence of the critical coupling g_c past which the multiplicative renormalization program breaks down is an example of the way in which perfectly sensible theories may be derivable from lattice theories, even when they cannot even be formulated sensibly within the framework of the usual perturbative approach to renormalization.

LECTURE 4. U(1)-GOLDSTONE MODEL

By this last lecture it has become painfully clear that due to lack of time I will not get to the one lattice gauge theory I had hoped to discuss--i.e. the lattice Higgs model. Rather than try and rush to include a discussion of this model, I will conclude this set of lectures with a discusssion of the theory which one gauges in order to obtain the Higgs model, namely, the U(1)-Goldstone model. However, I will try to keep the discussion of this model as brief as possible--without becoming totally cryptic--so as to leave time for a few remarks about what we do know about the Higgs model and Abelian gauge theories in general.

Introduction to the U(1) Model

The model we will discuss is the model of a complex scalar field which was first studied as the simplest example of the sponteneous breaking of a continuous symmetry. The theory is based upon the Lagrangian

$$\mathscr{L} = (\partial_\mu \phi^*)(\partial_\mu \phi) - \lambda(2\phi^*\phi - f^2)^2 \tag{4.1}$$

where

$$\phi = \frac{1}{\sqrt{2}}(\sigma + i\chi)$$

or alternatively, upon the Hamiltonian

$$H = \int dx \left(\Pi_{\phi^*}\Pi_\phi + \partial_1\phi^*\partial_1\phi + \lambda(2\phi^*\phi - f^2)^2 \right) \tag{4.2}$$

where we define

$$\Pi_{\phi^*} = \partial_0\phi \; ; \quad \Pi_\phi = \partial_0\phi^* \; .$$

Rather than discussing this continuum theory, let us directly transcribe it to a lattice theory and then discuss that. The lattice form we will use will

have a nearest neighbor form for the gradient since, as we saw in the first lecture, this makes no important difference for the scalar field and it simplifies the pedagogical part of our discussion. The lattice Hamiltonian we adopt is

$$H = \sum_j \left[\Pi_{\phi_j^*} \Pi_{\phi_j} + (\phi_{j+1}^* - \phi_j^*)(\phi_{j+1} - \phi_j) + \lambda(2\phi_j^* \phi_j - f^2)^2 \right] \qquad (4.3)$$

where

$$\phi_j = \frac{1}{\sqrt{2}} (\sigma_j + i\chi_j)$$

Having transcribed the continuum theory to a lattice theory, let us be sure that the important features of the continuum model have not been lost, at least at the classical level, or at the level of perturbation theory. The purpose of the discussion to follow is to reassure those familiar with the continuum Goldstone model that nothing is different, and to introduce those of you who have never studied the problem to the concepts and the level of discussion of these points which one usually encounters.

First, let us discuss the classical limit of this theory. This corresponding to ignoring the $\Pi_{\phi_j^*} \Pi_{\phi_j}$ term in (4.3) and treating the remaining terms as an expression for the "energy" associated with a function ϕ_j. Since the classical part of (4.3) is a sum of squares, it is a manifestly positive function of ϕ_j; thus, the functions which minimize H are those for which each of these terms separately equal zero. This means that $(\phi_{j+1} - \phi_j) = 0$ or $\phi_j = \phi_0$ independent of 'j', and that

$$2\phi_0^* \phi_0 - f^2 = 0. \qquad (4.4)$$

If we write $\phi_0 = r_0 e^{i\theta}$, then (4.4) implies that $r_0 = f/\sqrt{2}$ and so there is a one-parameter family of functions $\phi_0(\theta) = \frac{f}{\sqrt{2}} e^{i\theta}$ which have zero energy. In terms of the variables σ_j and χ_j, this result is simply that both σ_j and χ_j are independent of 'j' and that $\sigma_0 = f\cos\theta$ and $\chi_0 = f\sin\theta$. This situation, where there is a continuum of degenerate groundstate configurations, is what is meant by saying that the U(1) model exhibits spontaneous symmetry breaking, since a U(1) transformation rotates one of the degenerate states into another and so no single "groundstate" is rotationally invariant.

Turning from the classical picture, let us turn to the usual perturbation discussion of the quantum field theory. For this discussion it is convenient to go to the variables σ_j and χ_j. Substituting $\phi_j = \frac{1}{\sqrt{2}}(\sigma_j + i\chi_j)$ into (4.3),

$$H = \sum_j \left[\frac{1}{2}\Pi_{\sigma_j}^2 + \frac{1}{2}\Pi_{\chi_j}^2 + \frac{1}{2}(\sigma_{j+1} - \sigma_j)^2 + \frac{1}{2}(\chi_{j+1} - \chi_j)^2 + \lambda(\sigma_j^2 + \chi_j^2 - f^2)^2 \right] . \tag{4.5}$$

This Hamiltonian is usually analyzed by observing that in discussing perturbation theory one wants to do an expansion in small vibrations about a stable minimum of the classical potential. In this case, because any one of the field configurations $\sigma_0 = f\cos\theta$ and $\chi_0 = f\sin\theta$ are minima, we have to specify which one of these field configurations one is expanding about; however, the belief--backed up by analyses of models in 3 + 1 dimensions to any finite order in perturbation theory--is that one gets the same theory (in the sense of unitary equivalence) no matter which point we expand about. Assuming that this might be the case, let us choose to expand about $\sigma_0 = f$, $\chi_0 = 0$. In other words, let us define a small vibration field σ' by

$$\sigma = \sigma' + f \tag{4.6}$$

and rewrite H as

$$H = \sum_j \left[\frac{1}{2} \Pi^2_{\sigma'_j} + \frac{1}{2} \Pi^2_{\chi_j} + \frac{1}{2}(\sigma'_{j+1} - \sigma'_j)^2 + \frac{1}{2}(\chi_{j+1} - \chi_j)^2 \right.$$
$$\left. + m^2_{\sigma'} \sigma'^2 + \frac{m^2_{\sigma'}}{2f} \sigma'(\sigma'^2 + \chi^2) + \frac{m^2_{\sigma'}}{8f^2} (\sigma'^2 + \chi^2)^2 \right] \tag{4.7}$$

where $m^2_{\sigma'} \equiv 8\lambda f^2$.

Focusing attention on the quadratic part of (4.7), in order to define the propagators to be used in the perturbation expansion, we see that the σ'-field is a massive free field of mass $m_{\sigma'} = 2\sqrt{2}\ \lambda^{1/2} f$ and χ is a massless field--i.e., it is the infamous Goldstone boson. The usual perturbation analysis is then usually done by holding $m^2_{\sigma'}$ fixed, and expanding in powers of $1/f$. Hence, $f \to \infty$ is the weak coupling region of the theory and is the region where one expects the notion that the groundstate of the quantum theory has $\langle\sigma\rangle = f$ and a massless Goldstone boson to be good. In fact, in 2 + 1 and 3 + 1 dimensional theories one believes that the perturbation picture for $f >> 1$ is essentially correct; however, in the 1 + 1 dimensional theory things are different. This is because in one dimension a massless particle propagator causes infrared divergences so severe as to invalidate the entire perturbation analysis. One might of course conjecture that although the perturbative analysis in 1 + 1 dimensions breaks down, perhaps the general picture of the classical analysis survives. This, however, is known to be false. There is an exact theorem for the continuum theory due to S. Coleman,[8] and an earlier version of the same theorem for any lattice theory due to Mermin and Wagner,[9] which says that in 1 + 1 dimensions the groundstate

of the theory cannot have $\langle\phi\rangle \neq 0$. This brings us to the question of why we are so interested in studying this model.

Why the U(1) Model?

As I pointed out in the first lecture, there is every reason to believe that the ability to compute spontaneous breaking of continuous symmetries is one of the important requirements to put on a purportedly non-perturbative method for analyzing field theories. On the other hand, another important requirement for such a calculational scheme is that it does not fool you and predict spontaneous symmetry breaking when it does not exist. For this reason the 1 + 1 dimensional U(1)-Goldstone model is of great interest to us, as--despite the suggestions of the classical and overly naive perturbation theory analysis of the model--there is no Goldstone boson. More precisely, for the 1 + 1 dimensional theory it is impossible for $\langle\phi\rangle$ to be different from zero. Since we have already seen that our iterative calculational scheme does seem to predict phase transitions (or spontaneous symmetry breaking) when we know they do occur, we study this model to be sure that it does not predict them when they are known not to occur.

Another important reason for studying this model is that it provides an example (other than the free scalar field, which I will not discuss due to lack of time) of how to handle a boson field theory. This is important, since up until now we have only discussed systems which have a finite number of states associated with each lattice site. For the boson field there are an infinite number of states per site and so there is an interesting new feature of the iteration procedure to be investigated.

Finally, as I alluded to at the outset, gauging the U(1)-symmetry of this model gives the Higgs model, which has a rich structure in 1 + 1 dimensions, and I will have a few brief remarks to make about this at the end of this talk.

Non-Perturbative Analysis

At this point of my talk we have discussed why the U(1) model is interesting and why it is necessary to go beyond perturbation theory in order to properly analyze what is going on. Let us now see how this can be done. To begin, let us recast (4.5) as a gigantic Schroedinger problem. This is easily done, since

$$[\Pi_{\sigma_j}, \sigma_{j'}] = [\Pi_{\chi_j}, \chi_{j'}] = -i\delta_{jj'},$$

and so we can think of H as an operator on the space of square integrable functions of the $2(2N + 1)$ variables $(\sigma_{-N}, \chi_{-N}; \sigma_{-N+1}, \chi_{-N+1}; \ldots; \sigma_N, \chi_N)$, i.e., we think of the problem of diagonalizing H as equivalent to solving the Schroedinger problem

$$H\Psi(\sigma_{-N}, \chi_{-N} \ldots \sigma_N, \chi_N) = E\Psi(\sigma_{-N}, \chi_{-N}; \ldots \sigma_N, \chi_N) \tag{4.8}$$

where we use for H the form given in (4.5) with the substitutions

$$\Pi_{\sigma_j} = \left(\frac{1}{i} \frac{\partial}{\partial \sigma_j}\right); \quad \Pi_{\chi_j} = \left(\frac{1}{i} \frac{\partial}{\partial \chi_j}\right) \tag{4.9}$$

and we assume

$$\int d\sigma_{-N} \cdots \int \chi_N \, \Psi^*(\sigma_{-N}, \ldots, \chi_N) \, \Psi(\sigma_{-N}, \ldots \chi_N) = 1 \quad . \tag{4.10}$$

Having made this observation, we will now rewrite H in its Schroedinger form, collect all "single-site" terms, and show that if we take the limit $\lambda \to \infty$, f held fixed (i.e. we let $m_\sigma^2 \to \infty$ for fixed f), we greatly simplify the

problem to be analyzed without losing any essential features of the model. (To those familiar with the language, this limit corresponds to studying the non-linear version of the σ-model instead of the linear version, and at least in 2 + 1 and 3 + 1 dimensions one knows that the Goldstone boson exists in both versions.) To be precise, let us rewrite (4.5) as

$$H = \sum_j \left[-\frac{1}{2}\frac{\partial^2}{\partial\sigma_j^2} - \frac{1}{2}\frac{\partial^2}{\partial\chi_j^2} + \sigma_j^2 + \chi_j^2 + \lambda(\sigma_j^2 + \chi_j^2 - f^2)^2 \right] - \sum_j (\sigma_{j+1}\sigma_j + \chi_{j+1}\chi_j) \tag{4.11}$$

Note that the terms σ_j^2, χ_j^2, $\sigma_{j+1}\sigma_j$ and $\chi_{j+1}\chi_j$ come from the gradient term. Let us now focus attention on any one of the terms

$$h_j = \left(-\frac{1}{2}\frac{\partial^2}{\partial\sigma_j^2} - \frac{1}{2}\frac{\partial^2}{\partial\chi_j^2} + \sigma_j^2 + \chi_j^2 + \lambda(\sigma_j^2 + \chi_j^2 - f^2)^2 \right) \tag{4.12}$$

and try to solve the single-site problem

$$h_j \Psi(\sigma_j,\chi_j) = E\Psi(\sigma_j,\chi_j) \quad . \tag{4.13}$$

Now (4.13) is nothing but a two-variable Schroedinger equation and is invariant with respect to rotations in the (σ_j,χ_j) plane. Hence, it is convenient to change variables and define

$$r_j = \sqrt{\sigma_j^2 + \chi_j^2} \; ; \quad \theta_j = \tan^{-1}(\sigma_j/\chi_j) \; ; \quad -\pi \leq \theta_j \leq \pi \tag{4.14}$$

and rewrite h_j as

$$h_j = \left[-\frac{1}{2}\left(\frac{1}{r_j}\frac{\partial}{\partial r_j}\left(r_j\frac{\partial}{\partial r_j}\right)\right) - \frac{1}{2r_j^2}\frac{\partial^2}{\partial\theta_j^2} + r_j^2 + \lambda(r_j^2 - f^2)^2 \right] \tag{4.15}$$

and observe that in order to solve the problem one can separate variables and define

$$\Psi(r,\theta) = \sum_m \phi_m(r)\, e^{-im\theta} \tag{4.16}$$

In this case we see that the problem of finding eigenfunctions of h_j reduces to solving the 'm' dependent Schroedinger problem

$$\left[-\frac{1}{2r_j}\frac{\partial}{\partial r_j}\left(r_j \frac{\partial}{\partial r_j}\right) - \frac{m^2}{2r_j^2} + r_j^2 + \lambda(r_j^2 - f^2)^2 \right]\phi_m(r_j) = E\phi_m(r_j) \tag{4.17}$$

Clearly, as $\lambda \to \infty$ the potential

$$v(r_j) = -\frac{m^2}{2r_j^2} + r_j^2 + \lambda(r_j^2 - f^2)^2 \tag{4.18}$$

has its minimum at $r_j = f$ and has a curvature which goes like $(\sqrt{2}\ \lambda^{1/2} f)^2$. Hence, one can readily convince oneself that a Gaussian of the form $e^{-\gamma(r_j - f)^2/2}$, where $\gamma \approx \lambda^{1/2} f/\sqrt{2}$, provides a good groundstate wave function for any finite 'm', and so one can choose this for $\phi_m(r)$ for all 'm' and compute the expectation of H in states of this form. Clearly, since the Gaussians get narrower and narrower as $\lambda \to \infty$, this amounts to ignoring the term $-\frac{1}{2r_j}\frac{\partial}{\partial r_j}\left(r_j \frac{\partial}{\partial r_j}\right)$ in (4.17) and replacing 'r_j' everywhere by f. We therefore observe that in the limit $\lambda \to \infty$, (4.17) becomes

$$h_j^{(\lambda=\infty)} = \left(\theta_{\text{independent const.}}\right) + \frac{1}{2f^2}\left(\frac{1}{i}\frac{\partial}{\partial \theta_j}\right)^2 \tag{4.19}$$

where $h_j^{(\lambda=\infty)}$ is defined, by construction, on a space of functions $f(\theta)$ satisfying periodic boundary conditions; i.e., up to a constant--which we can ignore--h_j is the Hamiltonian of a rotor of moment of inertia $1/f^2$. Going back to (4.11) and making the corresponding substitutions

$$\sigma_j = f\cos(\theta_j) \quad ; \quad \chi_j = f\sin(\theta_j) \tag{4.20}$$

we can rewrite (4.11) in the limit $\lambda \to \infty$ as:

$$H^{(\lambda=\infty)} = \sum_j \left[\frac{1}{2f^2}\left(\frac{1}{i}\frac{\partial}{\partial\theta_j}\right)^2 - f^2\cos(\theta_{j+1} - \theta_j)\right] \tag{4.21}$$

where H is defined as a self-adjoint operator on the space of square integrable functions $\Psi(\theta_{-N},\ldots,\theta_N)$ satisfying periodic boundary conditions in each variable $-\pi \leq \theta_j \leq \pi$.

The previous argument tells us that in the $\lambda \to \infty$ or $M_\sigma \to \infty$ limit of the U(1)-Goldstone the theory goes over to a system of planar rotors of moment of inertia $1/f^2$ coupled to one another by an amount proportional to the difference between the directions in which they point. This same model is also a beloved model of "statistical mechanics" who cryptically call it the x-y model.

Have We Lost Anything?

Before discussing this model and the structure of the resulting theory as a function of 'f', let us observe that the theory specified by (4.21) has the same features, at the classical level, that the original theory had. Once again, the classical approximation is to drop the $\left(\frac{1}{i}\frac{\partial}{\partial\theta_j}\right)^2$ terms. If we do this, it is clear that the classical theory has a 1-parameter family of ground-states labeled by a parameter θ_0, namely the configuration $\theta_{j+1} - \theta_j = 0$, or $\theta_j = \theta_0$.

Having argued that the classical level of the theory is unchanged, let us now argue that the "small vibration analysis" of the quantum theory suggests that we are studying, for large f, the theory of a weakly interacting massless field. The easiest way to do this is to let

$$\theta'_j = f\theta_j \quad ; \quad -\pi f \leq \theta_j \leq \pi f \tag{4.22}$$

and rewrite H as

$$H = \sum_j \left[\frac{1}{2} \left(\frac{1}{i} \frac{\partial}{\partial \theta'_j} \right)^2 - f^2 \cos(\theta'_{j+1} - \theta'_j) \right] \tag{4.23}$$

If we now assume that $\theta'_j = \theta'_0 + \delta_j$ and expand H in terms of the small vibration field δ_j, we obtain

$$\begin{aligned} H = -f^2(2N)\mathbb{1} + \frac{1}{2} \sum_j \left[\left(\frac{1}{i} \frac{\partial}{\partial \delta_j} \right)^2 + (\delta_{j+1} - \delta_j)^2 \right] \\ + \sum_j \left[- \frac{(\delta_{j+1} - \delta_j)^4}{4! f^2} + \frac{(\delta_{j+1} - \delta_j)^6}{6! f^4} + \ldots \right] \end{aligned} \tag{4.24}$$

which as $f \to \infty$ goes over to the Hamiltonian of a free massless field. This, of course, is what we saw for the original U(1) model, i.e. that perturbation theory corresponded to an expansion in f^{-1} about the theory of a massless scalar field.

Hence, we see that our specialization to the $\lambda \to \infty$, f held fixed limit of the U(1) model (or the x-y model) loses none of the important features we wished to study. Our goal will be to show that our calculation for $f \ll 1$ agrees with the Mermin-Wagner (or lattice versions of Coleman's theorem) in that it predicts that

$$\langle e^{i\theta_j} \rangle \equiv \frac{1}{f} \langle \phi_j \rangle = 0 \quad ,$$

i.e., it is not $e^{i\theta_0}$, as we would expect from the classical argument.

Discussing the Lattice Model Variationally

Time will not permit me to give all the details of the iterative calculation for this model, nor is it very interesting to you to check my arithmetic.

I will spend the remainder of my time trying to give you a feeling for what we did and how a typical iteration looks.

To begin, let us consider some general physics associated with the Hamiltonian (4.4), or its rescaled form (4.23). I would like to point out that as f decreases towards zero there is a big difference between (4.21) and the theory obtained by replacing $\cos(\theta_{j+1} - \theta_j)$ by $\frac{1}{2}(\theta_{j+1} - \theta_j)^2$. This is true because if we study the theory

$$H = \sum_j \left[\frac{1}{2f^2} \left(\frac{1}{i} \frac{\partial}{\partial \theta_j} \right)^2 + \frac{f^2}{2} (\theta_{j+1} - \theta_j)^2 \right] \tag{4.25}$$

we see that we can always perform the canonical transformation

$$\theta_j = \theta'_j / f \quad ; \quad \left(\frac{1}{i} \frac{\partial}{\partial \theta_j} \right) = f \left(\frac{1}{i} \frac{\partial}{\partial \theta'_j} \right) \tag{4.26}$$

and rewrite H as a massless free field theory; hence the physics of (4.25) is independent of f. However, the physics of (4.21) is quite dependent upon f, as can be readily seen by studying H in the limit $f >> 1$. In this case if we define

$$H_0 = \frac{1}{2f^2} \sum_j \left(\frac{1}{i} \frac{\partial}{\partial \theta_j} \right)^2 \tag{4.27}$$

and treat

$$V = -f^2 \cos(\theta_{j+1} - \theta_j) \tag{4.28}$$

as a perturbation, we see that the eigenstates of H_0 are wave functions of the form

$$\Psi(\theta_{-N}, \ldots, \theta_N) = \prod_{j=-N}^{N} \left(e^{-im_j \theta_j} \right) \tag{4.29}$$

with eigenenergies

$$E(m_j) = \frac{1}{2f^2}\sum_j m_j^2 \tag{4.30}$$

Hence, the groundstate of H_0 is the unique state

$$\Psi(\theta_{-N},\ldots,\theta_N) = 1 \tag{4.31}$$

and the gap to the lowest excited state is $1/2f^2$, which becomes large as $f \to 0$. Clearly for 'f' sufficiently small the perturbation V has no way of wiping out a gap of order $1/2f^2$ and so the theory is a theory of massive excitations. The question is, how come the small 'f' limit of(4.21) is a massive theory, whereas the theory of (4.25) is a massless theory independent of the value of f? The answer is that the arguments, m_j, of (4.29) are integers because the $\psi(\theta_j)$ are defined to be periodic in the variables θ_j--and so for small enough 'f' the gap becomes large. Hence, it is the fact that the $\psi(\theta_j)$ know about boundaries of the defining region which allows the theory to go massive.

This comment is important because the gist of our iterative solution will be to show that for f's greater than some constant f_c, the groundstate wave function never sees the fact that the Hamiltonian is periodic, i.e., that θ_j runs over a finite range. To be specific, let us define $j = 2p + r$, $r = 0_r$, and rewrite H as

$$H = \sum_p \left[-\frac{1}{2f^2}\left(\frac{\partial}{\partial\theta_{2p}}\frac{\partial}{\partial\theta_{2p+1}}\right) - f^2\cos(\theta_{2p+1} - \theta_{2p})\right] - f^2\sum_p \cos(\theta_{2(p+1)} - \theta_{2p+1}) \quad , \tag{4.32}$$

and let us define

$$h_p = -\frac{1}{2f^2}\left(\frac{\partial^2}{\partial\theta_{2p}^2} + \frac{\partial^2}{\partial\theta_{2p+1}^2}\right) - f^2\cos(\theta_{2p+1} - \theta_{2p}) \tag{4.33}$$

in analogy to earlier iterations. As before, our next step is to analyze this 1-block problem in detail, identify the eigenstates of the system, and truncate away those combinations of fields corresponding to "high mass modes." To analyze (4.33), it is suggestive to introduce the variables

$$\begin{aligned} 2\psi_p &= (\theta_{2p} + \theta_{2p+1}) \quad , \\ 2\phi_p &= (\theta_{2p} - \theta_{2p+1}) \quad , \end{aligned} \tag{4.34}$$

and rewrite

$$h_p = -\frac{1}{2f^2}\left(\frac{\partial^2}{\partial\psi_p^2}\right) - \frac{1}{2f^2}\left(\frac{\partial^2}{\partial\phi_p^2}\right) - f^2\cos(2\phi_p) \tag{4.35}$$

Expanding the cos $(2\phi_p)$ and fixing attention on quadratic terms suggests that the variable ϕ_p behaves like an oscillar of frequency $\omega_\phi = 2$ and mass $m_\phi = f^2$, and the variable ψ_p acts like a rotor, in that $-\partial^2/\partial\psi_p^2$ is diagonalized by functions of the form $e^{ik\psi_p}$. Actually, this apparent decoupling of the 2-site Hamiltonian is deceptive, since the requirement that h_p act on wave functions $\Psi(\theta_{2p},\theta_{2p+1})$ which are periodic on the square $-\pi \leq \theta_{2p} \leq \pi$, $-\pi \leq \theta_{2p+1} \leq \pi$ requires that variables ψ_p and ϕ_p be coupled; since, for fixed ψ_p we have

$$-\pi \leq \phi_p \leq \pi \qquad \text{but} \qquad -(\pi - |\phi_p|) \leq \psi_p \leq (\pi - |\phi_p|) \tag{4.36}$$

This recoupling of the variables through the boundary conditions requires that one carefully handle the Schroedinger problem. Having said that one must be careful, let me promise you that we have been. The key point is that for $\omega_{\phi_p} = 2$ we have the groundstate wave function $e^{-m_\phi\omega_\phi(\phi_p^2)/2} = e^{-f^2\phi_p^2}$ and for large f (f >> 1) it is clear that the system doesn't see the boundary to

any great degree. Moreover, it is clear that the mean value of ϕ_p is in the range 1/f, hence the ψ_p variable (up to terms on the order of 1/f) can be considered to be a rotor with periodic boundary conditions on the interval $(-\pi,\pi)$. The aim of this brief discussion is to show why a caref study shows that for large f, naively treating h_p as the Hamiltonian of an uncoupled rotor and oscillator is O.K. The next step is to couple two such blocks together through the typical coupling

$$-f^2 \cos\left(\psi_{p+1} - \psi_p + (\phi_{p+1} - \phi_p)\right) \quad , \tag{4.37}$$

show that it becomes a system of one rotor and three oscillators, and then truncate away states generated by having the two oscillators of highest mass out of their groundstate. This truncation brings us back to an effective Hamiltonian of a system which is one rotor and one oscillator. All that will have changed is the coefficients of the various terms. Let me briefly sketch how this goes by noting that at the nth iteration, the truncated Hamiltonian takes the generic form

$$H_n = \sum_p \left[C_n \mathbb{1}_p + \frac{1}{2} \alpha_n^2 \left(\frac{-\partial^2}{\partial \psi_p^2} \right) + \frac{1}{2} \left(- \frac{\partial^2}{\partial \phi_p^2} + \omega_n^2 \phi_p^2 \right) \right.$$
$$\left. - \beta_n \cos\left(\psi_{p+1} - \psi_p + \delta_n(\phi_{p+1} - \phi_p)\right) \right] \tag{4.38}$$

and if we define a superblock of 4-sites by letting $p = 2\ell + s$, $s = 0,1$, then we get an effective superblock Hamiltonian

$$h_\ell = \left[2C_n \mathbb{1} - \frac{1}{2}\alpha_n^2\left(\frac{\partial^2}{\partial\psi_{2\ell}^2} + \frac{\partial^2}{\partial\psi_{2\ell+1}^2}\right) - \frac{1}{2}\left(\frac{\partial^2}{\partial\phi_{2\ell}^2} + \frac{\partial^2}{\partial\phi_{2\ell+1}^2}\right)\right.$$
$$\left. + \frac{\omega_n^2}{2}\left(\phi_{2\ell}^2 + \phi_{2\ell+1}^2\right) - \beta_n \cos\left(\psi_{2\ell+1} - \psi_{2\ell} + \delta_n(\phi_{2\ell+1} + \phi_{2\ell})\right)\right]. \tag{4.39}$$

If we have β_n/α_n^2 sufficiently large, then we can justify expanding the cosine and keeping only quadratic terms. This reduces the h_ℓ problem to a system of one rotor and three coupled oscillators; diagonalizing this and truncating away the two highest oscillators gives us C_{n+1}, α_{n+1}^2, β_{n+1}, ω_{n+1}^2 and δ_{n+1}. The results of one such iteration for two values of $x_0 = 1/f$ are given in Tables 8 and 9. The variable K_n is defined by $\beta_{n+1} = \beta_n e^{-K_n x_0^2 \beta_n}$. Although I will not prove it now, it is easy to convince oneself that $\lim_{n\to\infty}(\beta_n^{1/2})$ is the expectation value, in the variationally constructed groundstate, of $\frac{1}{f}\, e^{i\theta_j}$. From the fact that K_n goes to a constant after a few iterations, it follows that after a few iterations we can write

$$\beta_n^{1/2} = \bar{\beta}^{1/2}\, e^{-(.368\ldots)n/2f^2} \tag{4.40}$$

Since the volume of the block under consideration gives as $2(2^n)$, we have that

$$n = \frac{\ell n(v)}{\ell n(z)} - 1 \tag{4.41}$$

or

$$\beta_n^{1/2} = \bar{\beta}'\, e^{-(.368..)\ell n(v)/f^2}$$
$$= \bar{\beta}'\left(\frac{1}{v}\right)^{(.368..)/f^2} \tag{4.42}$$

Hence, we find that as the volume of the world goes to infinity we predict that $\langle e^{i\theta(j)} \rangle$ tends to zero as a constant times $(1/V)^{(0.36821/f^2)}$. This is the result predicted from analyses based on a treatment of $\theta(j)$ as a massless free field and is not easily reproduced from a first principles calculation.

Summary

What we have shown in this discussion is that the same iterative procedure we have used to discuss theories describing systems having finite number of states at a site works well in describing bosonic systems which have an infinite number of states at a site--if we truncate so as to preserve a full field operator per site. Moreover, we have seen that the physical insight obtained from the Hamiltonian picture makes it trivial to see that the theory undergoes a transition to a massive phase for f sufficiently small and that it describes a massless theory for $f \ll 1$. Although we have not discussed the transition region in these lectures, the same iteration procedures--with greater care spent upon the recoupling of oscillators in a block through the boundary conditions--could be used to calculate for all 'f' and study the nature of the transition. We have not chosen to do so, simply because little is known in the way of precise information about the behavior at the critical point, and we are using this model to test our method. This brings us to the final point: we have shown that the naive truncation procedure is powerful enough to predict a transition to a Goldstone phase when it occurs, and to predict that $f\langle e^{i\theta_j} \rangle = \langle \phi(j) \rangle = 0$ in one dimension, where--contrary to classical arguments--the transition to a Goldstone phase is known not to occur.

Closing Remarks

Although I will have no time to speak about it, we are now actively studying the application of these same methods to gauge theories. At present, we have focused attention on Abelian theories in all dimensions in order to find out whether or not the prescriptions given by Wilson and Kogut and Susskind correctly describe quantum electrodynamics for any range of couplings. This is an important question, since for Abelian theories can be formulated in a different manner than that given by Wilson and this formulation manifestly reproduces QED; hence, if the Wilson prescription winds up confining electrons for all couplings in 1 + 3 dimensions, one will worry whether or not confinement--as discussed within the framework of lattice theories--has anything to do with properties (such as asymptotic freedom) associated with the continuum theory.

To date, we have found that these methods can be straightforwardly extended to gauge theories once one properly understands the significance of local gauge invariance. In particular, Helen Quinn and I have used this method to analyze the 1 + 1-dimensional Higgs model--obtained by gauging the U(1) Goldstone model just discussed. We have shown that from the Hamiltonian point of view one can trivially, in a strictly physical way, obtain all of the results of analyses based upon instantons.

Up to this point we have seen no reason to believe that the same methods will not extend in a straightforward manner to non-Abelian gauge theories. However, that work is currently in progress and so we have no solid statements we would care to make about it at this time.

Table 1*

State_p	Energy_p	Gap from lowest state
$(\|\downarrow\downarrow\rangle + a_0\|\uparrow\uparrow\rangle)/\sqrt{1+a_0^2}$	$-\sqrt{\epsilon_0^2+\Delta_0^2}$	0
$(\|\downarrow\uparrow\rangle + \|\uparrow\downarrow\rangle)/\sqrt{2}$	$-\Delta_0$	$\sqrt{\epsilon_0^2+\Delta_0^2}-\Delta_0$
$(\|\downarrow\uparrow\rangle - \|\uparrow\downarrow\rangle)/\sqrt{2}$	$+\Delta_0$	$\sqrt{\epsilon_0^2+\Delta_0^2}+\Delta_0$
$(-a_0\|\downarrow\downarrow\rangle + \|\uparrow\uparrow\rangle)/\sqrt{1+a_0^2}$	$\sqrt{\epsilon_0^2+\Delta_0^2}$	$2\sqrt{\epsilon_0^2+\Delta_0^2}$

*$a_0 = (\sqrt{\epsilon_0^2+\Delta_0^2}-\epsilon_0)/\Delta_0$

Table 2

State	$Q_j = n_b(j) - n_d(j)$	$Q_5(j) = n_b(j) + n_d(j) - 1$
$\|0_j\rangle$	0	−1
$\|+_j\rangle \equiv b_j^\dagger \|0_j\rangle$	+1	0
$\|-_j\rangle \equiv d_j^\dagger \|0_j\rangle$	−1	0
$\|\pm_j\rangle \equiv b_j^\dagger d_j^\dagger \|0_j\rangle$	0	+1

Table 3. Q=0 sector

Q_5	State	iCP Transform
-3	$\|000\rangle$	$-\|000\rangle$
-1	$\|\pm 00\rangle$	$\|00\pm\rangle$
	$\|00\pm\rangle$	$\|\pm 00\rangle$
	$\|0\pm 0\rangle$	$\|0\pm 0\rangle$
	$\|+-0\rangle$	$\|0+-\rangle$
	$\|0+-\rangle$	$\|+-0\rangle$
	$\|+0-\rangle$	$\|+0-\rangle$
	$\|-+0\rangle$	$\|0-+\rangle$
	$\|0-+\rangle$	$\|-+0\rangle$
	$\|-0+\rangle$	$\|-0+\rangle$
+1	$\|0\pm\pm\rangle$	$-\|\pm\pm 0\rangle$
	$\|\pm\pm 0\rangle$	$-\|0\pm\pm\rangle$
	$\|\pm 0\pm\rangle$	$-\|\pm 0\pm\rangle$
	$\|-+\pm\rangle$	$-\|\pm-+\rangle$
	$\|\pm-+\rangle$	$-\|-+\pm\rangle$
	$\|-\pm+\rangle$	$-\|-\pm+\rangle$
	$\|+-\pm\rangle$	$-\|\pm+-\rangle$
	$\|\pm+-\rangle$	$-\|+-\pm\rangle$
	$\|+\pm-\rangle$	$-\|+\pm-\rangle$
+3	$\|\pm\pm\pm\rangle$	$\|\pm\pm\pm\rangle$

Table 4. Q=1 sector

Q_5	State	P Transform
-2	$\|+00\rangle$	$-\|\pm\pm+\rangle$
	$\|00+\rangle$	$-\|+\pm\pm\rangle$
	$\|0+0\rangle$	$-\|\pm+\pm\rangle$
0	$\|+0\pm\rangle$	$-\|0\pm+\rangle$
	$\|0\pm+\rangle$	$-\|+0\pm\rangle$
	$\|0+\pm\rangle$	$-\|0+\pm\rangle$
	$\|+\pm0\rangle$	$-\|\pm0+\rangle$
	$\|\pm0+\rangle$	$-\|+\pm0\rangle$
	$\|\pm+0\rangle$	$-\|\pm+0\rangle$
	$\|+-+\rangle$	$-\|+-+\rangle$
	$\|++-\rangle$	$-\|-++\rangle$
	$\|-++\rangle$	$-\|++-\rangle$
2	$\|+\pm\pm\rangle$	$-\|00+\rangle$
	$\|\pm\pm+\rangle$	$-\|+00\rangle$
	$\|\pm+\pm\rangle$	$-\|0+0\rangle$

Table 5. g=0

Iteration (n)	$X_n(1)$	g_{eff}	$X_{norm}(j=1,..,5)$	$(A1)_n/X_n(1)$	$(A2)_n/X_n(1)$
1	−.53333	0	1.00000 .81897 .81187 .81071 .81040	−.625	−.18519
2	−.31376	0	1.00000 .60875 .60204 .60100 .60072	−.46176	−.13878
3	−.18805	0	1.00000 .49614 .49103 .49024 .49003	−.38047	−.10942
15	-2.9897×10^{-4}	0	1.00000 .31256 .30977 .30934 .30922	−.24386	−.06528
20	-1.9376×10^{-5}	0	1.00000 .30931 .30655 .30612 .30601	−.24138	−.06455

Table 6. g=.1

Iteration (n)	$X_n(1)$	g_{eff}	$X_{norm}(j=1,..,5)$	$(A1)_n/X_n(1)$	$(A2)_n/X_n(1)$
1	−.53265	.076589	1.00000 .81896 .81187 .81072 .81040	−.62498	−.18521
2	−.31313	.058668	1.00000 .60872 .60201 .60098 .60070	−.46173	−.13878
3	−.18760	.046113	1.00000 .49611 .49100 .49021 .49001	−.38044	−.10942
15	-2.9803×10^{-4}	.005397	1.00000 .31256 .30977 .30934 .30922	−.24386	−.06528
20	-1.9315×10^{-5}	.002389	1.00000 .30931 .30655 .30612 .30601	−.24138	−.06455

Table 7. g=2

Iteration (n)	$X_n(1)$	g_{eff}	$X_{norm}(j=1,..,5)$	$(A1)_n/X_n(1)$	$(A2)_n/X_n(1)$
1	−.37452	2.9147	1.00000 .81734 .81111 .81034 .81022	−.61755	−.19282
2	−.12898	5.4885	1.00000 .58901 .58319 .58255 .58244	−.43935	−.14330
3	−.032991	16.712	1.00000 .44220 .43814 .43778 .43778	−.32930	−.10871
10	-1.6136×10^{-9}	-3.0816×10^{8}	1.00000 .00119 .00120 .00123 .00124	-6.2370×10^{-4}	-6.4155×10^{-4}
15	4.1286×10^{-15}	$1.2044\times10^{+14}$	1.00000 -2.0661×10^{-5} -2.7470×10^{-5} -3.0416×10^{-5} -3.1892×10^{-5}	1.6849×10^{-5}	1.7855×10^{-5}

Table 8. Iteration for $x_0 = .1$. The notation in this table conforms to that given in Eq (3.15) except for the definition of K_n. This is defined by the relation $\beta_{n+1} = e^{-k_n x_0^2} \beta_n$. Note in particular that ω_n^2 decreases by a factor of 2^2 for each iteration and hence $\omega_N \propto \frac{1}{2}^N \propto (\text{volume})^{-1}$.

Iteration (n)	$(\alpha_n)^2$	$(\omega_n)^2$	β_n	δ_n	K_n	c_n
0	.01	2	50	.1		-50
1	.005	.76393	49.906	.097325	.18831	-149.08
2	.0025	.22226	49.77	.077379	.27347	-347.38
3	.00125	.059722	49.612	.05778	.31801	-744.05
8	3.9062×10^{-5}	6.3112×10^{-5}	48.733	.010803	.36635	-25336
9	1.9531×10^{-5}	1.5802×10^{-5}	48.554	.0076468	.36727	-50721
10	9.7656×10^{-6}	3.9535×10^{-6}	48.376	.0054099	.36774	-101490
14	6.1035×10^{-7}	1.5454×10^{-8}	47.669	.0013531	.36819	-1624562
15	3.0518×10^{-7}	3.8637×10^{-9}	47.493	9.5682×10^{-4}	.3682	-324171
16	1.5259×10^{-7}	9.6594×10^{-10}	47.319	6.7658×10^{-4}	.36821	-6498389
17	7.6294×10^{-8}	2.4149×10^{-10}	47.145	4.7841×10^{-4}	.36821	-12996826

Table 9. Iteration for $x_0 = .01$. This table is included to show that for both $x_0 = .1$ and $x_0 = .01$ the iteration is basically the same up to a scale factor. The fact that α_n^2 and ω_n^2 both drop rapidly with respect to β_n tells us that the oscillator approximation is valid at all stages.

Iteration$_{(n)}$	$(\alpha_n)^2$	$(\omega_n)^2$	β_n	δ_n	K_n	c_n
0	.0001	2	5000	.01		-5000
1	5×10^{-5}	.76393	4999.9	.0097325	.18831	-14999
2	2.5×10^{-5}	.22233	4999.8	.0077385	.27346	-34998
3	1.25×10^{-5}	.059757	4999.6	.0057788	.31801	-74994
9	1.9531×10^{-7}	1.5813×10^{-5}	4998.5	7.6469×10^{-4}	.36717	-5114571
10	9.7656×10^{-8}	3.9562×10^{-6}	4998.3	5.4099×10^{-4}	.36763	-10234139
11	4.8828×10^{-8}	9.8941×10^{-7}	4998.2	3.8264×10^{-4}	.36786	-20473277
14	6.1035×10^{-9}	1.5465×10^{-8}	4997.6	1.3531×10^{-4}	.36807	-1.64×10^{8}
15	3.0518×10^{-9}	3.8663×10^{-9}	4997.4	9.5682×10^{-5}	.36808	-3.2765×10^{8}
16	1.5259×10^{-9}	9.6658×10^{-10}	4997.2	6.7658×10^{-5}	.36809	-0.65530×10^{9}
17	7.6294×10^{-10}	2.4165×10^{-10}	4997.1	4.7841×10^{-5}	.36809	-1.31060×10^{9}

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FIGURE CAPTIONS

1. Two climbers attempting to scale what appears to be the same peak.
2. A bird's eye view may reveal that the peaks are, in fact, different.
3. A road map of some well travelled theoretical routes to the black box which presumably holds the secret of confined quarks.
4. Could there be a free and hungry colored quark in the black box?
5. Dispersion relation for cut-off free scalar field theory (solid curve) and latticized free scalar field theory (dashed curve).
6. The dispersion relation for the free fermion case.
7. Simultaneous eigenstates of $\sigma_x(j)$ and $H_{\epsilon_o=0}$.
8. Simultaneous eigenstates of $\sigma_z(j)$ and $H_{\Delta o=0}$.
9. The graph of $y_{n+1}-y_n$ ($\equiv R(y_n)$) vs. y.
10. Exact and variational calculation values of ϵ_o as a function of y.
11. Exact and variational calculation values of $\langle\sigma_x\rangle$ as a function of y.
12. Exact and variational calculation values of $-\partial^2\epsilon_o/\partial y^2$ as a function of y.
13. The quantity $\sqrt{Z_2(g)/Z_2(0)}$ is shown as a function of g.

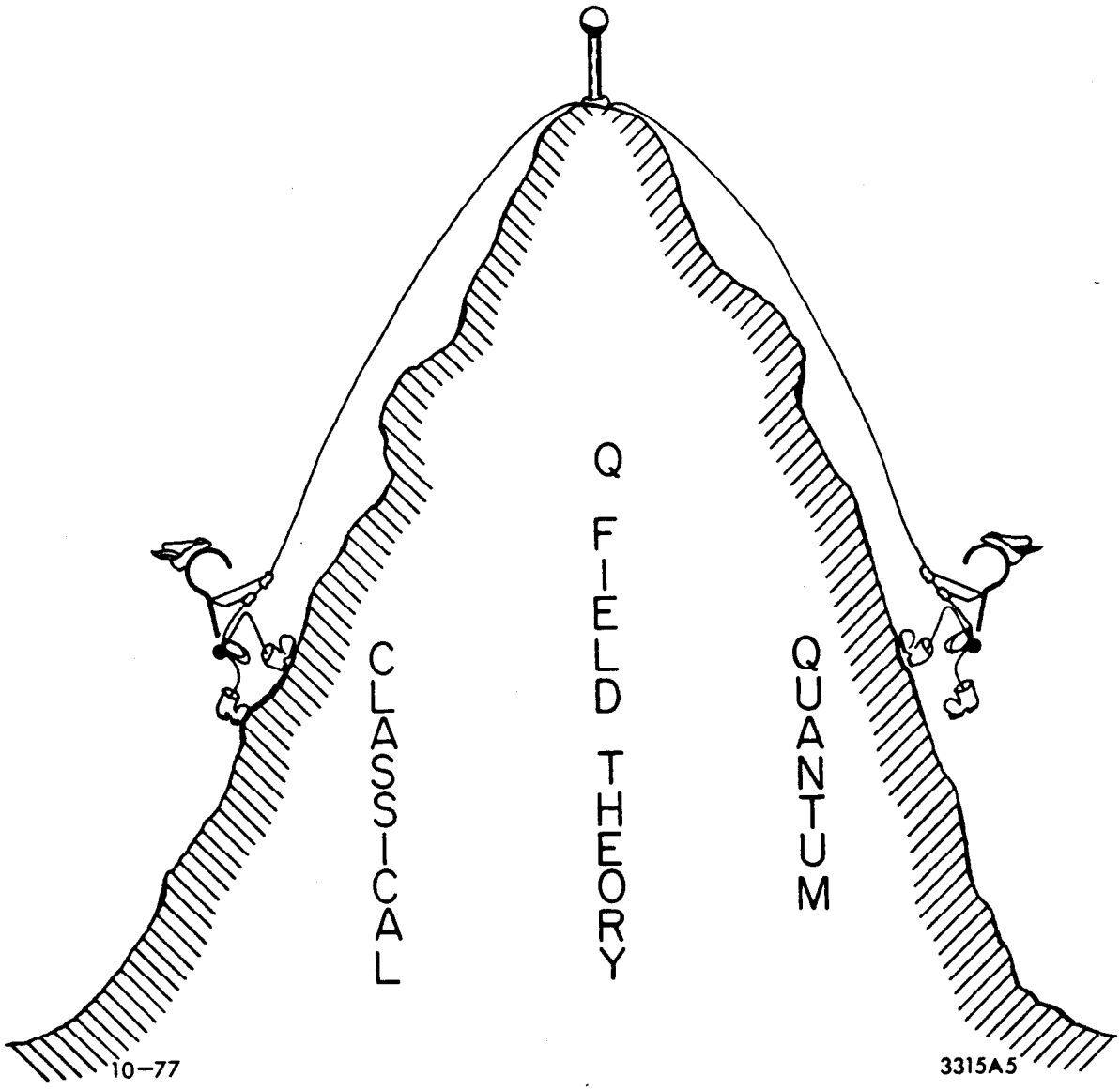


Fig. 1

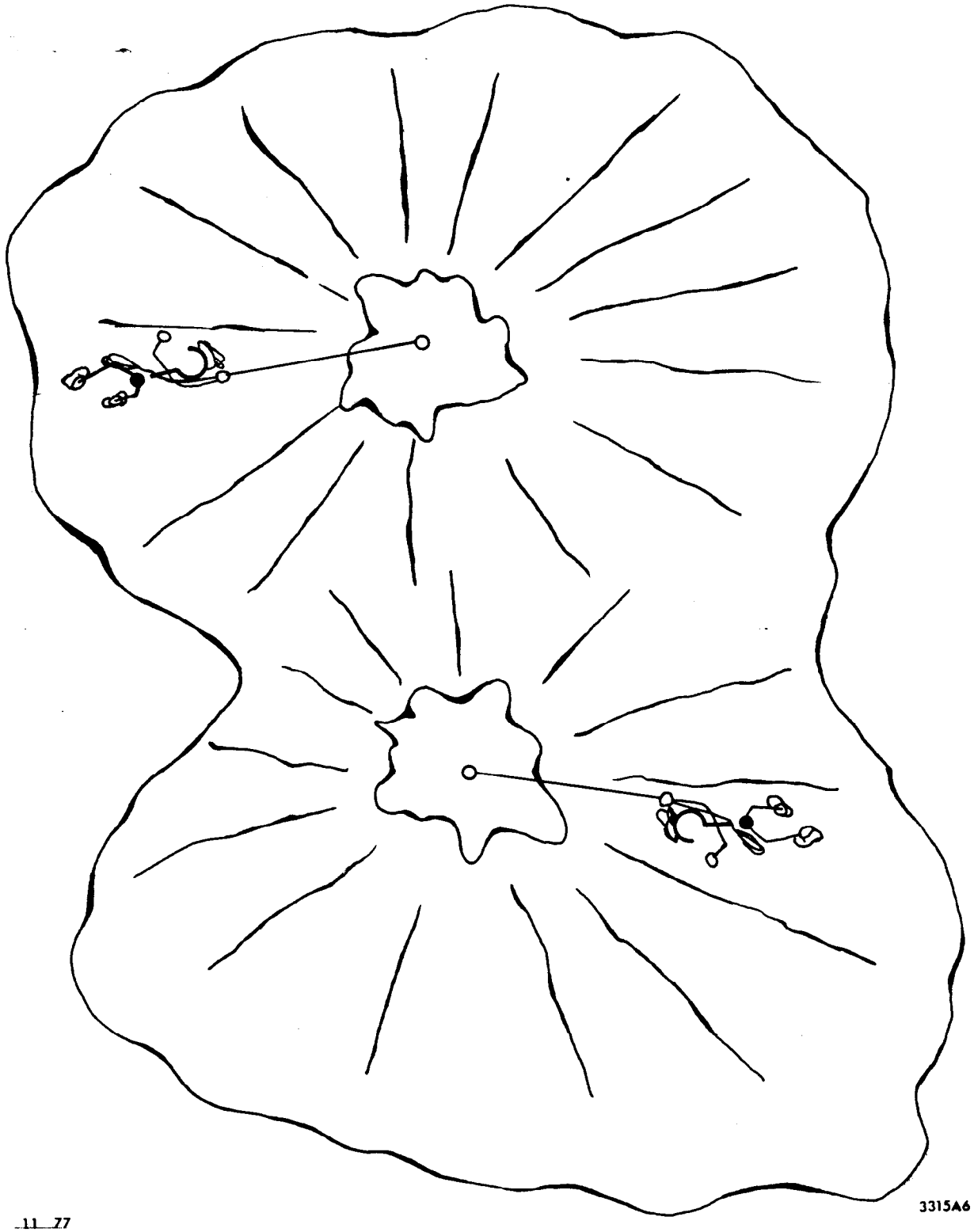


Fig. 2

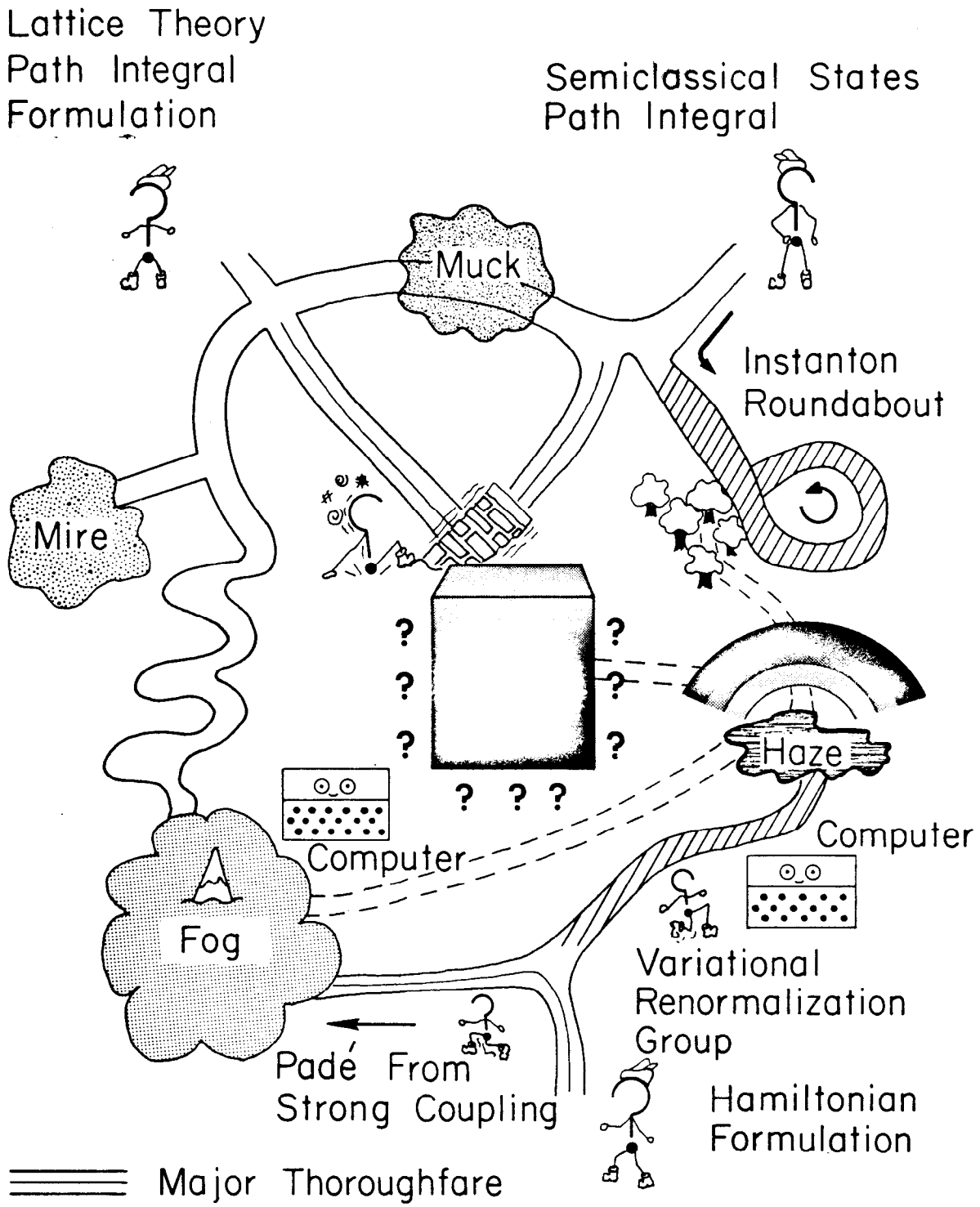

Lattice Theory
Path Integral
Formulation
Semiclassical States
Path Integral
Muck
Instanton
Roundabout
Mire
Haze
Computer
Computer
Fog
Variational
Renormalization
Group
Padé From
Strong Coupling
Hamiltonian
Formulation
Major Thoroughfare
Divided Highway
Main Road - Well Traveled
Country Road
Under Construction
11-77
3315B7

Fig. 3

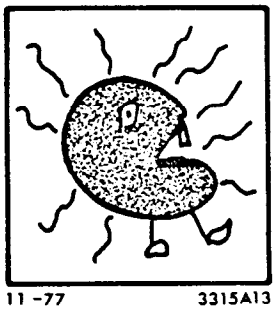

11-77
3315A13

Fig. 4

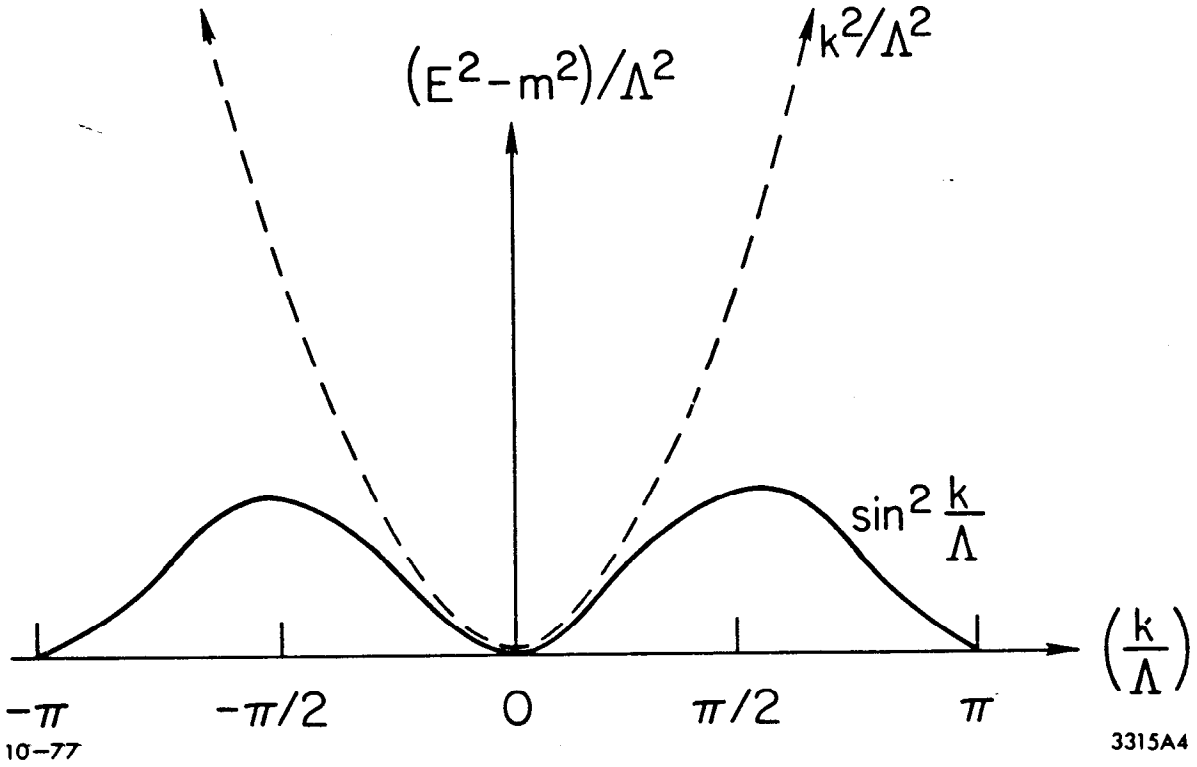

$(E^2-m^2)/\Lambda^2$
k^2/Λ^2
$\sin^2 \frac{k}{\Lambda}$
$\left(\frac{k}{\Lambda}\right)$
$-\pi$
$-\pi/2$
0
$\pi/2$
π
10-77
3315A4

Fig. 5

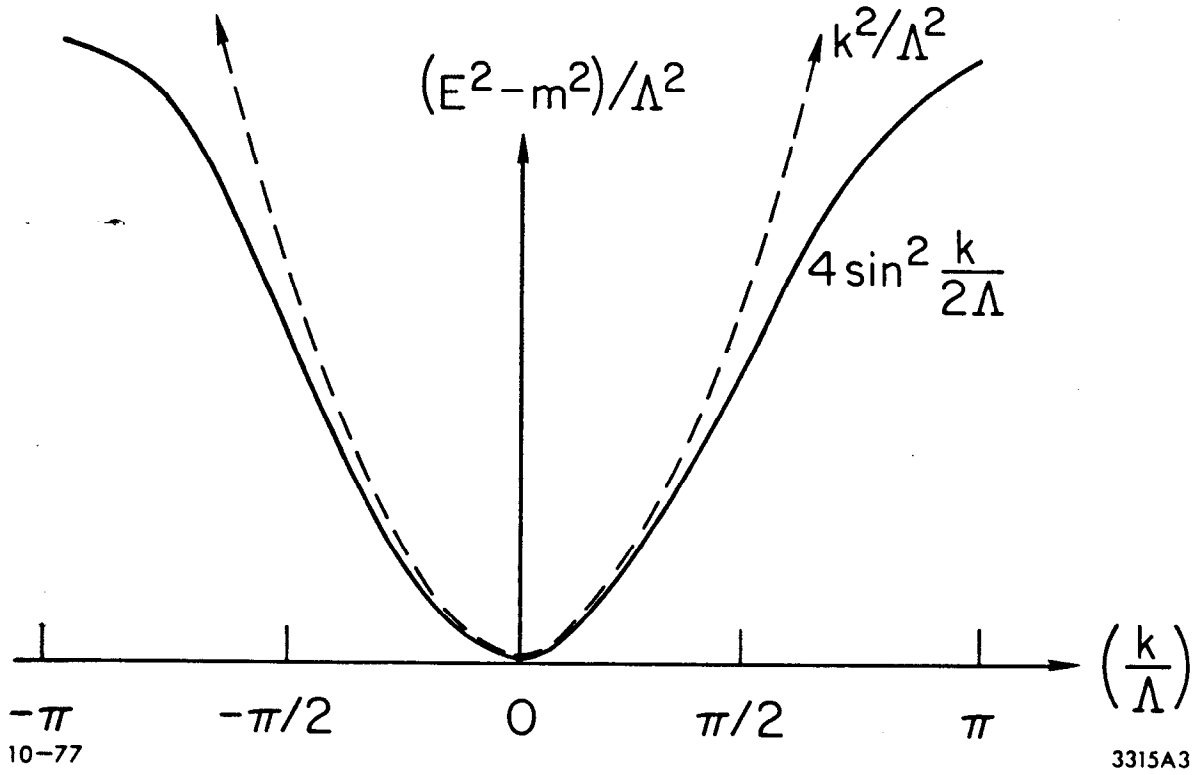


Fig. 6

$$\sigma_x(j)\,|\rightarrow\rangle_j = +|\rightarrow\rangle$$

$$\sigma_x(j)\,|\leftarrow\rangle = -|\leftarrow\rangle$$

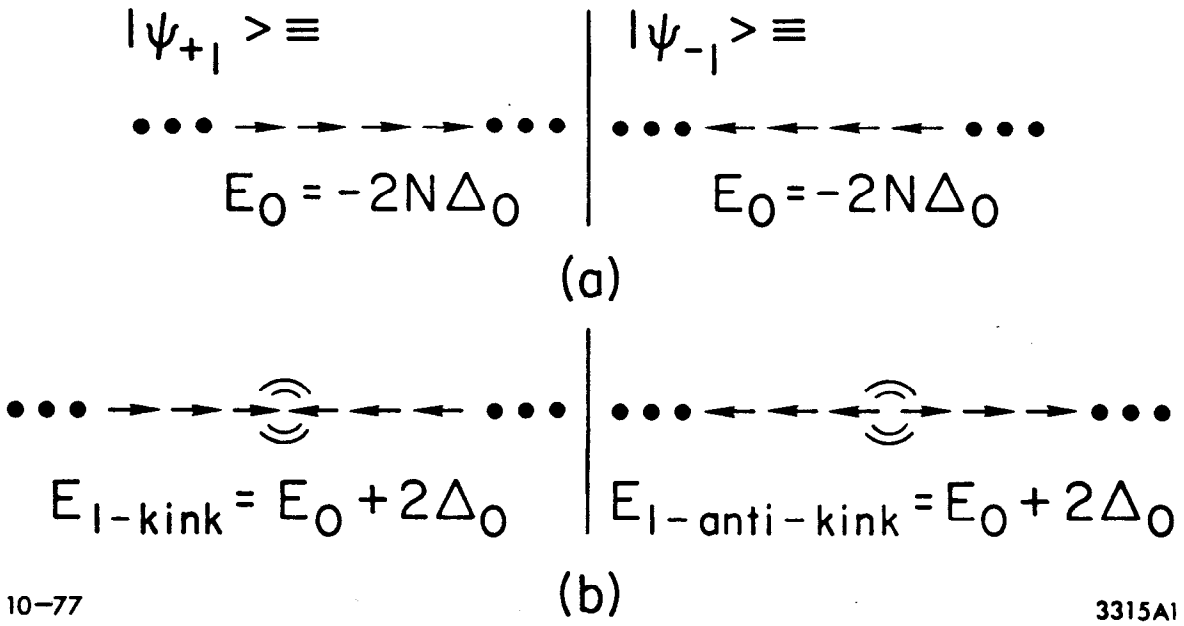


Fig. 7

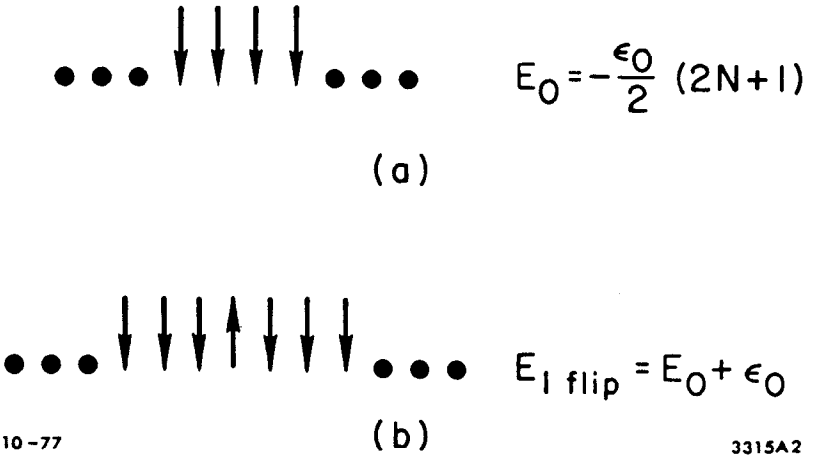

$E_0 = -\frac{\epsilon_0}{2}(2N+1)$
(a)
$E_{1\ flip} = E_0 + \epsilon_0$
10-77
(b)
3315A2

Fig. 8

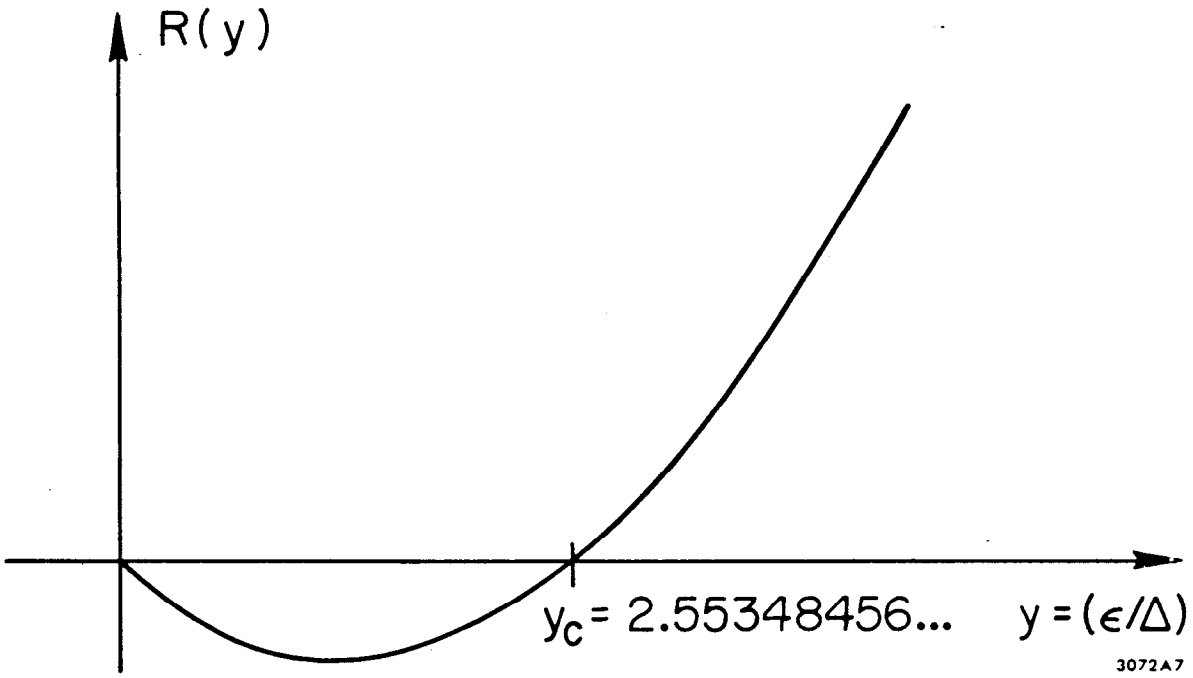

R(y)
$y_c = 2.55348456...$
$y = (\epsilon/\Delta)$
3072A7

Fig. 9

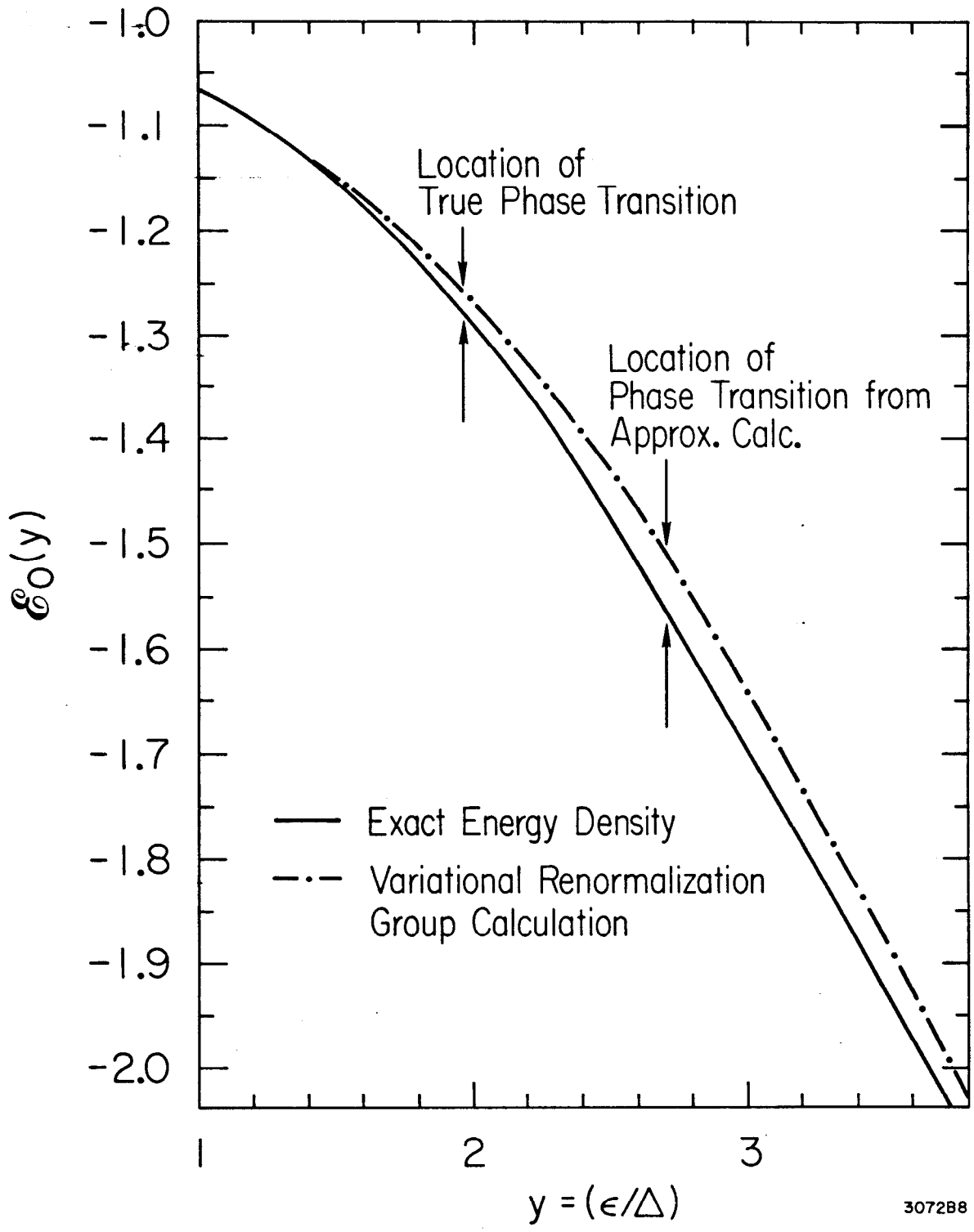

-1.0
-1.1
-1.2
-1.3
-1.4
-1.5
-1.6
-1.7
-1.8
-1.9
-2.0
$\mathcal{E}_0(y)$
Location of
True Phase Transition
Location of
Phase Transition from
Approx. Calc.
Exact Energy Density
Variational Renormalization
Group Calculation
1
2
3
$y = (\epsilon/\Delta)$
3072B8

Fig. 10

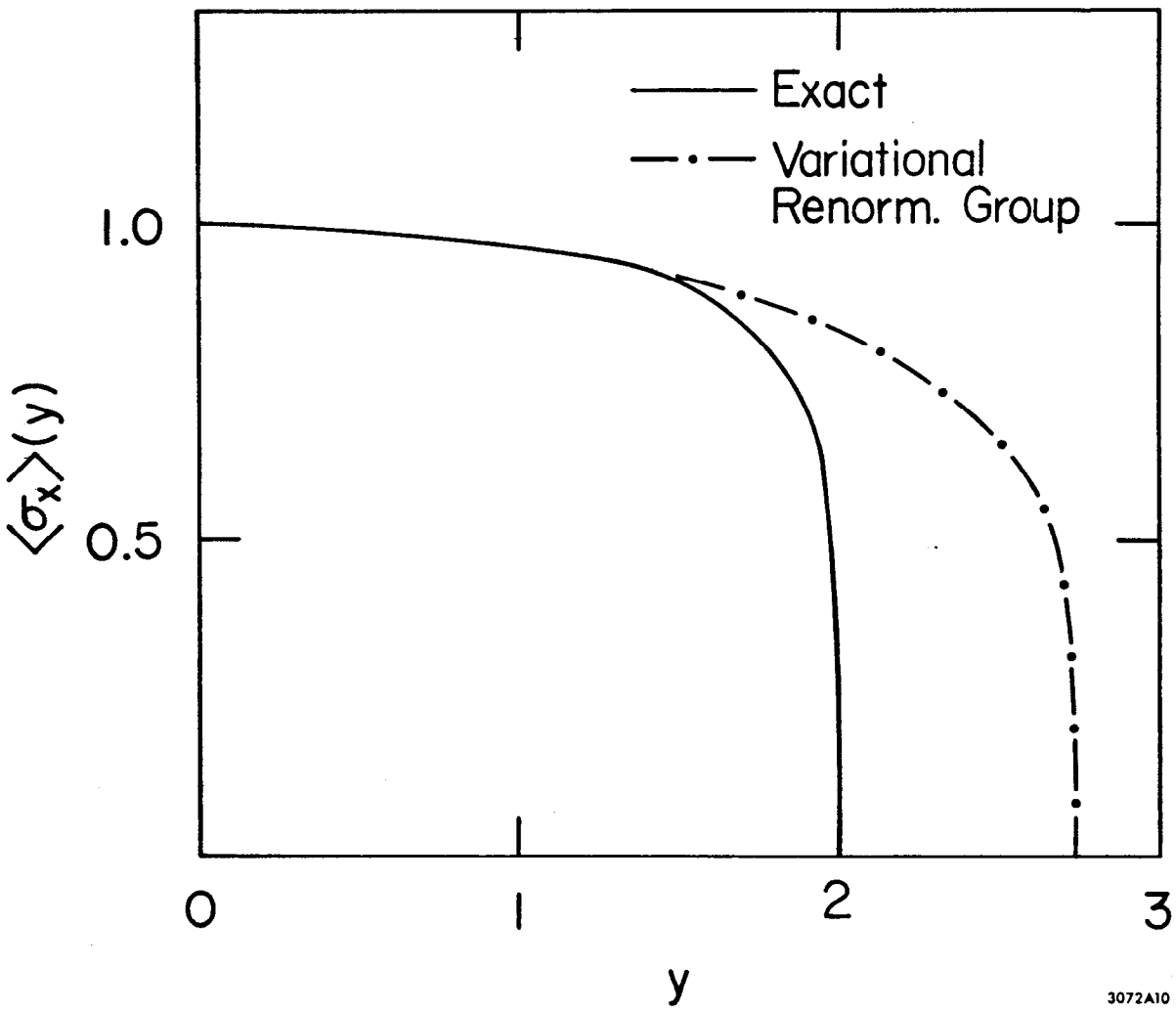

Exact
Variational
Renorm. Group
1.0
0.5
$\langle\sigma_x\rangle(y)$
0
1
2
3
y
3072A10

Fig. 11

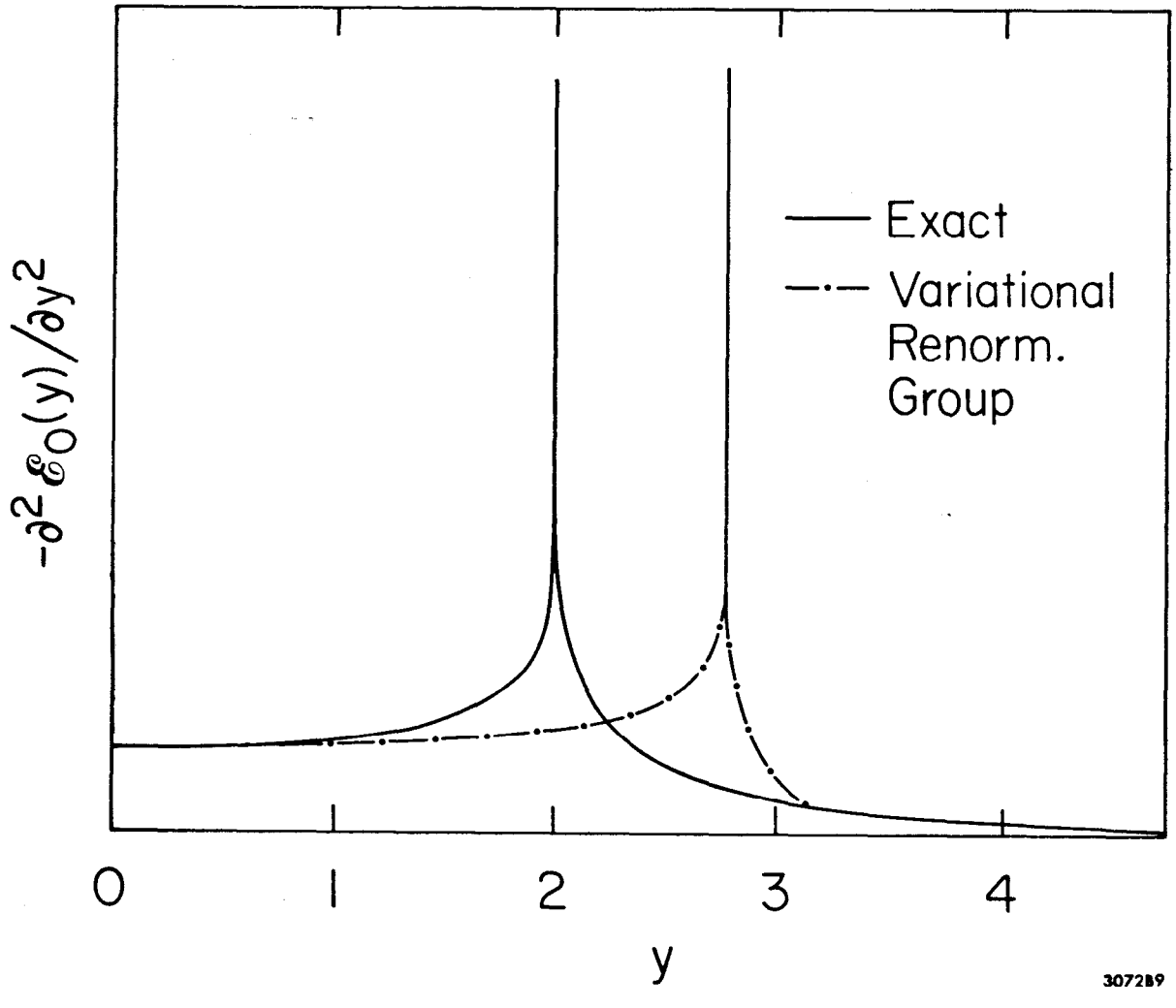
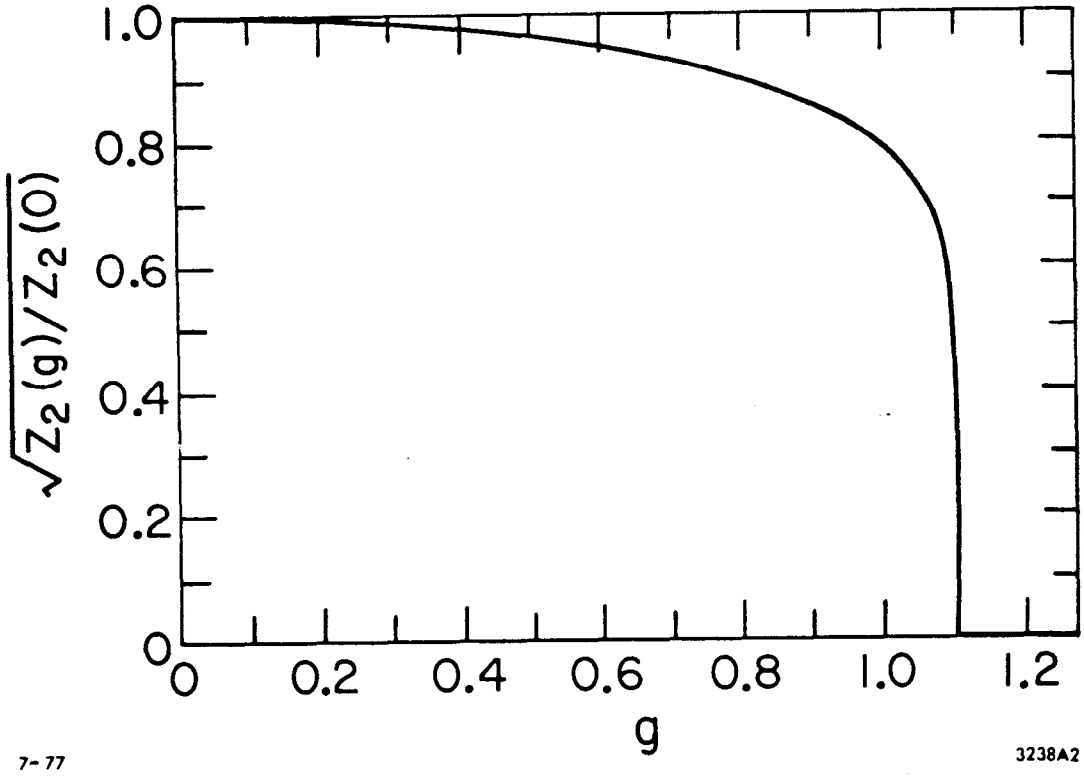

Exact
Variational
Renorm.
Group
$-\partial^2 \mathcal{E}_0(y)/\partial y^2$
0
1
2
3
4
y
307289

Fig. 12



7-77

3238A2

Fig. 13