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DIVERGENCE OF THE BETHE-SALPETER WAVE FUNCTION AT THE ORIGIN*

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ABSTRACT

We point out an error in the proof of Nishijima et al. that the Bethe-Salpeter wave function for composite particles, lying on Regge trajectories, is finite. We describe a simple counter-example drawn from non-relativistic quantum mechanics and cite others from field theory.

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In recent years it has become apparent that knowledge of the behavior of Bethe-Salpeter (BS) wave functions at short distances is important in the analysis of processes involving high transverse momentum.[1] In particular it is important to determine whether or not the wave functions are finite at the origin. Nishijima et al.[2] argue that the BS wave function can be infinite at x = 0 only if the composite particle is a fixed pole. This contradicts results obtained by Appelquist et al.[3] using the renormalization group. Their analysis suggests that the finiteness of the wave function depends crucially upon the anomalous dimensions of the field theory, at least when the theory is asymptotically free at short distances. The particular field theory they consider (ϕ^3 in 6 dimensions) actually has divergent wave functions for particles lying on Regge trajectories. Here we describe a crucial flaw in the proof of Nishijima et al. which renders their conclusions suspect and resolves the conflict with renormalization group analyses.

We first review the argument of Nishijima et al.[2] They consider the 3-point function describing the bound state-constituent-constituent vertex (on and off mass-shell). Adopting the HNZ construction for bound state interpolating fields, this vertex function Γ_n can be expressed in terms of the 4-point, constituent Green's function G and the BS wave function ϕ_n (assuming scalar constituents):

$$\Gamma_n(x, P) = \lim_{\xi \to 0} \frac{G(x, \xi, P)}{\phi_n(\xi)^*} \qquad (1)$$

$$\to \frac{\phi_n(x)}{P^2 - M_n^2} \text{ as } P^2 \to M_n^2$$

Here P is the total 4-momentum of the system and x, ξ are relative coordinates of the constituents. The BS equation for G is then used to write an equation for Γ_n:

$$\Delta_F^{-1}\,\Delta_F^{-1}\;\Gamma_n\,(x,\,P) = \lim_{\xi\to 0}\frac{\delta^4\,(x-\xi)}{\phi_n^*\,(\xi)} + \int d^4y\;K\,(x,\,y,\,P)\;\Gamma_n\,(y,\,P) \qquad (2)$$

Nishijima et al. argue that if $\phi_n^*\,(\xi)$ diverges as $\xi\to 0$, the inhomogeneous term in (2) (i.e., $\lim \delta^4\,(x-\xi)/\phi_n^*(\xi)$) vanishes and Γ_n satisfies an homogeneous equation (the BS wave function equation). Assuming appropriate boundary conditions, solutions of the homogeneous equation exist only at discrete values of the total energy (the bound state energies) when P^2 is below the continuum threshold. However, Γ_n, being a Green's function, is well defined and non-trivial for all energies and in particular for energies below threshold and intermediate between bound state energies. This apparent contradiction suggests that the BS wave function must be finite at x = 0.

In fact the root of this problem lies not in the behavior of the wave function, but rather in dropping the inhomogeneous term. When $\phi_n^* \to \infty$, the distribution

$$\lim_{\xi\to 0}\;\frac{\delta^4\,(x-\xi)}{\phi_n^*\,(\xi)}$$

clearly vanishes for $x \neq 0$ but at $x = 0$ it is ill-defined. That it does not vanish there is evident if we convolute it with some function f(x) which is as divergent as ϕ_n^* or more so:

$$\lim_{\xi\to 0}\int d^4x\;f(x)\;\frac{\delta^4\,(x-\xi)}{\phi_n^*\,(\xi)} = \lim_{\xi\to 0}\;\frac{f\,(\xi)}{\phi_n^*\,(\xi)} \neq 0$$

Equation (2) can be formally rewritten as

$$\Gamma_n = \lim_{\xi\to 0}\;\frac{1}{1-\Delta_F\,\Delta_F\,K}\;\Delta_F\,\Delta_F\;\frac{\delta^4\,(x-\xi)}{\phi_n^*\,(\xi)}$$

$$= \lim_{\xi \to 0} \int d^4x' \, G(x, x', P) \, \frac{\delta^4(x' - \xi)}{\phi_n^*(\xi)}$$

As Γ_n is well defined, the operator in brackets must be precisely as divergent as $\phi_n^*(\xi)$. Thus the inhomogeneous part of (2) does not vanish even though $1/\phi_n^*$ may. The equation must be solved first and then the limit $\xi \to 0$ taken.

The problem is illustrated by a simple counter-example drawn from non-relativistic quantum mechanics. Consider bound states described by a Hamiltonian

$$H = H_0 + V$$

$$H_0 = \frac{P^2}{2M} \qquad V = -\frac{\alpha}{r} - \frac{\gamma}{2mr^2} \qquad 0 < \alpha, \ \gamma \ll 1$$

When $\gamma = \alpha^2$ the spectrum of this Hamiltonian is very similar to that of positronium or of the hydrogen atom. The bound states lie on Regge trajectories and are not fixed poles, and yet all s-state wave functions diverge at the origin:[4]

$$(H - E_n)\,\psi_n = 0 \longrightarrow (P^2 - \frac{\gamma}{r^2})\;\psi_n(r) = 0 \quad \text{as } r \to 0$$

$$\Longrightarrow \psi_n(r) \sim r^s \quad \text{as} \quad r \to 0$$

$$\text{where } s = \frac{-1}{2} + \sqrt{\frac{1}{4} - \gamma} < 0$$

As above, we define a Green's function and s-state vertex function:

$$(E - H_0)\; G(\vec{r}\, \vec{r}'\, E) = \delta^3(\vec{r} - \vec{r}') + V(r)\; G(\vec{r}\, \vec{r}' E) \tag{3}$$

$$\Gamma_n(\vec{r}\, E) = \lim_{\xi \to 0} \frac{G(\vec{r}\, \vec{\xi}\, E)}{\psi_n(\xi)^*} \longrightarrow \frac{\psi_n(\vec{r})}{E - E_n} \quad \text{as } E \to E_n \tag{4}$$

The short distance behavior of both G and ψ_n is determined by the kinetic and $1/r^2$ terms in H. It is energy independent and is identical in both functions. Thus Γ_n is a finite non-trivial function of r for all $E \neq E_n$. Combining (3) and (4) we obtain

$$(E - H_0)\,\Gamma_n(\vec{r}\,E) = \lim_{\xi \to 0} \frac{\delta^3\,(r - \xi)}{\psi_n(\xi)^*} + V(r)\,\Gamma_n(\vec{r}\;\;E) \qquad (5a)$$

$$\Longrightarrow \Gamma_n = \lim_{\xi \to 0}\left(\frac{1}{1 - (E - H_0)^{-1}\,V}\;(E - H_0)^{-1}\right)\frac{\delta^3\,(\vec{r} - \vec{\xi})}{\psi_n\,(\xi)^*} \qquad (5b)$$

$$= \lim_{\xi \to 0} \int d^3r' \;\; G(\vec{r}\;\vec{r}\,'E)\;\; \frac{\delta^3\,(\vec{r}\,' - \vec{\xi})}{\psi_n(\xi)^*}$$

If, following Ref. 2, we drop the inhomogeneous term in (5a), we find that $\Gamma_n(\vec{r},\ E)$ must grow exponentially as $r \to \infty$ for any $E(\neq E_n)$ below zero. This behavior is quite unphysical. Again the problem is resolved when we note that the operator in brackets ($=G$) in (5b) is as divergent as $\psi_n(\xi)^*$ and thus it is incorrect to omit the inhomogeneous part of (5a). This argument is still more compelling if we introduce the spectral decomposition of $\delta^3(\vec{r} - \vec{\xi})$ into the inhomogeneous term:

$$\lim_{\xi \to 0} \frac{\delta^3\,(\vec{r} - \vec{\xi})}{\psi_n(\xi)} = \lim_{\xi \to 0} \sum_j \psi_j(\vec{r})\,\frac{\psi_j(\xi)^*}{\psi_n(\xi)^*}$$

$$\equiv \sum_j c_j\;\psi_j(\vec{r})$$

$$\neq 0$$

The constants c_j are finite and non-zero when ψ_j describes an s-state ($c_n = 1$). They vanish otherwise.

This sort of behavior occurs in other systems. The Dirac equation with a Coulomb kernel has divergent wave functions. The BS equation, with a kernel containing all irreducible ladder and cross ladder Coulomb interactions, almost certainly has divergent wave functions because it reduces to the Dirac-Colomb equation when the mass of one constituent is made infinite. Models calculations by Guth and Soper,[5] for massive vector meson exchange in the ladder approxim ation, also exhibit a wave function which diverges at the origin.

In principle then, there seems to be no reason why the BS wave function for particles lying on Regge trajectories must be finite at the origin so long as it can be normalized. It should be noted however when they diverge, wave functions in (U. V.) asymptotically free theories seem to diverge only logarithmically.[3] Thus dimensional counting predictions for form factors, etc.[1] are modified only by logarithms.

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