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MULTIPLE VACUA IN A LATTICE FORMULATION OF THE TWO-DIMENSIONAL ABELIAN HIGGS MODEL*

Helen R. Quinn†
Stanford Linear Accelerator Center
and
Department of Physics, Stanford University

Marvin Weinstein
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

ABSTRACT

This paper presents a lattice formulation of the Abelian one-space-one-time dimensional Higgs model. The emphasis of our study is to show that this formulation correctly gives all the physics of a theory with multiple vacua. We find that these effects have a simple physical interpretation in terms of a background E-field. Comparison is made to results from a path integral calculation including instanton contributions.

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I. INTRODUCTION

In this note we study a lattice formulation of the one-space-one-time dimensional Abelian Higgs model. This model is of particular interest because its continuum version shares with the higher dimensional non-Abelian gauge theories the property that there exist solutions to its Euclidean field equations which carry a non-vanishing topological charge.[1] The existence of such solutions implies that the naive perturbation theory vacuum is not the true vacuum of the theory, and that there are in fact an infinity of distinct possible worlds, each based on a different vacuum labelled by an angle Θ, $-\pi \leq \Theta \leq \pi$.[2] The question which we address in this paper is whether or not the lattice formulation of this theory and its analysis by variational methods will automatically take into account the existence of these multiple Θ-vacua.

As we will show, not only does a direct reformulation of the continuum theory as a lattice gauge theory allow us to discover this effect, but it is forced upon us as a consequence of the gauge invariance of both the Hamiltonian and the algebra of physical observables. It will be made clear in the discussion to follow that while the specific source of Θ-parameter labeling the different versions of the theory varies somewhat in different gauges, the existence of the phenomenon is easily understood in any gauge without reference to the instanton. In particular, we will see that in the $A_0 = 0$ version of the theory the existence of the Θ-vacua is obvious once one properly understands the physical significance of the residual gauge invariance of the Hamiltonian. We believe that the reformulation of the question in these terms is particularly physical.

Our subsequent discussion will proceed as follows: In Section 2 we will define the lattice theory of interest and following the procedure of Ref. 3 reduce it to a simpler Schroedinger problem. This section presents discussions in both axial-gauges $A_0 = 0$ and $A_1 = 0$ in order to show the significance of the parameter Θ in each case. We will establish the relationship between Θ and the possibility of having a background electric field in 1-dimension which cannot be completely shielded by producing pairs of positive and negative charges which migrate to the boundaries of the defining volume. This phenomenon is exactly the same as that discussed by S. Coleman for the massive Schwinger model.[4] Section 3 will be devoted to a simple analysis of the lattice theory for certain ranges of parameters. The discussion here is for the purpose of elucidating certain interesting features of the theory and is not meant to be complete; in particular, the variational analysis presented is sketchy and for full details the reader is referred to Ref. 5 where a similar procedure is fully discussed for the U(1) Goldstone model (the $e = 0$ limit of this Higgs model). Section 4 of this paper makes some comments about the way in which one expects these results to generalize to Yang-Mills theories in higher dimensions.

II. THE LATTICE HIGGS MODEL

We begin with a formulation of the $A_0 = 0$ lattice theory and follow this discussion with the $A_1 = 0$ or "Coulomb gauge" formulation of the same theory. The approach we adopt is directed towards a Hamiltonian formulation of the problem and so, from a Lagrangian viewpoint time is a continuous variable and space is taken to be discrete. Our introduction of the fields (A_0, A_1) to enable us to define gauge covariant derivatives is standard, except that we will allow the fields A_0 or A_1 to take values over the range $-\infty \leq A_0, A_1, \leq \infty$. Our discussion of the procedure for the introduction of these concepts on a lattice will be quite brief and the reader is referred to Refs. 3 and 5 for further details.

$A_0 = 0$ Gauge

In defining $A_0 = 0$ we remove the freedom of making time-dependent gauge transformations and so it is only necessary to define covariant derivatives with respect to spatial directions. The continuum Lagrangian is taken to be

$$\mathscr{L} = \tfrac{1}{2}(\partial_0 A_1)^2 + (\partial_0\phi^*)(\partial_0\phi) - (\partial_1 + ieA_1)\,\phi^*\,(\partial_1 - ieA_1)\,\phi - \lambda(2\phi^*\phi - f^2)^2 \tag{2.1}$$

In going to the lattice we discretize the space variable and allow 'x' to take the values $x_j = j/\Lambda$ where 'j' runs over the integers from -N to +N. Hence the "volume" of our system is $L = (2N+1)/\Lambda$. Following common practice[6] we introduce the complex fields $\phi(j)$ and try to write a lattice Lagrangian which is invariant with respect to local gauge transformations

$$\phi(j) \longrightarrow \phi_\theta(j) \equiv e^{i\frac{e}{\Lambda}\theta(j)}\phi(j) \tag{2.2}$$

This is done by first introducing a field, $A_1(j)$ defined on the link joining 'j' to 'j+1', which transforms under this gauge transformation as

$$A_1(j) \longrightarrow A_\theta(j) \equiv A(j) + \theta(j+1) - \theta(j) \tag{2.3}$$

where $A_1(j)$ can take all real values from $-\infty$ to $+\infty$. We then define the covariant derivative of $\phi(j)$ as

$$(D\phi)_\Lambda(j) = \Lambda\left[e^{\frac{-ie}{\Lambda}A_1(j)}\phi(j+1) - \phi(j)\right] \tag{2.4}$$

which under a gauge transformation generated by a function $\theta(j)$ becomes

$$(D\phi)^\theta_\Lambda(j) = \Lambda\left\{e^{\frac{-ie}{\Lambda}\left[A(j)+\theta(j+1)-\theta(j)\right]}e^{\frac{ie}{\Lambda}\theta(j+1)}\phi(j+1) - e^{\frac{ie}{\Lambda}\theta(j)}\phi(j)\right\} = e^{\frac{ie}{\Lambda}\theta(j)}(D\phi)_\Lambda(j) \tag{2.5}$$

Having introduced the notation (2.2) - (2.5) it is simple to write down gauge invariant expressions. In particular there is an obvious transcription of the gauge continuum action to a discretized gauge invariant action, namely:

$$\int dt \int_{-N/\Lambda}^{N/\Lambda} dx\,\mathscr{L}(x,t) \Rightarrow \int \frac{dt}{\Lambda} \sum_{j=-N}^{N} \Bigg\{ \frac{(\partial_0 A_1(j,t))^2}{2} + (\partial_0 \phi^*(j,t))(\partial_0 \phi(j,t))$$
$$- (D\phi^*)_\Lambda(j,t)\,(D\phi)_\Lambda(j,t) - \lambda(2\phi^*\phi - f^2)^2 \Bigg\} \tag{2.6}$$

where we have adopted the convention that $D\phi(N) = 0$. Substituting (2.4) into (2.6) yields the gauge invariant classical lattice Lagrangian which we quantize i.e.:

$$\int dx\,\mathscr{L}(x,0) = \frac{1}{\Lambda} \sum_j \Big(\tfrac{1}{2}(\partial_0 A_1(j))^2 + (\partial_0 \phi^*(j))(\partial_0 \phi(j)) - \lambda(2\phi^*(j)\phi(j) - f^2)^2 \Big)$$
$$+\Lambda \Bigg[\phi^*(-N)\phi(-N) + \phi^*(N)\phi(N) + \Lambda \sum_{j=-(n-1)}^{N-2} 2\phi^*(j)\phi(j)$$
$$-\Lambda \sum_{J=-N}^{N-1} \Big(\phi^*(j+1)\, e^{\frac{+ie}{\Lambda}A(j)} \phi(j) + \phi^*(j)\, e^{\frac{-ie}{\Lambda}A(j)} \phi(j+1) \Big) \Bigg] \tag{2.7}$$

We remark here that we have chosen a formulation for which the gauge field varies over a non-compact range, i.e., in (2.7) we allow A_1 to take any real value $-\infty \le A_1 \le \infty$. In this we differ from previous formulations of lattice gauge theories[6] which always choose to limit A_1 to a compact range. As we shall see in the subsequent discussion, this difference is unimportant in this one-space dimensional case; we could arrive at the same results with a compact formulation and a careful treatment of boundary conditions.

To quantize the theory specified by (2.7) we proceed in the usual way; that is, we identify the momentum conjugate to each of the fields $\phi(j)$ and $A(j)$, impose canonical commutation relations, and construct the Hamiltonian of the theory in terms of fields defined at $t = 0$. (This procedure guaranties that the Euler-Lagrange equations derived from (2.7) will hold as operator equations of motion.) The Hamiltonian of the lattice theory obtained this way is

$$H = \frac{1}{\Lambda}\sum_{j=-N}^{N}\left\{\frac{1}{2}E^2(j) + P_\phi{}^*(j)P_\phi(j) + \lambda\left(2\phi^*(j)\phi(j) - f^2\right)^2 + 2\Lambda^2\phi^*(j)\phi(j)\right\} + \Lambda\left[\phi^*(-N)\phi(-N) + \phi^*(N)\phi(-N)\right]$$
$$-\Lambda\sum_{j=-N}^{N-1}\left\{\phi^*(j+1)e^{\frac{ie}{\Lambda}A(j)}\phi(j) + \text{h.c.}\right\} \tag{2.8}$$

where

$$E(j) = P_A(j) = \partial_0 A_1(j)$$

$$P_\phi(j) = \partial_0\phi^*(j) : P_{\phi^*}(j) = \partial_0\phi(j) \tag{2.9}$$

and we have imposed the commutation relations

$$\left[E(j),\, A(j')\right] = -i\Lambda\delta_{jj'}$$

$$\left[P_\phi(j),\, \phi(j')\right] = -i\Lambda\delta_{jj'}$$

$$\left[P_{\phi^*}(j), \phi^*(j)\right] = i\Lambda\delta_{jj'} \tag{2.10}$$

with all other commutators set equal to zero. Note, the link variable $A(j)$ is only defined for $j = -N, \ldots, N-1$ whereas ϕ is defined for $j = -N, \ldots, N$. Equations (2.8) - (2.10) completely define the lattice version of our quantum theory and our problem is to diagonalize H in the Fock space of states generated by in the usual way by applying polynomials in $A(j)$ and $\phi(j)$ to some "vacuum" state.

While we could proceed to analyze this theory directly for all values of the parameters e, λ and f, it is not necessary to do so. In fact, as in Ref. 5 it is just as interesting and somewhat simpler to study this theory in the limit $\lambda \longrightarrow \infty$ with e and f held finite. This essentially freezes out radial excitations. To be precise, we introduce dimensionless variables

$$\tilde{e} = e/_{\Lambda}\,, \quad \tilde{\lambda} = \lambda/_{\Lambda^2}\,, \quad \tilde{P}_{\phi} = P_{\phi^*}/_{\Lambda} \quad \tilde{P}_{\phi^*} = P_{\phi^*}/_{\Lambda}\,, \quad \mathscr{A}(j) = eA(j)\,,$$

$$\text{and} \qquad \phi(j) = \frac{1}{\sqrt{2}}\rho(j)e^{i\chi(j)}$$

Since our Fock space is just the space of square integrable functions in the variables

$$-\infty \leq \mathscr{A}(j) \leq \infty\,, \quad 0 \leq \rho(j) \leq \infty \quad \text{and} \quad -\pi \leq \chi(j) \leq \pi \tag{2.11}$$

We can replace $P_{\phi^*}(j)P_{\phi}(j)$ by

$$P_{\phi^*}(j)\,P_{\phi}(j) = \frac{1}{2}\left[-\frac{1}{\rho(j)}\frac{\partial}{\partial\rho(j)}\,\rho(j)\frac{\partial}{\partial\rho(j)} - \frac{1}{\rho^2(j)}\,\frac{\partial^2}{\partial\chi(j)^2}\right] \tag{2.12}$$

Our Hamiltonian can then be rewritten as

$$\begin{aligned} H = \Lambda\Bigg\{ & \sum_{j=-N}^{N-1}\left(\frac{-\tilde{e}^2}{2}\,\frac{\partial^2}{\partial\mathscr{A}(j)^2}\right) + \sum_{j=-N}^{N}\tfrac{1}{2}\left(-\frac{1}{\rho(j)}\frac{\partial}{\partial\rho(j)}\,\rho(j)\,\frac{\partial}{\partial\rho(j)} - \frac{1}{\rho^2(j)}\,\frac{\partial^2}{\partial\chi(j)^2}\right) \\ & + \sum_{J=-N}^{N}\left(\tilde{\lambda}\,(\rho^2(j)-f^2)^2 + \rho^2(j)\right) - \tfrac{1}{2}\,(\rho^2(-N) + \rho^2(N)) \\ & - \tfrac{1}{2}\sum_{j=-N}^{N-1}\left(\rho(j+1)\rho(j)e^{i\left(\mathscr{A}(j)+\chi(j)-\chi(j+1)\right)} + \text{h.c.}\right)\Bigg\} \end{aligned} \tag{2.13}$$

a form which makes explicit the fact that one is dealing with a many degree of freedom Schroedinger problem. Having reduced the problem to this form, simple inspection of the single-site terms allows us to argue that in the limit $\tilde{\lambda} \longrightarrow \infty$ the multiplication operator $\rho(j)$ can be replaced by f. Then, up to a c-number, the Hamiltonian can be simplified to

$$\frac{H^{\tilde{\lambda}\to\infty}}{\Lambda} = \sum_{j=-N}^{N-1} \frac{\tilde{e}^2}{2}\left(\frac{1}{i}\frac{\partial}{\partial \mathscr{A}(j)}\right)^2 + \frac{1}{2f^2}\sum_{j=-N}^{N}\left(\frac{1}{i}\frac{\partial}{\partial \chi(j)}\right)^2$$

$$- \sum_{j=-N}^{N-1} \frac{f^2}{2}\left(e^{\,i(\mathscr{A}(j)+\chi(j)-\chi(j+1))} + \text{h.c.}\right) \qquad (2.14)$$

Since by construction the variable $\mathscr{A}(j)$ runs over $-\infty \leq \mathscr{A}(j) \leq \infty$, and $\chi(j)$ is a periodic variable running over $-\pi \leq \chi(j) \leq \pi$ the system described by (2.13) is a set of rotors defined at each site 'j' coupled to oscillators defined on the links joining the point 'j' to 'j + 1'. Hence, if we choose to work in the basis where $\epsilon(j) = \frac{1}{i}\frac{\partial}{\partial \mathscr{A}(j)}$ and $\frac{1}{i}\frac{\partial}{\partial \chi(j)}$ are diagonal, our space of state is spanned by a basis whose members are specified by giving two quantum numbers for each 'j' ; i.e., $|y_j; n_j>$, where n_j takes all possible integer values and y_j runs over all real numbers. In terms of the variables $\mathscr{A}(j)$ and $\chi(j)$ such a state is of the form

$$|y_j; n_j> \equiv \prod_{j=-N}^{N}\left\{e^{iy_j \mathscr{A}(j)}\, e^{in_j \chi(j)}\right\} \Big|0;0> \qquad (2.15)$$

At this juncture, having specified our Hamiltonian and Hilbert space of states, we must turn to the general problem of diagonalizing the Hamiltonian and identifying the subclass of states which we may call "physical."

One needs to specify the class of physical states because in $A_0 = 0$ gauge the equation $\nabla \cdot E(x) = j_0(x)$ or its lattice equivalent $E(j+1) - E(j) = j_0(j)$ is not one of the Heisenberg equations of motion. In fact, it is easy to see that the space of states we are considering includes states for which this equation is totally false, and that the presence of such states is intimately related to the residual gauge invariance of the theory. This is most easily seen if we construct the operators which generate the gauge transformation corresponding to a given function $\theta(j)$. The problem is to find an operator $U(\theta)$ such that

$$U(\theta)\mathcal{A}(j)\,U^{+}(\theta) = \mathcal{A}(j) + \theta(j+1) - \theta(j)$$

$$U(\theta)\,\phi(j)\,U^{+}(\theta) = e^{i\theta(j)}\phi(j) \tag{2.16}$$

or

$$U(\theta)\rho(j)\,U^{+}(\theta) = \rho(j) \qquad U(\theta)e^{i\chi(j)}U^{+}(\theta) = e^{i[\chi(j)+\theta(j)]} \tag{2.17}$$

where we recall that although $\phi(j)$ is defined for $-N \leq j \leq N$, $\mathcal{A}(j)$ as a link variable is defined only for $-N \leq j \leq N-1$. Since $\epsilon(j)$ is the momentum conjugate to $\mathcal{A}(j)$ and $\left(\frac{1}{i}\frac{\partial}{\partial\chi(j)}\right)$ is conjugate to $\chi(j)$ it is clear that $U(\theta)$ is correctly given by

$$U(\theta) = e^{i\left\{\sum_{j=-N}^{N-1}\epsilon(j)(\theta(j+1)-\theta(j)) + \sum_{j=-N}^{N} j_0(j)\,\theta(j)\right\}} \tag{2.18}$$

where

$$\epsilon(j) = \frac{1}{i}\frac{\partial}{\partial\mathcal{A}(j)} \quad \text{and} \quad j_0(j) \equiv \frac{1}{i}\frac{\partial}{\partial\chi(j)} \quad .$$

Collecting terms referring to the same 'j' we can rewrite $U(\theta)$ as

$$U(\theta) = e^{i(j_0(-N)-\epsilon(-N))\theta(-N)}\; e^{i(\theta(N-1)+j_0(N))\theta(N)} \prod_{j=-N+1}^{N-1} e^{i\theta(j)[j_0(j)+\epsilon(j-1)-\epsilon(j)]} \tag{2.19}$$

From (2.19) it follows that any state which satisfies the Maxwell equation

$$\epsilon(j) - \epsilon(j-1) = j_0(j) \quad \text{for } -N+1 \leq j \leq N-1 \tag{2.20}$$

is a singlet (i.e., invariant) under all gauge transformations generated by functions $\theta(j)$ which vanish at $j = \pm N$. (Note, requiring (2.20) for $j = \pm N$ is not a meaningful thing to do since this corresponds to a choice of boundary conditions and is not part of Maxwell equations.)

Since $U(\theta) H U^{+}(\theta) = H$ for all $\theta(j)$ it follows that classifying the set of states in our Hilbert space into irreducible representations of the group of gauge transformations is a useful preliminary step to diagonalizing the Hamiltonian, H. Moreover, in order to have our theory describe a quantized version of the classical theory for which (2.20) is true we have seen we must restrict our attention to the set of states which are singlets under gauge transformations generated by arbitrary $\theta(j)$ which vanish at the end points. As can readily be seen from (2.19), for this set of states the irreducible representations of the group of all gauge transformations are specified by giving the eigenvalues of the operators

$$-\epsilon_L = j_0(-N) - \epsilon(-N)$$

$$\epsilon_R = \epsilon(N-1) + j_0(N) \tag{2.21}$$

Alternatively, we can--for the set of gauge invariant states--specify ϵ_L and the total charge $Q = \sum_{j=-N}^{N} j_0(j)$ since from (2.20) and (2.21) it follows that

$$\epsilon_R = \epsilon_L + Q \tag{2.22}$$

Hence, we see that the eigenstates of H are all labelled by a value of Q and the single parameter $-\infty \leq \epsilon_L \leq \infty$.

If we observe that for a state $|y_j ; n_j>$, $\frac{1}{i}\frac{\partial}{\partial \chi_j}$ is just multiplication by n_j, these general considerations tell us that the set of all physical states correspond to the space of states spanned by linear combinations of states $|y_j; n_j\rangle$ for which

$$y_j - y_{j-1} = n_j \qquad \text{for} \qquad -N+1 \le j \le N-1 \tag{2.23}$$

and

$$\sum n_j = Q \; ; \; -\epsilon_L = -y_{-N} + n(-N) \tag{2.24}$$

It follows, therefore, that for a given ϵ_L we can make the replacement $y_j = m_j + \epsilon_L$ where the m_j take integer values only. Clearly, in the $Q = 0$ sector $\epsilon_L = \epsilon_R$ and ϵ_L just describes a background electric field coming in from the left of the lattice and leaving from the right. To see that the background field is the same as the Θ-parameter which labels the vacua in the path integral formulation of the theory we must do two things. First, we must establish that physics is periodic in the variable ϵ_L and it is therefore only meaningful to label theories with $-\frac{1}{2} \le \epsilon_L \le \frac{1}{2}$. Second, we must show that the labelling of sets of physical states by the parameter ϵ_L is equivalent to the θ labelling of vacua in the instanton way of looking at things.

The first point is made clear by observing that the states $|\epsilon_L, Q, m_j\rangle$ and $|\epsilon_L + 1, Q, m_j - 1\rangle$ are identical in all ways except that the latter has an additional negative charge at $j = -N$ and an additional positive charge at $j = +N$. Hence, modulo end effects--which vanish as $N \to \infty$--the expectation value of the Hamiltonian and all gauge invariant observables will be the same in these states. Physically, this periodicity in ϵ_L corresponds to the possibility of cancelling out a background field by polarization of pairs out of the vacuum. If we were to start with a state having a background $|\epsilon_L| > \frac{1}{2}$ on interior links a pair of charges will materialize and migrate to the edges of the lattice so as to reduce the E-field by a unit on every interior link. Since only integer charges exist this reduction can only take place in integer steps and so background fields will all be reduced into the range $-\frac{1}{2} \le \epsilon_L \le \frac{1}{2}$. To see that this migration will occur we need only follow S. Coleman[4] and compute the energy of a pair of charges in a background field $\epsilon_L > \frac{1}{2}$. Assuming there are 2N links on the lattice the energy of a state with all $m_j = 0$ is $\frac{1}{2}(2N)\epsilon_L^2$. Now if a pair separates by 'S' links then $y_j = \epsilon_L$ on all links except the 'S-links' between the charges where it is $\epsilon_L - 1$. In this case the energy as a function of S is

$$E(S) = \tfrac{1}{2}(2N-S)\,\epsilon_L^2 + \tfrac{1}{2}\,S(\epsilon_L - 1)^2$$

$$= \tfrac{1}{2}\,2N\,\epsilon_L^2 + S(\tfrac{1}{2} - \epsilon_L) \tag{2.25}$$

It therefore follows that for $\epsilon_L < \frac{1}{2}$ the charges are attracted to one another whereas for $\epsilon_L > \frac{1}{2}$ there is a gain in energy of $2N(\frac{1}{2} - \epsilon_L)$ to be obtained by allowing the charges to separate to the ends of the lattice.

To see that this angle gives us the same information as the Θ parameter in the instanton way of computing let us first divide H (Eq. 12-14) into two parts

$$H^{\text{classical}} = \sum_{j=-1}^{N-1} f^2 \cos\left[\chi(j+1) - \chi(j) - \mathcal{A}(j)\right] \tag{2.26}$$

and

$$V = \sum_{j=-N}^{N} \left[\frac{\tilde{e}^2}{2}\,\frac{1}{i}\,\frac{\partial}{\partial \mathcal{A}(j)}\right]^2 + \frac{1}{2f^2}\left[\frac{1}{i}\,\frac{\partial}{\partial \chi(j)}\right]^2 \tag{2.27}$$

and restrict attention to $H^{\text{classical}}$. Clearly, the states of absolutely lowest energy correspond to classical configurations such that

$$2\pi p_j + \chi(j+1) - \chi(j) = \mathcal{A}(j) \tag{2.28}$$

where p_j is an arbitrary integer. This is nothing but the $A_0 = 0$ latticized form of the $t = \pm\infty$ form of an instanton. It follows from (2.28) that

$$2\pi P + \sum_{j=-N}^{N-1} \left[\chi(j+1) - \chi(j)\right] = \sum_{j=-N}^{N-1} \mathscr{A}(j)$$

where

$$P = \sum_{J=-N}^{N-1} p_j \tag{2.29}$$

or

$$2\pi P' + \delta = \sum_{j=-N}^{N-1} \mathscr{A}(j) \tag{2.30}$$

where δ is a number between $\pm\pi$. Hence, each classical ground state has an integer P' associated with it. Moreover, this integer cannot be changed by a gauge transformation $U(\theta)$ generated by a function $\theta(j)$ for which $\theta(-N) = \theta(+N) = 0$ since under this transformation $\mathscr{A}(j)$ becomes

$$\mathscr{A}_\theta(j) = \mathscr{A}(j) + \theta(j+1) - \theta(j)$$

and so

$$\sum \mathscr{A}_\theta(j) = \sum \mathscr{A}(j) + \theta(N) - \theta(-N) = 2\pi P' + \delta$$

This is an important result since this means that an integer P', as well as a $\delta \in [-\pi, \pi]$ can be associated with every classical ground state which satisfies Maxwell's equations, since these states are formed from any of the ones under discussion by taking any given configuration and summing over all configurations which can be generated from it by gauge transformations of this type. Hence, the integer P' which is the direct analogue of the continuum variable $\int dx \nabla\chi = 2\pi P' + \delta$ characterizes the gauge invariant ground states; however, these states correspond neither to irreducible representations of the full gauge group nor to the configurations which the path integral analysis classifies as Θ-vacua. To get to the Θ-vacua we note that the instanton prescription is to limit attention to "gauge invariant" states having definite values of $\chi(+N)$ and $\chi(-N)$. One the sums over these configurations to form states of definite Θ as follows:

$$|\Theta> = \sum_P e^{-iP\Theta} \quad |P, \delta> \tag{2.31}$$

Similarly one can take the states labelled by P and δ and project them onto irreducible representations of the gauge group, which is now taken to include transformations which do not vanish at $j = \pm N$. Since such transformations can change a state characterized by two numbers P, δ where P is an integer and $-\pi \leq \delta \leq \pi$ to a $|P', \delta'>$. Moreover, all such states are transformed in the same way by a single gauge transformation since $2\pi P+\delta \longrightarrow 2\pi P+\delta+\theta(N)-\theta(-N)$ for all configurations; hence, the problem of finding irreducible representations of the gauge group coincides with the problem of finding irreducible representations of the translation operator on the real numbers. This is of course done by making use of the functions e^{ikx}, and since in our case $x = 2\pi P+\delta$ we find that a state of definite ϵ_L is just

$$|\epsilon_L> \propto \sum_P e^{-i\epsilon_L(2\pi P)} \int_{-\pi}^{\pi} d\delta e^{i\epsilon_L\delta} \, |P, \delta> \tag{2.32}$$

which just averages in a well defined way over the configurations of definite $[\chi(N)-\chi(-N)]$ specified in the instanton prescription. Comparison of the two prescriptions leads to the identification of Θ with $2\pi\epsilon_L$. Note that the prescriptions coincide trivially if we limit ourselves to forming irreducible representations of the gauge group including only those functions

$$\theta(N)-\theta(-N) = 2\pi(\text{integer}) \quad .$$

This completes our discussion of the general properties of the $A_0 = 0$ gauge formulation of the theory and the identification of the ambiguity in the definition of the physical vacua of this theory with the possibility of having a background electric field of absolute magnitude less than one-half (in units of e). The remainder of this section is devoted to a brief discussion of the formulation of the same theory in $A_1 = 0$, or Coulomb, gauge in order to see the same background field occurs in a natural way as part of the quantization procedure.

In $A_1 = 0$ gauge, the variable A_0 is not a dynamical variable but satisfies the constraint equation

$$- \partial_1^2 A_0 = eJ_0 = ef\partial_0\chi \tag{2.33}$$

This is just Maxwell's equation. Furthermore recognizing that

$$\partial_1^2 A_0 = \Lambda^2\left[A_0(j+1) + A_0(j-1) - 2A_0(j)\right] \tag{2.34}$$

summing over all sites 'j' gives

$$Q = \sum n(j) = 0 \tag{2.35}$$

so that in this gauge we can only consistantly quantize the $Q = 0$ sector of the theory! The background E-field appears, as in the continuum case, as a constant of integration when one inverts (2.33) to obtain

$$A_0(j) = \sum_j \left|j - j'\right| n(j) + \epsilon_L j + C \quad .$$

This is exactly the same feature as has been observed in the Schwinger model as discussed in this gauge by Sydney Coleman.[4]

We remark at this point that the need to choose irreducible representations of the gauge group is clearly quite unrelated to the existence or otherwise of classical Euclidean solutions of various topologies and must be faced for any two dimensional QED theory. We have chosen to examine the Higgs model because in the path integral formulation it has an additional complication due to the existence of classical solutions (or approximate solutions) for the $q \neq 0$ sectors. However, we find that for a Minkowski space Hamiltonian lattice formalism such as ours the theory is in no sense more complicated than the Schwinger model. Both theories are properly described in terms of an additional parameter which has the physical interpretation of a background E-field and which labels the subspaces of the space of states which are irreducible representations of the full gauge group. Whether those subspaces corresponding to different background fields are physically distinguishable depends on the operators which are included in the algebra of observables. For example, in the massless Schwinger model they are not, provided we insist that all observables are chirally invariant. If we introduce as an observable any operator of definite chirality, then the different sectors are physically distinct, whether or not there is a mass-term in the Lagrangian.

III. SOLUTION OF THE THEORY IN VARIOUS COUPLING REGIONS

We can insert the constraints (2.23) and (2.24) in the Hamiltonian (2.13) to find H in the physical subspace labelled by (ϵ_L, Q) This gives

$$\frac{H}{\Lambda} = \frac{\tilde{e}^2}{2} \sum_{j=-N}^{N-1} (m_j+\epsilon_L)^2 + \frac{1}{2f^2} \left\{ \sum_{j=-N}^{N-2} (m_{j+1}-m_j)^2 + m_{-N}^2 + (m_{N-1}-Q)^2 \right\} - f^2 \sum_{j=-N}^{N-1} \cos\alpha_j \tag{3.1}$$

where $\alpha(j) = \mathscr{A}(j) - \chi(j+1) + \chi(j)$ is the variable conjugate to $m_j + \epsilon_L$. Clearly the properties of the theory (3.1) depend on the relative sizes of the dimensionless quantities $\tilde{e}^2$ and $\frac{1}{f^2}$. We will now identify various regions and comment upon them. While we have not carried out a full variational calculation, nor calculated renormalization constants, and so our answers involve the arbitrary scale Λ in a non-trivial fashion, even a simple-minded examination of the theory is enough to convince us that we are obtaining the same physics as the dilute instanton gas calculation (at least in that range of parameters for which we expect both calculations to be valid). In the dilute instanton gas approximation the classical action S_0 is proportional to f^2 and the instanton radius of order '1/ef'. The dilute gas approximation is thus expected to be reasonable when $e^{-f^2}\left(\frac{1}{ef}\right)^2$ is small, that is for large f and ef not too small. The scale on which the dimensionful quantity ef is measured is not well defined in this treatment, and in fact factors of ef are introduced in a somewhat arbitrary fashion to provide the correct dimensions of various quantities. On the other hand in our calculation everything is expressed in terms of the dimensionless couplings $\tilde{e}$ and f and the dimensionful factors will always appear as powers of the cut-off of Λ. To remove the cut-off dependence requires renormalization of the theory which we have not done, since we are only interested in answering a limited set of questions.

Let us begin by examining the region, $f \gg 1$; $\tilde{e} \ll 1$. In this region the cosine terms dominate the Hamiltonian and force α_j to be small in the ground state. Hence, we can reasonably expand $\cos \alpha_j \simeq 1 - \alpha_j^2/2$ provided we also remember that we are in fact dealing with a periodic potential. At this point we observe that if we re-interpret α_j as a momentum and $\frac{1}{i}\frac{\partial}{\partial \alpha_j}$ as the co-ordinate conjugate to that momentum, then, up to an overall c-number, our Hamiltonian is exactly that for a free particle of mass of ef. This naive interpretation is, however, not the full story, as it neglects two important fact; one is that the potential is periodic and the other is the fact that $\frac{1}{i}\frac{\partial}{\partial\alpha_j} = m_j + \epsilon_L$ is shifted away from integer values by an amount proportional to the background E-field.

To include the effect of these constraints we must set up the variational calculation and examine its behavior. The program we follow is identical to that discussed for the Goldstone model by Drell and Weinstein.[5] The reader is referred to this paper and to Ref. 3 for details. Here we will simply state some results and try to show heuristically how they arise.

We consider first the single site Hamiltonian obtained from (3.1) by dropping the terms $\left(1/f^2\right)\left(m_j\, m_{j+1}\right)$ which couple neighboring sites. At each site j we then solve the problem given by this Hamiltonian

$$\frac{H(j)}{\Lambda} = \left(\frac{\tilde{e}^2}{2} + \frac{1}{f^2}\right)\left(\frac{1}{i}\frac{\partial}{\partial\alpha_j}\right)^2 - f^2 \cos\alpha_j \tag{3.2}$$

where the fact that

$$\frac{1}{i}\frac{\partial}{\partial\alpha_j} = m_j + \epsilon_L \quad \text{with } m_j = \text{integer} \tag{3.3}$$

is taken care of by requiring that the solution of the problem must be in terms of functions of the form $\psi(\alpha_j) = e^{i\epsilon_L\alpha_j}\,\phi(\alpha_j)$, where the $\phi(\alpha_j)$ satisfy periodic boundary conditions. Thus our problem is equivalent to the Bloch wave problem for a periodic potential $\cos(\alpha_j)$ with $-\infty \le \alpha_j \le \infty$, where we identify ϵ_L as the momentum of the particle. To a good approximation for low-lying states n the energy of this particle is given by[7]

$$E_n = \tilde{E}_n + A_n \cos(2\pi\epsilon)$$

where $\tilde{E}_n$ is the energy of the n^{th} excited state of the single well and A_n is the transition amplitude for a particle in the state 'n' in one well to tunnel into the neighboring well. In order to simplify the computation of E_n and A_n we will make some modification of the $\cos(\alpha_j)$ term in the Bloch problem but it will not be important for the case $f \gg 1$. The substitution will be to replace $\cos\alpha$ within each region $(2n+1)\pi \leq \alpha \leq (2n+3)\pi$ by the expansion about its midpoint, i.e., for $(2n+1)\pi \leq \alpha \leq (2n+3)\pi$ we take

$$-f^2 \cos(\alpha) \Rightarrow -f^2 + \frac{f^2}{2} \left(\alpha - 2(n+1)\right)^2 \quad . \tag{3.4}$$

Clearly, this modifies our problem to some extent but for $f \gg 1$, since the ground-state wave function must concentrate about the regions $\alpha_n = 2(n+1)\pi$, the use of (3.4) is sufficiently accurate for our purposes.

To find the lowest state of the problem (3.4) having momentum ϵ_L we observe that for large 'f' we have a system of weakly coupled wells. In the absence of tunneling between the wells there would be a set of degenerate ground state levels corresponding to having the n = 0 oscillator of mass $m = f^2(2+e^2f^2)^{-1}$ and frequency $\omega = (2+e^2f^2)^{1/2}$ confined to any one well. Obviously, from (3.4) the energy associated with such a state is

$$\tilde{E} = -f^2 + \frac{f^2}{2}(2+e^2f^2)^{1/2} \quad .$$

To get the ground state of the system, we observe that the Hamiltonian in (3.4) mixes levels corresponding to oscillators in different wells, and we can estimate the mixing coefficient by taking the expectation value of H between the Gaussian wave packets

$$\psi_n = \propto \exp\left(-\frac{1}{2} f^2 (2+e^2f^2)^{1/2} \left(\alpha - 2(n+1)\right)^2\right)$$

and

$$\psi_{n+1} \propto \exp\left(-\frac{1}{2} f^2 (2+e^2f^2)^{1/2} \left(\alpha - 2(n+2)\right)^2\right)$$

to obtain

$$A_0 = e^{-f^2 \pi^2 (2 + e^2 f^2)^{1/2}} \left[1 + 0\left(\frac{1}{f^2}\right)\right] \tag{3.5}$$

It follows from our previous discussion that to a good approximation eigenstates of (3.4) are given by

$$\psi_\epsilon = \sum e^{i\, 2\pi \epsilon\, n}\, \psi_n$$

whose energy is proportional to

$$\cos\,(2\,\pi\,\epsilon_L)\, e^{-f^2 \pi^2 / \sqrt{2}} \quad . \tag{3.6}$$

for each of the 2N single site term in the Hamiltonian (3.2). Thus we see that just as in the dilute instanton gas calculation we find a contribution to the ground state energy density which is proportional to $\cos\,(\Theta)\, e^{-f^2 K}$ where K is a number. Obviously a more accurate determination of K would be obtained by solving the Schroedinger equation for the boundary problem more carefully than we have done here.

The Hamiltonian (3.1) also allows us to determine the ground state expectation value of the E-field. We observe that the E-field is simply given by

$$\tilde{e}\, \left\langle m_j + \epsilon_L \right\rangle = \frac{1}{\tilde{e}} \left\langle \frac{\partial\,(H/\Lambda)}{\partial\, \epsilon_L} \right\rangle \tag{3.7}$$

Thus we find, again in agreement with the dilute instanton gas calculation, that the E-field density in the vacuum is proportional to $\sin\,(2\pi\epsilon_L)\, e^{-f^2 K}$. Of course, we have not calculated a meaningful constant of proportionality; to do so would require performing the wave-function renormalization.

Since for large 'f' the shift in energy is the same for the first few excited levels, the remainder of the calculation can be done as if we were dealing with oscillators at each site. Hence, if one restores the coupling between adjacent sites and diagonalizes the resulting coupled oscillator problem, one obtains the spectrum of a particle of mass 'ef'. Since now we have taken the periodicity of the potential and the effect of ϵ_L into account, this is a correct version of our previous argument.

It is interesting also to look at the model in the region $\tilde{e} \gg f^2 \gg 1$. Naively, one might expect that the dilute gas calculation should also be valid here, but this is the limit $e \to \infty$ in which the classical configurations can no longer be expected to dominate the path integral. In this region the lattice version of the theory looks quite different. The momentum terms dominate the Hamiltonian and one expects the ground state to be close to the state $|\epsilon_L, Q, m_j = 0 \text{ all } j\rangle$. This will be true for all values $|\epsilon_L| < \frac{1}{2}$ and $\tilde{e}^2$ sufficiently large. The case $\epsilon_L = \frac{1}{2}$ is peculiar, here as in the Schwinger model, since for $\epsilon_L = \frac{1}{2}$ excitations which consist of a pair of charges aligned to exactly reverse the sign of the E-field between them exist at an energy cost of only $\frac{1}{f^2}$, independent of the charge separation. These are the "half-asymptotic states" which exist also in the Schwinger model. They will not change the vacuum energy density, but they will reduce the expectation value of the E-field density; the field strength remains constant at $e/2$ but its direction fluctuates. Presumably, an iterative calculation would allow us to calculate the value of the ground state expectation value of the electric field, but it has not been carried out.

III. CONCLUDING COMMENTS

We can ask what happens to our conclusions when we look at higher dimensions. Although we have not carried out a detailed analysis of such theories, certain points are immediately obvious. The first is that much of the physics of the Abelian Higgs model observed here will also apply in higher dimensions. Using Maxwell's equations will allow us to eliminate the scalar field as a dynamical variable and reduce the problem to that of a massive photon. A priori we will still obtain a background E-field in higher dimensions, but the physical ground state in the infinite volume limit will always be that with $E_B = 0$. This is because even though the charges are integers they can cancel any value of the E-field by arranging charge distributions on the surface at infinity. Thus in this case there exists a unique physical vacuum; the multiplicity of possible background E-fields is a peculiarity of the one-dimensional case (as is the existence of the topological charge $q = \int d^2x\, \epsilon^{\mu\nu} F_{\mu\nu}$).

The lesson that we learn from our study of the two-dimensional Higgs model is that the question of finding the correct vacuum is in fact readily addressed without reference to the Euclidean solutions and topological invariants. It is in fact simply the question of finding those sub-spaces of the Hilbert space of physical states which form irreducible representations of the gauge group. We remark that this exercise was straightforward with our non-compact formulation of the theory, but it would be the same for a compact formulation in terms of a gauge field $\mathscr{A}$ such that $-\pi \leq \mathscr{A} \leq \pi$, even though at first glance it would seem that there is no room for a background field ϵ. This is because there is an ambiguity in defining the operator $\frac{1}{i}\frac{\partial}{\partial\mathscr{A}}$ as a self-adjoint operator on a compact interval. The key point is that

$$\int_{-\pi}^{\pi} f^*(\mathscr{A})\,\frac{1}{i}\,\frac{\partial}{\partial\mathscr{A}}\, g(\mathscr{A})d\mathscr{A} = \int_{-\pi}^{\pi}\left(\frac{1}{i}\,\frac{\partial}{\partial\mathscr{A}}f(\mathscr{A})\right)^* g(\mathscr{A})d\mathscr{A}$$

if and only if

$$f^*(\pi)\,g(\pi) - f^*(\pi)\,g(-\pi) = 0$$

and if the choice of boundary conditions for g forces f to satisfy the same boundary conditions (i.e., the domain of the adjoint operator $\left(\frac{1}{i}\frac{\partial}{\partial \mathscr{A}}\right)^+$ must equal the domain of $\left(\frac{1}{i}\frac{\partial}{\partial \mathscr{A}}\right)$). Clearly, while these conditions are trivially satisfied if we require $g(\pi) = g(-\pi)$, they are also satisfied $g(\pi) = e^{i2\pi\epsilon}g(-\pi)$ for arbitrary 'ϵ'. Hence, there is a one-parameter family of definitions of $\left(\frac{1}{i}\frac{\partial}{\partial \mathscr{A}}\right)$ which it self-adjoint and therefore a one-parameter set of compact gauge field formulations of one-dimensional QED.

It follows from this discussion that there is no difference in the physics of the one-dimensional theories for the two formulations. In higher dimensions there is a difference for Abelian theories and no one has successfully written a non-compact lattice formulation of non-Abelian gauge theories. One thing which is clear is that the problem of finding the sub-spaces of the Hilbert space which form irreducible representations of the gauge group for four-dimensional non-Abelian gauge theories must be discussed in a similar manner to our discussion of the two-dimensional Higgs model. The requirement that the physical states satisfy the non-Abelian equivalent of Maxwell's equations is fulfilled by choosing states which are singlets under all gauge transformations generated by functions, $\theta(x)$, which vanish at spatial infinity. The further requirement of irreducibility under gauge transformations which go to a constant at infinity is presumably satisfied by choosing the Θ-basis of the standard instanton treatment although there remains the intriguing possibility that the remaining class of gauge transformations, namely those which have non-trivial behavior at infinity, may alter the classification.

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