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FIELD THEORY ON A LATTICE: ABSENCE OF GOLDSTONE BOSONS IN THE U(1) MODEL IN TWO DIMENSIONS*

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ABSTRACT

We study the U(1)-Goldstone Model in two dimensions. We formulate this model on a one-dimensional spatial lattice and show that Coleman's theorem (i.e., there exist no Goldstone bosons in two dimentions) is satisfied by the solution found by the variational approach of dissecting the lattice into small (4-site) blocks and iteratively constructing an effective truncated Hamiltonian.

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In a series of papers[1] we have developed relatively simple variational techniques for solving quantum field theories on a lattice and have applied them successfully to construct low lying physical states and to find phase transitions. Toward the eventual goal of understanding quark confinement and calculating the observed hadronic states on the basis of non-Abelian color gauge theories (or QCD) we have thus far applied these methods to simple two-dimensional models with known exact properties that were successfully reproduced. These applications include, in addition to free massless bosons and fermions, the Ising model in a transverse magnetic field and the massless Thirring model on a lattice. We conclude this phase of our program by studying the U(1)-Goldstone model in two dimensions. This model is of particular interest in view of Coleman's theorem[2] for the continuum theory which says that there is no Goldstone boson in one space-one time dimensions, in spite of the predictions of the naive classical analysis to the contrary. The lattice analogue of this theorem was proved first by Mermin and Wagner.[3] Having shown in Paper III that our techniques successfully reproduce phase transitions known to occur in the Ising model, we now further demostrate that they also do not predict them when they are known not to occur.

The basic idea of our variational renormalization group approach is to dissect the lattice into small blocks, each containing a few lattice sites which are coupled to one another by the gradient terms in the Hamiltonian.[4] The Hamiltonian for the resulting few-degree of freedom problem within each block is diagonalized and the degrees of freedom "thinned" by keeping an appropriate set of low-lying states. We then construct an effective Hamiltonian by computing the matrix elements of the original Hamiltonian in the space of states spanned by eigenvectors having the lowest energy eigenvalues in each block.[5] The process is then repeated for the new effective Hamiltonian, whose coupling parameters change at each step. The procedure is iterated until we enter a regime that can be handled simply by perturbation theory, either for very weak or very strong effective couplings.

In Section II we formulate the U(1)-Goldstone model in two dimensions on a one-dimensional spatial lattice. In Section III we prove by the variational approach that this model has no false Goldstone bosons. In Section IV by perturbative calculations we provide a heuristic explanation of why they are absent.

II. U(1)-GOLDSTONE MODEL ON A LATTICE

The continuum model is specified by the Lagrangian density

$$\mathscr{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - \lambda(2\phi^*\phi - f^2/2)^2 ; \quad \lambda,\ f^2 > 0 \tag{2.1}$$

The corresponding Hamiltonian, in d = p + 1 dimensions, is

$$H = \int d^p x \left\{ \pi^* \pi + \vec{\nabla}\phi^* \cdot \vec{\nabla}\phi + \lambda(2\phi^*\phi - f^2/2)^2 \right\} \tag{2.2}$$

$$\pi = \dot{\phi}^* ; \ \pi^* = \dot{\phi}$$

In the classical limit, $\pi = \pi^* = 0$, there is a one-parameter family of degenerate ground states represented by

$$\phi_0(\underline{x}) = \frac{1}{2} f e^{i\theta} \tag{2.3}$$

with the constant phase angle θ arbitrary in the interval $(-\pi, \pi)$, corresponding to the minimum of H with zero energy. The naive approach to the quantum theory (2.2) expands the field ϕ about ϕ_0, specifying $\theta = 0$ for convenience

$$\phi(x) \equiv \frac{1}{\sqrt{2}} \left[\sigma(x) + i\chi(x) \right] = \frac{1}{\sqrt{2}} \left[\sigma'(x) + i\chi(x) + \frac{f}{\sqrt{2}} \right] \tag{2.4}$$

This leads to

$$H = \int d^p x \left\{ \frac{1}{2}\,\pi_{\sigma'}^2 + \frac{1}{2}\,\pi_{\chi}^2 + \frac{1}{2}\,(\nabla\sigma')^2 + \frac{1}{2}\,(\nabla\chi)^2 + \frac{1}{2}\,m_{\sigma'}^2\,\sigma'^2 \right. \tag{2.5}$$

$$\left. + \frac{m_{\sigma'}^2}{\sqrt{2}\,f}\,\sigma'(\sigma'^2 + \chi^2) + \frac{m_{\sigma'}^2}{4f^2}\,(\sigma'^2 + \chi^2)^2 \right\}$$

where

$$m_{\sigma'}^2 \equiv 4\lambda f^2 \tag{2.6}$$

The form of (2.5) suggests that for fixed mass $m_{\sigma'}$ and large $f^2 \gg 1$, we can make a perturbation expansion about the free modes for a massive σ' field, with mass (2.6), and a massless χ field, the "Goldstone boson." If this were valid, we would expect that for $f \gg 1$, $\langle\sigma'\rangle = \langle\chi\rangle = 0$, and hence from (2.4),

$$\langle\phi\rangle = f/2 \neq 0 \tag{2.7}$$

This result is believed to be valid for $p = 2$ or 3. However, it is known to be false for $p = 1$. This is the so-called Coleman Theorem[2] which requires that

$$\langle\phi\rangle = 0 \tag{2.8}$$

for all finite values of f, no matter how large. The failure in the naive analysis is the result of infrared divergence in the propagator of the χ field when $p = 1$. This expresses the fact that the quantum fluctuations are uncontrollably large, leading to (2.8). For fixed $m_{\sigma'}^2$, and f^2 arbitrarily large, $\lambda \to 0$ according to (2.6), and we are in the weak coupling region. In the opposite strong coupling extreme of $m_{\sigma'}^2 \to \infty$, and f^2 fixed but arbitrary, so that $\lambda \to \infty$, we expect from (2.5) that the infinitely massive σ' excitations will be "frozen out," so that presumably only the false Goldstone bosons survive. Study of this region provides the most severe test of our methods and therefore we shall analyze (2.2) in this limit.[6]

First we transcribe (2.2) to a lattice in terms of dimensionless variables by writing,[7] with p = 1,

$$\phi(x) \to \phi_j \equiv \frac{1}{\sqrt{2}}(x_j + i y_j) \qquad \pi(x) \to \pi_j \equiv \frac{\Lambda}{\sqrt{2}}(p_{x_j} - i p_{y_j})$$

$$\phi^*(x) \to \phi_j^* \equiv \frac{1}{\sqrt{2}}(x_j - i y_j) \qquad \pi^*(x) \to \pi_j^* \equiv \frac{\Lambda}{\sqrt{2}}(p_{x_j} + i p_{y_j}) \tag{2.9}$$

$$H = \Lambda \sum_{j=-N}^{N} \left[\frac{1}{2} p_{x_j}^2 + \frac{1}{2} p_{y_j}^2 + \lambda \left(x_j^2 + y_j^2 - f^2/2 \right)^2 \right]$$

$$+ \Lambda \sum_{j=-N}^{N-1} \left[\frac{1}{2}(x_{j+1} - x_j)^2 + \frac{1}{2}(y_{j+1} - y_j)^2 \right] \tag{2.10}$$

The canonical commutators are

$$\left[p_{x_j} , x_{j'} \right] = -i \delta_{j,j'}$$

$$\left[p_{y_j} , y_{j'} \right] = -i \delta_{j,j'} \tag{2.11}$$

with all other commutators vanishing. We observe that diagonalizing (2.10) is equivalent to solving the 2(2N+1)-degree of freedom Schroedinger problem

$$H \Phi \left(x_{-N}, y_{-N}, \ldots\ldots\ldots x_j, y_j, \ldots\ldots x_N, y_N \right) = E \Phi .$$

where

$$p_{x_j} = \frac{1}{i}\frac{\partial}{\partial x_j} \quad \text{and} \quad p_{y_j} = \frac{1}{i}\frac{\partial}{\partial y_j} \; .$$

We attack this problem by going to polar variables

$$r_j \equiv \sqrt{x_j^2 + y_j^2} \quad ; \quad r_j \geq 0$$

$$\theta_j \equiv \tan^{-1} x_j/y_j \quad ; \quad -\pi \leq \theta_j \leq \pi \tag{2.12}$$

and rewrite (2.10)

$$H = \Lambda \sum_{j=-N}^{N} \left[\frac{-1}{2} \frac{\partial^2}{\partial r_j^2} - \frac{1}{2r_j}\frac{\partial}{\partial r_j} - \frac{1}{2r_j^2}\frac{\partial^2}{\partial \theta_j^2} + r_j^2 + \lambda\left(r_j^2 - f^2/2\right)^2 \right] \tag{2.13}$$

$$- \Lambda \sum_{j=-N}^{N-1} r_j\, r_{j+1} \cos(\theta_j - \theta_{j+1}) - \frac{\Lambda}{2}\left(r_{-N}^2 + r_N^2\right)$$

In the limit $\lambda f^2 \to \infty$ it is apparent that the radial modes in (2.13) are frozen[8] at $r_j = f/\sqrt{2}$ and the Hamiltonian reduces to

$$H_{\left(\lambda^{1/2} f \to \infty\right)} = \Lambda \left[\sum_{j=-N}^{N} \frac{-1}{f^2}\frac{\partial^2}{\partial \theta_j^2} - \sum_{j=-N}^{N-1} f^2/2 \cos\left(\theta_j - \theta_{j+1}\right) \right] + \text{const.} \tag{2.14}$$

The set of basis states

$$|m_{-N} \ldots m_N\rangle \equiv \prod_j |m_j\rangle$$

with

$$|m_j\rangle = e^{i m_j \theta_j} \tag{2.15}$$

provides a convenient representation for H when we identify

$$J_3(j)\,|m_j\rangle = \frac{1}{i}\frac{\partial}{\partial\theta_j}\,|m_j\rangle$$

$$= m_j\,|m_j\rangle \tag{2.16}$$

and

$$J_\pm(j)\,|m_j\rangle = e^{\pm i\theta_j}\,|m_j\rangle = |m_j \pm 1\rangle$$

In this representation

$$H_{\left(\lambda^{1/2} f\to\infty\right)} = \Lambda \sum_j \left\{ \frac{1}{f^2} J_3^2(j) - \frac{1}{4} f^2 \left[J_+(j)J_-(j+1) + J_-(j)J_+(j+1) \right] \right\} \tag{2.17}$$

which describes a rigid planar rotor with all possible integer values m_j of the angular momenta at each lattice point j. The rotors have moments of inertia $f^2/2$ and adjacent sites are coupled by a force proportional to the angle between the rotors. For infinite moments of inertia, $f^2 \to \infty$, one finds the classical limit of the lowest energy state; i.e., at all sites $\theta_j = \theta_0$, with $-\pi < \theta_0 < \pi$. This is the same as described in Section I for the classical continuum limit. Our interest is to show by our variational procedure how the $\frac{1}{f^2}$ term in (2.17) modifies this conclusion and removes the false Goldstone boson for finite f^2. For simplicity we shall keep $\frac{1}{f^2} \equiv x_0^2$ finite but small so that we can expand to leading order in $x_0^2 \ll 1$.

III. VARIATIONAL ANALYSIS

Our first step is to divide the lattice into blocks each containing two sites, rewriting the sum over sites as

$$j = 2p + s$$

where

$$p = -\frac{1}{2}N, \ldots\ldots, +\frac{1}{2}N \quad \text{and} \quad s = 0, 1, \tag{3.1}$$

and introducing new angle variables within each block:

$$\begin{aligned} 2\psi_p &\equiv \theta_{2p} + \theta_{2p+1} \quad ; \qquad -\pi \le \psi_p \le +\pi \\ 2\phi_p &\equiv f(\theta_{2p} - \theta_{2p+1}) \, ; \qquad -(\pi - \psi_p) \le \frac{\phi_p}{f} \le (\pi - \psi_p) \end{aligned} \tag{3.2}$$

Substituting in (2.14) gives (up to an irrelevant constant)

$$\begin{aligned} H = \Lambda \sum_p &\left\{ -\frac{1}{2f^2}\frac{\partial^2}{\partial\psi_p^2} - \frac{1}{2}\frac{\partial^2}{\partial\,\phi_p^2} - \frac{f^2}{2}\cos\frac{2}{f}\phi_p \right\} \\ &- \Lambda \sum_p \frac{f^2}{2}\cos\left(\psi_{p+1} - \psi_p + \frac{1}{f}\left[\phi_{p+1} + \phi_p\right]\right) \end{aligned} \tag{3.3}$$

In the small $x_0^2 \equiv \frac{1}{f^2} \ll 1$ region the single block terms describe an uncoupled rotor and oscillator. In particular up to corrections $0(x_0^2)$ the motion in ϕ_p describes a simple harmonic oscillator of frequency $\omega_0 = \sqrt{2}$ and mass unity. Since the ground state wave function for this motion varies as

$$e^{-\phi_p^2/\sqrt{2}} = e^{-\frac{1}{4\sqrt{2}}f^2(\theta_{2p} - \theta_{2p+1})^2}, \tag{3.4}$$

(3.4) shows that for large $f^2 \gg 1$, the variable $(\theta_{2p} - \theta_{2p+1})$ is restricted to be very nearly zero. This justifies the quadratic approximation to (3.3) as well as the application of periodic boundary conditions to ψ_p in the interval $(-\pi, \pi)$. We can now rewrite the block Hamiltonian (3.3) in this approximation

$$H = \Lambda \sum_p \left\{ -\frac{1}{2} x_0^2 \frac{\partial^2}{\partial \psi_p^2} - \frac{1}{2} \frac{\partial^2}{\partial \phi_p^2} + \frac{1}{2} \omega_0^2 \phi_p^2 + c_0 \, \mathbb{1} \right\}$$

$$- \Lambda \sum_p \frac{1}{2x_0^2} \cos \left[(\psi_{p+1} - \psi_p) + x_0 (\phi_{p+1} + \phi_p) \right] \tag{3.5}$$

where

$$\omega_0^2 = 2, \; c_0 = -1/2 \, x_0^2 \; ,$$

and

$$-\pi \leq \psi_p \leq \pi$$

$$-\pi/x_0 \leq \phi_p \leq \pi/x_0$$

We are now ready to initiate the procedure of iteratively forming higher blocks and thinning degrees of freedom. Our basic truncation algorithm is to retain the lowest oscillator degree of freedom plus the trivial rotor in each block. Proceeding next to coupling two blocks together in (3.3) we will find the rotor plus three oscillator degrees of freedom and our thinning procedure consists of retaining just the rotor plus the lightest oscillator per superblock, always truncating away the two higher oscillators in constructing the new effective Hamiltonian. This truncation is accomplished by taking the ground state expectation values of H with respect to the degrees of freedom of these higher oscillators. Formally these steps imitate those described in Papers III and IV and will be simply sketched. They lead to recursion relations from which we find a soluble fixed form for H which describes a theory with no gap in its spectrum but with $\langle J_+(j)\rangle = \langle e^{i\theta_j}\rangle = 0$ in accord with Coleman's theorem, where $\langle\ \rangle$ denotes the ground state expectation value for the higher oscillators.

The general form of (3.5) after n iterations and truncations is written

$$\frac{1}{\Lambda} H_{(n)} = \sum_p \Bigg\{ c_n \mathbb{1}_p + \frac{1}{2}\alpha_n^2 \left(\frac{-\partial^2}{\partial\psi_p^2}\right) + \frac{1}{2}\left(\frac{-\partial^2}{\partial\phi_p^2} + \omega_n^2 \phi_p^2\right) - \beta_n \cos\left[\psi_{p+1} - \psi_p + \delta_n(\phi_{p+1} + \phi_p)\right]\Bigg\} \tag{3.6}$$

Next we perform the (n+1)st iteration, defining $p = 2\ell + s$; $s = 0, 1$, where $p_{max} = N/2^{n+1}$, and rewrite H_n as

$$\frac{1}{\Lambda} H_{(n)} = \sum_\ell \Bigg\{ 2c_n \mathbb{1}_\ell + \frac{1}{2}\alpha_n^2 \left(\frac{-\partial^2}{\partial\psi_{2\ell}^2} - \frac{\partial^2}{\partial\psi_{2\ell+1}^2}\right) + \frac{1}{2}\left[\frac{-\partial^2}{\partial\phi_{2\ell}^2} - \frac{\partial^2}{\partial\phi_{2\ell+1}^2} + \omega_n^2\left(\phi_{2\ell}^2 + \phi_{2\ell+1}^2\right)\right] - \beta_n \cos\left[\left(\psi_{2\ell+1} - \psi_{2\ell}\right) + \delta_n\left(\phi_{2\ell+1} + \phi_{2\ell}\right)\right]\Bigg\} \tag{3.7}$$

$$- \sum_\ell \beta_n \cos\left[\psi_{2(\ell+1)} - \psi_{2\ell+1} + \delta_n\left(\phi_{2(\ell+1)} + \phi_{2\ell+1}\right)\right]$$

We introduce new angle variables within each superblock 'ℓ', with Ψ_ℓ and Φ_ℓ defined as in (3.2):

$$2\Psi_\ell = \psi_{2\ell} + \psi_{2\ell+1} \quad ; \quad r_\ell \equiv \frac{1}{\sqrt{2}}\left(\phi_{2\ell+1} + \phi_{2\ell}\right)$$

$$2\Phi_\ell = \frac{\sqrt{2}}{\alpha_n}\left(\psi_{2\ell} - \psi_{2\ell+1}\right) \quad ; \quad v_\ell \equiv \frac{1}{\sqrt{2}}\left(\phi_{2\ell+1} - \phi_{2\ell}\right) . \qquad (3.8)$$

In terms of (3.8), (3.7) becomes

$$\frac{1}{\Lambda} H_{(n)} = \sum_\ell \left\{ 2 c_n \mathbb{1}_\ell + \frac{\alpha_n^2}{4}\left(-\frac{\partial^2}{\partial \Psi_\ell^2}\right) + \frac{1}{2}\left(\frac{-\partial^2}{\partial \Phi_\ell^2} - \frac{\partial^2}{\partial r_\ell^2} - \frac{\partial^2}{\partial v_\ell^2}\right)\right.$$

$$\left. + \frac{1}{2}\omega_n^2\left(r_\ell^2 + v_\ell^2\right) - \beta_n \cos\left(-\sqrt{2}\,\alpha_n \Phi_\ell + \sqrt{2}\,\delta_n r_\ell\right)\right\} \qquad (3.9)$$

$$-\sum_\ell \beta_n \cos\left[\Psi_{\ell+1} - \Psi_\ell + \frac{\alpha_n}{\sqrt{2}}\left(\Phi_{\ell+1} + \Phi_\ell\right) + \frac{\delta_n}{\sqrt{2}}\left(r_{\ell+1} + r_\ell + v_{\ell+1} - v_\ell\right)\right]$$

We again make the quadratic approximation to the single ℓ-block terms in (3.9) and diagonalize the resulting system of coupled oscillators. As discussed above (3.4) this procedure is justified if, in terms of the original θ parameters, the oscillators to be frozen out have sufficiently narrow ground state wave functions. We verify this ā posteriori at each step of the iterative calculation for $f^2 \gg 1$.

In this approximation the single block terms in (3.9) describe coupled oscillators in the r_ℓ, Φ_ℓ variables. To find the normal modes we rotate the coordinates

$$\chi_\ell \equiv r_\ell \cos\xi_n - \Phi_\ell \sin\xi_n$$

$$\theta_\ell \equiv r_\ell \sin\xi_n + \Phi_\ell \cos\xi_n \qquad (3.10)$$

so that we can rewrite the H_n as

$$\frac{1}{\Lambda} H_n = \sum_{\ell} \left\{ (2 c_n - \beta_n) \mathbb{1}_\ell + \frac{\alpha_n^2}{4} \left(- \partial^2/\partial\Phi_\ell^2 \right) + \frac{1}{2} \left(\frac{-\partial^2}{\partial v_\ell^2} + \omega_n^2 v_\ell^2 \right) \right.$$

$$+ \frac{1}{2} \left(- \frac{\partial^2}{\partial \theta_\ell^2} + \omega_{n+1}^2 \theta_\ell^2 \right)$$

$$\left. + \frac{1}{2} \left(- \frac{\partial^2}{\partial \chi_\ell^2} + \bar{\omega}_n^2 \chi_\ell^2 \right) \right\}$$

$$- \sum_{\ell} \beta_n \cos \left[\Psi_{\ell+1} - \Psi_\ell + \left(\frac{\alpha_n}{\sqrt{2}} \cos \xi_n + \frac{\delta_n}{\sqrt{2}} \sin \xi_n \right) \left(\theta_{\ell+1} + \theta_\ell \right) \right.$$

$$+ \left(\frac{\delta_n}{\sqrt{2}} \cos \xi_n - \frac{\alpha_n}{\sqrt{2}} \sin \xi_n \right) \left(\chi_{\ell+1} + \chi_\ell \right)$$

$$\left. + \frac{\delta_n}{\sqrt{2}} \left(v_{\ell+1} - v_\ell \right) \right] \tag{3.11}$$

where

$$\omega_{n+1}^2 \equiv \Omega_-^2$$

$$\bar{\omega}_n^2 \equiv \Omega_+^2 \tag{3.12}$$

$$\Omega_\pm^2 \equiv \frac{\omega_n^2}{2} + \beta_n \left(\delta_n^2 + \alpha_n^2 \right)$$

$$\pm \sqrt{ \beta_n^2 \left(\delta_n^2 + \alpha_n^2 \right)^2 + \beta_n \left(\delta_n^2 - \alpha_n^2 \right) \omega_n^2 + \frac{1}{4} \omega_n^4 }$$

and

$$\tan \xi_n \equiv \frac{1}{\delta_n}\left(\alpha_n - \frac{\omega_{n+1}^2}{2\beta_n \alpha_n}\right) \tag{3.13}$$

We now complete the reduction of H_n to the truncated Hamiltonian H_{n+1} for the $(n+1)^{st}$ iteration by "freezing out" the two higher frequency oscillators in the variables v_ℓ and χ_ℓ with frequencies ω_n and $\bar{\omega}_n$, respectively, both of which are greater than ω_{n+1}. Specifically we take the expectation value of (3.11) using normalized ground state oscillator wave functions of the form

$$\prod_\ell e^{-\frac{1}{2}\bar{\omega}_n \chi_\ell^2} e^{-\frac{1}{2}\omega_n v_\ell^2} \quad . \tag{3.14}$$

Comparing the result with H_n in (3.6) we see that the truncated H_{n+1} has the same form with

$$\alpha_{n+1}^2 = \alpha_n^2/2$$

$$\beta_{n+1} = \beta_n \, e^{-\left[\frac{\delta_n^2}{8\omega_n} + \frac{\left(\alpha_n \sin \xi_n - \delta_n \cos \xi_n\right)^2}{8\bar{\omega}_n}\right]}$$

$$\delta_{n+1} = \frac{1}{\sqrt{2}}\left(\alpha_n \cos \xi_n + \delta_n \sin \xi_n\right)$$

$$c_{n+1} = 2(c_n) - \beta_n + \frac{1}{2}(\omega_n + \bar{\omega}_n) \tag{3.15}$$

and

$$\alpha_0^2 \equiv x_0^2 = \delta_0^2$$

$$\omega_0^2 = 2$$

$$\beta_0 = \frac{1}{2x_0^2} \tag{3.16}$$

Equations (3.12), (3.13), and (3.15) provide us with an analytic recursion relation for the parameters in the truncated Hamiltonian. The results of a numerical solution of this relation are given in Table I for $x_0 = 0.1$. As we can see, the truncated Hamiltonian rapidly iterates to a fixed form which is, to all intents and purposes, the fixed form to which the massless free field would iterate. In particular, the oscillator frequency, ω_n, goes to zero like $1/2^N$ or $(\text{volume})^{-1}$. Thus we find that the gap to the lowest lying state above the ground state of this model vanishes as 1/(volume), and we are dealing with a massless theory. It is apparent from (3.11) and (3.12) that the higher oscillators have frequencies in the ratio $\omega_n^2/\omega_{n+1}^2 \sim 2$ and $\overline{\omega}_n^2/\omega_{n+1}^2 \sim 2^n$ to the lowest one that we retain.

In order to verify that our procedure satisfies Coleman's theorem, we must show that

$$\langle e^{i\theta_j} \rangle_N = 0 ,$$

where $\langle \; \rangle_N$ denotes the ground state expectation value (3.14) with respect to the higher oscillators after the N^{th} iteration. We find by the same steps leading to (3.15),

$$\langle e^{i\theta_j} \rangle_N = \beta_N/\beta_0 \approx e^{-N \cdot K x_0^2}$$

and K is a constant. The dependence on x_0^2 is explicitly shown as deduced from (3.15) and (3.16), in the sixth column of Table I for $x_0 = 0.1$, and in Table II for $x_0 = 0.01$: $K \rightarrow 0.368$. Since the volume is proportional to 2^N, we have

$$\frac{\beta_N}{\beta_0} \approx e^{-c\{\ell n L\} x_0^2}$$

which is the result one obtains heuristically by evaluating the expectation value of $c^{i\theta(j)}$ for a massless free field in one space-one time dimension.[9] The variational analysis described here applies to the $f^2 \gg 1$ limit of (2.14). In the opposite limit of $f^2 \ll 1$, the theory switches over to one of massive excitations—a result which has recently[10] been obtained by going to a dual lattice formulation of the X-Y model and then discussing vortices in the path integral formulation of this theory.

The occurrence of such a transition to a massive theory is evident from an inspection of (2.14) or (2.17) since, in the limit $f^2 \to 0$, the single site terms proportional to $1/f^2$ dominate $H_{(\lambda^{1/2} f \to \infty)}$. Therefore the groundstate is the unique state $(m_j = 0)$ by (2.15) and (2.16). The first excited states of the theory have a mass $2/f^2$, corresponding to any one m_j equal to ± 1. We discuss the perturbation about this limit in the next section. From the point of view of our variational analysis the breakover to this phase occurs when $\beta_{n+1} < \beta_n/2$ so that the single site terms grow in strength relative to the coupling terms. Table III shows that this happens for $x_0 \gtrsim 1$. In this region we have to be more careful in taking account of boundary effects since the Gaussian approximation (3.4) to (3.3) breaks down but the analysis is still straightforward.

IV. HEURISTIC PERTURBATION TREATMENT

Having shown that our lattice techniques do not predict a non-vanishing expectation value for ϕ_j, in this section we carry through a perturbation treatment in order to provide a better heuristic understanding of the mechanism that prevents this occurrence. In particular, we want to show why Goldstone bosons disappear for arbitrarily large but finite f^2, although the $f^2 \to \infty$ limit is a classical theory with Goldstone bosons. In other words, why can't we do perturbation theory in $1/f^2 \ll 1$?

For $f^2 \ll 1$ which is normally called the strong coupling limit, since the gradient term in H is small compared with the single-site potential terms, we have a unique ground state to the theory. It corresponds by (2.17) to the eigenstate of

$$H_0 = \Lambda \sum_{j=-N}^{N} \frac{1}{f^2} J_3^2(j) \tag{4.1}$$

with all rotors in their ground state—i.e.,

$$m_j = 0 \text{ for each } j \tag{4.2}$$

To first order the perturbation

$$H_I = -\frac{1}{4} \Lambda f^2 \sum_{j=-N}^{N-1} \left[J_+(j)\, J_-(j+1) + J_-(j)\, J_+(j+1) \right] \tag{4.3}$$

has no effect on the ground state section. It does, however, lead to a first order shift in the 2(2N+1) fold degenerate sector of first excited states, in which any one rotor is excited to $m_j = \pm 1$. By standard degenerate perturbation theory we find these states split into two degenerate momentum bands with

$$E(\pm k_p) = \Lambda \left\{ \frac{1}{f^2} - \frac{1}{2} f^2 \cos k_p + \ldots\ldots\ldots \right\} \quad \text{for } f^2 \ll 1 \,;\; k_p \equiv \frac{2\pi p}{2N+1} \tag{4.4}$$

Thus the theory has a mass gap

$$\Delta E \approx \Lambda \left\{ \frac{1}{f^2} - f^2/2 \right\} \tag{4.5}$$

and the ground state remains unique for $f^2 \ll 1$, so that

$$\langle \Psi_0 | \phi_j | \Psi_0 \rangle = f \langle \Psi_0 | e^{i\theta_j} | \Psi_0 \rangle = 0 \tag{4.6}$$

and there is no Goldstone boson. Note, however, the gap begins to narrow in (4.5) with increasing f.

Turning to the limit $f \gg 1$, we first show the failure of a perturbation expansion in terms of (4.1). A convenient product basis for diagonalizing (4.3) is in terms of

$$\Psi \equiv \prod_j |\theta_j\rangle \tag{4.7}$$

where

$$|\theta_j\rangle \equiv \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{im\theta_j} |m\rangle_j \tag{4.8}$$

and

$$\langle\theta_j'|\theta_j\rangle = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{-im(\theta_j - \theta_j')} = \delta(\theta_j - \theta_j') \quad .$$

Since

$$J_\pm(j)|\theta_j\rangle = e^{\pm i\theta_j} |\theta_j\rangle \tag{4.9}$$

we find

$$H_1 |\Psi\rangle = -\left\{ \frac{\Lambda}{2} f^2 \sum_{j=-N}^{N-1} \cos(\theta_{j+1} - \theta_j) \right\} |\Psi\rangle \tag{4.10}$$

In this limit we have a classical description, since the conjugate variable to θ_j, i.e., $-i\partial/\partial\theta_j$ has been dropped with the neglect of (4.1). Equation (4.10) tells us that there is a one-parameter family of ground states

$$\Psi_0(\tilde{\theta}) \text{ with } \theta_j = \tilde{\theta} \text{ for all j} \tag{4.11}$$

where

$$-\pi \leq \tilde{\theta} < \pi$$

These are "classical" states with energy

$$E_I = -(2N)\Lambda f^2/2 \tag{4.12}$$

and with a non-vanishing value for

$$\frac{1}{\langle \Psi_0(\tilde{\theta}) | \Psi_0(\tilde{\theta}) \rangle} \langle \Psi_0(\tilde{\theta}) | e^{i\theta_j} | \Psi_0(\tilde{\theta}) \rangle = e^{i\tilde{\theta}}$$

in apparent contradiction to Coleman's theorem. When we now treat (4.1) as a perturbation on these states, we find

$$\frac{1}{\langle \Psi_0(\tilde{\theta}), \Psi_0(\tilde{\theta}) \rangle} \langle \Psi_0(\tilde{\theta}) | H_0 | \Psi_0(\tilde{\theta}) \rangle = \frac{\Lambda}{f^2} \sum_j \langle \tilde{\theta} | J_3^2(j) | \tilde{\theta} \rangle \frac{1}{\langle \tilde{\theta} | \tilde{\theta} \rangle}$$

$$= \frac{\Lambda}{f^2}(2N+1) \frac{\sum_{m=-\infty}^{\infty} m^2}{\sum_{m=-\infty}^{\infty} 1} = \infty \tag{4.13}$$

which evidently diverges. Thus the effects of correcting the classical $f^2 = \infty$ limit cannot be perturbatively analyzed for the theory as formulated. We can, however, proceed perturbatively if we cut off the sum over rotor excitations at a finite M_{max} and study the disappearance of the Goldstone bosons in the $M_{max} \to \infty$ limit.

A finite M_{max} cutoff on the sum over m can be imposed simply by appending to (4.8) and (4.9) the definitions

$$|M_{max} + 1\rangle_j \equiv |-M_{max}\rangle_j$$
$$|-M_{max} - 1\rangle_j \equiv |+M_{max}\rangle_j \qquad (4.14)$$

so that

$$J_{\pm}(j)\,|m\rangle_j = |m \pm 1\rangle_j \quad \text{for } |m| < M_{max}$$
$$J_{\pm}(j)\,|\pm M_{max}\rangle_j = |\mp M_{max}\rangle_j \qquad (4.15)$$

This requirement is equivalent to discretizing the angle variable θ_j at each site so that it can only take on $(2M_{max} + 1)$ values θ_p—i.e., in place of (4.8) we write

$$|\theta_p(j)\rangle \equiv \frac{1}{\sqrt{2M_{max} + 1}} \sum_{m=-M_{max}}^{+M_{max}} e^{im\left(\frac{2\pi p}{2M_{max}+1}\right)} |m\rangle_j \qquad (4.16)$$

where p is an integer: $-M_{max} \leq p \leq M_{max}$. The classical $f^2 \to \infty$ limiting results of (4.8) - (4.10) still obtain except that there is a discrete set of $(2M_{max} + 1)$ permitted values of the parameter

$$\text{i.e.} \quad \tilde{\theta} \to \frac{2\pi p}{2M_{max} + 1} \quad . \qquad (4.17)$$

For the $(2M_{max} + 1)$-fold degenerate ground states the integers p_j assume the same value at each site. In the limit of $M_{max} \to \infty$ the ground state is infinitely degenerate and we retrieve the classical Goldstone picture. The lowest lying excitations correspond to "micro-sharp kinks" for which p jumps by its minimum step

$$\delta\theta \equiv \frac{2\pi}{2M_{max} + 1} \qquad (4.18)$$

at one lattice point ℓ. These configurations are generated from the degenerate vacua by the operators

$$\phi^{\pm}(\ell) \equiv e^{\pm i \delta\theta \sum_{j=\ell}^{N} J_3(j)} \tag{4.19}$$

i.e.,

$$\Psi^{\pm}(\ell) \equiv \phi^{\pm}(\ell) \, |\Psi_0\rangle$$

and have energies, to leading order in f,

$$E(\ell) = -f^2 \Lambda N + \frac{f^2 \Lambda}{2} \left(1 - \cos \frac{2\pi}{2M_{max} + 1}\right) \tag{4.20}$$

$$= E_0 + \frac{f^2 \Lambda}{4} \left(\frac{2\pi}{2M_{max} + 1}\right)^2 + 0\left(f^2 / M_{max}^4\right)$$

Next we compute the energy shifts due to (4.1) for the ground state Ψ_0 and the excited kink states. Analogously to (4.13) we find that the ground state is shifted up by

$$\Delta E_0 = \frac{\Lambda}{f^2} (2N+1) \frac{\left\{\sum_{m=-M_{max}}^{+M_{max}} m^2\right\}}{\left\{\sum_{m=-M_{max}}^{+M_{max}} 1\right\}} \equiv \frac{\Lambda}{f^2} (2N+1) D(0)$$

$$\approx \frac{\Lambda}{f^2} (2N+1) \frac{1}{3} M_{max}^2 \quad \text{for } M_{max} \gg 1 . \tag{4.21}$$

To calculate the energy shift for the kink state, we compute first the matrix elements of (4.1) among the degenerate states for different values of ℓ:

$$\left\langle \Psi^{\pm}(\ell') \right| \frac{\Lambda}{f^2} \sum_{j=N}^{N} J_3^2(j) \left| \Psi^{\pm}(\ell) \right\rangle = \frac{\Lambda}{f^2} \left\{ (2N+1) D(0)\, \delta_{\ell\ell'} + D(\delta\theta) \left[\delta_{\ell,\ell'+1} + \delta_{\ell,\ell'-1} \right] \right\}$$

where

$$D(\delta\theta) \equiv \frac{1}{2M_{max}+1} \sum_{m=-M_{max}}^{M_{max}} m^2 e^{i m \delta(\theta)} \tag{4.22}$$

The perturbation is diagonalized in a momentum basis

$$\left| \Psi_k \right\rangle \equiv \sum_{\ell=-N}^{N} \frac{1}{\sqrt{2N+1}} e^{-ik\ell} \left| \Psi_\ell \right\rangle \tag{4.23}$$

whose first order eigenvalues are

$$E(k) = E_0 + f^2 \frac{\Lambda}{2} (1 - \cos\delta\theta) + \frac{\Lambda}{f^2} \left[(2N+1) D(0) + 2D(\delta\theta) \cos k \right] \tag{4.24}$$

From (4.20) - (4.24) we find, to order $1/f^2$, that the gap between the micro-kink and the ground state is

$$\Delta E(k) = f^2 \frac{\Lambda}{2} (1 - \cos\delta\theta) + \frac{2\Lambda}{f^2} D(\delta\theta) \cos k \quad \text{for } f^2 \gg 1 . \tag{4.25}$$

For $\delta\theta$ as given by (4.18) for large M_{max}, (4.22) becomes

$$D(\delta\theta) \approx -2 M_{max}^2 / \pi^2$$

and the mass gap, $\Delta E(k=0)$ in (4.25) is

$$\Delta E(0) \cong \Lambda \left[\frac{\pi^2 f^2}{4 M_{max}^2} - 4 M_{max}^2 / \pi^2 f^2 \right] \tag{4.26}$$

Equation (4.26) shows that for

$$f^4 < \frac{16\, M_{max}^4}{\pi^4} \tag{4.27}$$

the micro-kink states cross below the (2N+1)-degenerate would-be ground states. As $M_{max} \to \infty$ (4.27) is satisfied for <u>all</u> finite f^2. In fact, one can construct micro-kink states that lie still lower than the single kind (4.19) by applying $\phi^+(\ell_i)$ many times and building up a series of steps of the form (4.18). An example is the class of states built from (4.19) with r micro-kinks, i.e.,

$$|\Psi(r)\rangle = \sum_{\ell_1 \ldots \ell_r} \phi^\dagger(\ell_1) \ldots \phi^\dagger(\ell_r) |\Psi_0\rangle \ .$$

Furthermore when $r \geq 2\, M_{max} + 1$ the evaluation of $\langle \Psi(r) | e^{i\phi_j} | \Psi(r) \rangle$ gives zero since each phase contributes, and the sum of the roots of unity vanishes.

As a final comment we note that these low lying micro-kink states which cross below the vacuum states are unique to one dimension. In higher dimensions they would require a line or surface of kinks rather than just one single step and therefore would be higher in energy by an amount diverging as $L \to \infty$.

CONCLUSION

We have demonstrated that our iterative procedure of constructing an effective lattice Hamiltonian when applied to the U(1)-Goldstone model in two dimensions leads to a solution in accord with Coleman's theorem. Adding this to its earlier success in reproducing known exact features of the transverse Ising and Thirring models we believe this technique is now ready for application to gauge theories in three and four dimensions in quest of answering whether or not QCD can provide a basis for understanding quark confinement. Our interest lies in this direction. There is of course much more work that should still be done with these calculational tools in further analyzing the cutoff models that we have already discussed in this and preceding papers. For example, analyses of the equations of motion, operator product expansions, current algebra relations, Lorentz invariance, and of an SU-2 Goldstone model are of considerable interest in their own rights. We leave these problems as "exercises for the reader."

References

1. S. D. Drell, M. Weinstein, and S. Yankielowicz, Phys. Rev. D14, 487 (1976); D14, 1627 (1976); "Quantum Field Theories on a Lattice: Variational Methods for Arbitrary Coupling Strengths and the Ising Model in a Transverse Magnetic Field," Stanford Linear Accelerator Preprint SLAC-PUB-1942 (1977), to appear in Phys. Rev.; and S. D. Drell, B. Svetitsky, and M. Weinstein, "Fermion Field Theory on a Lattice: Variational Analysis of the Thirring Model," Stanford Linear Accelerator Preprint SLAC-PUB-1999 (1977), to appear in Phys Rev.; hereinafter these are referred to as Papers I, II, III, and IV, respectively.
2. S. Coleman, Math. Phys. 31, 259 (1973).
3. N. D. Mermin and H. Wagner, Phys. Rev. Letters 17, 1133 (1966).
4. This is described in detail and applied in Ref. 1.
5. In a more accurate treatment they can be variationally determined as shown in Paper III.
6. In this limit this theory has the same physics as the X-Y model that has been extensively studied by many-body physicists, and so this limit has the virtue that much is known about its properties. See Lieb, Schulz, and Mattis, Ann. Phys. (N.Y.) 16, 407 (1961).
7. For simplicity we use the difference operator in defining the gradient. See Ref. 1 for details.
8. The lowest eigenstates of the single site Schrödinger problem are arbitrarily well approximated in this limit by Gaussian packets of the form

$$e^{-\lambda^{1/2} f (r - r_{Min})^2} \quad \text{where} \quad r_{Min} = \frac{1}{\sqrt{2}} f \cdot \left[1 + 0\left(\frac{1}{\lambda f^2}\right)\right].$$

The last term on the right-hand side of (2.13) contributes only $o\left(\frac{1}{2N+1}\right)$ to the energy density, and we will ignore it in what follows.

9. S-K. Ma and R. Rajaraman, Phys. Rev. D11, 170 (1975).

10. For a general discussion and extensive references see Jose, Kadanoff, Kirkpatrick and Nelson, "Renormalization, Vortices, and Symmetry-Breaking Perturbations in the Two-Dimensional Planar Model," Phys. Rev. B (to appear).

Table I: Iteration for $x_0 = .1$

Iteration$_{(n)}$	$(\alpha_n)^2$	$(\omega_n)^2$	β_n	δ_n	K_n	c_n
0	.01	2	50	.1		-50
1	.005	.76393	49.906	.097325	.18831	-149.08
2	.0025	.22226	49.77	.077379	.27347	-347.38
3	.00125	.059722	49.612	.05778	.31801	-744.05
8	3.9062×10^{-5}	6.3112×10^{-5}	48.733	.010803	.36635	-25336
9	1.9531×10^{-5}	1.5802×10^{-5}	48.554	.0076468	.36727	-50721
10	9.7656×10^{-6}	3.9535×10^{-6}	48.376	.0054099	.36774	-101490
14	6.1035×10^{-7}	1.5454×10^{-8}	47.669	.0013531	.36819	-1624562
15	3.0518×10^{-7}	3.8637×10^{-9}	47.493	9.5682×10^{-4}	.3682	-324171
16	1.5259×10^{-7}	9.6594×10^{-10}	47.319	6.7658×10^{-4}	.36821	-6498389
17	7.6294×10^{-8}	2.4149×10^{-10}	47.145	4.7841×10^{-4}	.36821	-12996826

The notation in this table conforms to that given in Eq. (3.15) except for the definition of K_n. This is defined by the relation $\beta_{n+1} = e^{-K_n x_0^2} \beta_n$. Note in particular that ω_n^2 decreases by a factor of 2^2 for each iteration and hence $\omega_N \propto 1/2^N \propto (\text{volume})^{-1}$.

Table II: Iteration for $x_0 = .01$

Iteration$_{(n)}$	$(\alpha_n)^2$	$(\omega_n)^2$	β_n	δ_n	K_n	c_n
0	.0001	2	5000	.01		-5000
1	5×10^{-5}	.76393	4999.9	.0097325	.18831	-14999
2	2.5×10^{-5}	.22233	4999.8	.0077385	.27346	-34998
3	1.25×10^{-5}	.059757	4999.6	.0057788	.31801	-74994
9	1.9531×10^{-7}	1.5813×10^{-5}	4998.5	7.6469×10^{-4}	.36717	-5114571
10	9.7656×10^{-8}	3.9562×10^{-6}	4998.3	5.4099×10^{-4}	.36763	-10234139
11	4.8828×10^{-8}	9.8941×10^{-7}	4998.2	3.8264×10^{-4}	.36786	-20473277
14	6.1035×10^{-9}	1.5465×10^{-8}	4997.6	1.3531×10^{-4}	.36807	-1.64×10^{8}
15	3.0518×10^{-9}	3.8663×10^{-9}	4997.4	9.5682×10^{-5}	.36808	-3.2765×10^{8}
16	1.5259×10^{-9}	9.6658×10^{-10}	4997.2	6.7658×10^{-5}	.36809	-0.65530×10^{9}
17	7.6294×10^{-10}	2.4165×10^{-10}	4997.1	4.7841×10^{-5}	.36809	-1.31060×10^{9}

This table is included to show that for both $x_0 = .1$ and $x_0 = .01$ the iteration is basically the same up to a scale factor. The fact that α_n^2 and ω_n^2 both drop rapidly with respect to β_n tells us that the oscillator approximation is valid at all stages.

Table III: Iteration for $x_0 = 2$

Iteration$_{(n)}$	$(\alpha_n)^2$	$(\omega_n)^2$	β_n	δ_n	K_n	c_n
0	4	2	.125	2		-.125
1	2	.76393	.058855	1.9465	.18831	.506
2	1	.18527	.019007	1.4701	.28258	1.6092
3	.5	.036168	.0049983	1.0152	.33393	3.46305
4	.25	.0059441	.0010923	.67625	.38020	7.0305
5	.125	7.8728×10^{-4}	1.7597×10^{-4}	.44295	.45645	14.1015
6	.0625	7.3716×10^{-5}	1.5154×10^{-5}	.28813	.61300	28.2175
8	.015625	3.6305×10^{-8}	4.6693×10^{-11}	.12638	2.18685	112.88

This table shows that when $x_0 \gtrsim 1$ we can no longer apply our oscillator approximation. It breaks down in the sense that β_n tends to zero faster than α_n^2 and ω_n^2, which means the Gaussians see the boundaries of the region in θ and one must use Mathieu functions to iterate. The columns $(\alpha_n)^2$ and $(\omega_n)^2$ are not significant except insofar as they show this effect, and the large positive values of c_n for $n > 1$ show that the Gaussian approximation is quite poor.

Table Captions

I. The notation in this table conforms to that given in Eq. (3.15) except for the definition of K_n. This is defined by the relation $\beta_{n+1} = e^{-K_n x_0^2} \beta_n$. Note in particular that ω_n^2 decreases by a factor of 2^2 for each iteration and hence $\omega_N \propto 1/2^N \propto (\text{volume})^{-1}$.

II. This table is included to show that for both $x_0 = .1$ and $x_0 = .01$ the iteration is basically the same up to a scale factor. The fact that α_n^2 and ω_n^2 both drop rapidly with respect to β_n tells us that the oscillator approximation is valid at all stages.

III. This table shows that when $x_0 \gtrsim 1$ we can no longer apply our oscillator approximation. It breaks down in the sense that β_n tends to zero faster than α_n^2 and ω_n^2, which means the Gaussians see the boundaries of the region in θ and one must use Mathieu functions to iterate. The columns $(\alpha_n)^2$ and $(\omega_n)^2$ are not significant except insofar as they show this effect, and the large positive values of c_n for $n > 1$ show that the Gaussian approximation is quite poor.