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CHIRAL SOLITONS AND CURRENT ALGEBRA*

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ABSTRACT

We investigate the possibility of a realistic hadrodynamics based solely on observable currents. The basic idea is to exploit the soliton generation in bosonic chiral theories as a mechanism for finding the fermionic representations of current algebra. A prototype realization is Skyrme's O(4) invariant theory of pions and nucleons. A comprehensive reexamination of this model in the context of chiral dynamics suffices to reveal a strikingly self-consistent dynamical picture. First a differential geometric formulation gives the proper framework for a chiral invariant quantum theory of solitons and allows a compact derivation of Skyrme's main results. While no exact analytic solution is found, the solitons are sufficiently localized so that their singularities can be properly isolated out for analysis. Using Witten's ansatz, a determination of the form of the 1-soliton singularity is obtained from the field equations. It is given by Cayley's stereographic projection from S^3 to $R^3 \cup \{\infty\} \approx S^3$; a most suitable form for the proof of spinor structure. Williams' proof that the quantized 1-soliton sector gives rise to fermionic spin states is recalled. It is argued that the topological dynamics of this sector induce an invariance group $K = SU(2)_I \times SU(2)_J$ and its associated strong coupling isobaric spectrum for the nucleons. The associated current algebra is derived and resolves the main difficulties of the Sugawara-Sommerfield program. The signature of a field theoretical bootstrap is clear: massive nucleons as soliton bound states of Nambu-Goldstone bosons illustrate a dynamical mechanism dual to that of Nambu and Jona-Lasinio.

I. CHIRAL SOLITONS AS HADRONS

Strong interaction physics have long been in need of both new conceptual and nonperturbative computational methods. While a colored gauge theory of quarks and gluons is held as a paradigm [1], in actual practice the complexities of the strong coupling regime have thus far empeded any real confrontation with the data on hadrons. In the infrared limit of QCD where the hadronic spectrum is to emerge there exists the issue of the proper collective coordinates to use in the resulting effective description of hadrons. Thus one has the boundary conditions of current algebra, PCAC etc — that is one expects a dynamical derivation of chiral dynamics.

A possibly viable alternative or supplement to QCD, at least until the latter yields physical results, is the soliton approach to hadrodynamics [2]. Here one observes that the lack of progress toward a field theory of hadrons could well be due to the old belief in the crucial role of quantum mechanics in having stationary bound states to relativistic field theories. That it need not be the case has been demonstrated for solitons, which are in fact bound states already at the classical level. Whence they can be treated as a nonperturbative Born term dominating a suitable weak coupling expansion of the S-matrix. For a many body system like a hadron, the prospect of a more intuitive and reliable semi-classical description is preferrable to a still intractable quark dynamics.

Currently, the existence and properties of exact pseudoparticle solutions underscores the importance of the aesthetic geometries of Yang-Mills and gravitational theories [3]. Formally akin to general relativity, chiral dynamics is intrinsically nonlinear and endowed with a very appealing geometric structure. The methodology of chiral dynamics is clear; it aspires to be a quantum field theory of strong interactions and lays no claim to be a fundamental theory of matter. It only purports to describe a limited range of low energy phenomena by dealing with the observable degrees of freedom, with the epiphenomena of possibly more basic hadronic interactions. A casual comparison with solid and liquid state physics readily identifies chiral dynamics as the relativistic analogs of the mean field, Landau-Ginzburg equations describing the physical realizations of spontaneous symmetry breaking. Instead of inquiring into the origin of the symmetry breakdown, one simply postulates a dynamical symmetry and explores its consequences. From the technical side, while the domain of applicability of phenomenological Lagrangians is usually restricted to the semi-classical approximation, the empirical success at the tree graph level has induced some to take chiral dynamics more seriously [4], to develop for it a consistent quantization scheme [5]. Chiral dynamics is not explicitly renormalizable, yet it has been hoped that its geometric nonlinearities might be such that a mechanism of compensation of divergences may render physical results finite. Support for this idea is indicated by the good agreement with the data given by computations at the one and two loop level in a superpropagator regularization quantum chiral theory [6].

From the above perspective, it seems important to test how far chiral dynamics in fact can be pushed, what physics besides the tree graph results can be extracted from the geometry. Specifically, the phenomenological relevance of the semi-classical approximation, a rather unique case in strong interaction physics, naturally invites excursion into the soliton sectors, if any, of nonlinear chiral theories. Just as the usual chiral dynamics is analogous to the dynamics of the homogeneous phase of ordered media, be they superfluids or liquid crystals [7,8], the study of chiral solitons should parallel the classification and dynamics of the defects or inhomogeneities in these media. In the latter, such observed singularities have been successfully accounted for by the mean field hydrodynamic description. It is pertinent to inquire whether chiral dynamics similarly generate singularities which upon quantization may be identified with observable hadronic states.

Before summarizing our work, it is helpful to briefly recall the salient features of topological chiral solitons which remind us of hadrons [9].

1) They carry an exactly conserved homotopic charge, a consequence of vacuum degeneracy. This dynamical charge is localized arbitrarily and provides an ideal candidate for a baryon number.

2) They are extended and interact strongly.

3) They are very massive compared to the masses of the fields in the Lagrangian. These features coupled with the existence of a rich spectrum of bound states emerging from one or a few fields have been the key advantages of a soliton approach to hadrons.

In a series of pioneering papers, Skyrme [10] first saw in the soliton generation a unified mechanism to have both mesons and nucleons from a quantized bosonic field theory. The latter are to be bound states of the former in contrast to the Heisenberg-Pauli [11] and Nambu-Jona-Lasinio philosophy [12]. To Faddeev [13], who espouses a more fundamental view of chiral fields, a local gauge invariant extension of Skyrme's model is taken as a good prototype theory where the strong interactions emerge from the weak and electromagnetic forces. Our own goal is more modest. We only aim to test the overall consistency and the scope of the physics of chiral solitons at the semi-classical level. Such a preliminary step seems advisable particularly in light of the a priori very intricate mathematical structure of a quantum soliton expansion [6] and/or more realistic models involving gauge fields [13]. For our purpose, it suffices to critically examine Skyrme's original model in light of the objectives and current understanding of chiral dynamics. Keenly aware of the mutual exclusivity between mathematical rigor and realistic models, we have chosen a possibly optimal attack of the problem. Namely our main effort is to extract as exact and coherent a dynamical picture of Skyrme's model as its rich topological structure can divulge at the semi-classical level. At the same time past questions and answers in chiral dynamics, dynamical groups and current algebra are brought to bear on the problem. As will be apparent, a synthetic view is gained already at the level of a still preliminary investigation.

In succinct terms, Skyrme's model is an $SU(2)\otimes SU(2) \approx O(4)$ invariant chiral theory of pions. While the standard nonlinear pion Lagrangian is quadratic in the group currents, Skyrme's has an additional piece, quartic in these currents, which allows the evasion of "Derrick's theorem" on static three dimensional finite energy solutions [14]. We identify this quartic term as a possible term in one loop quantum correction in the context of Slavnov's chiral quantum theory. By use of differential geometry à la Cartan, we first reformulate Skyrme's essential results in a compact and transparent form ideally suitable for an eventual quantization. While the model does not appear to admit exact analytic solutions, we can still extract key physical features. By exploiting the geometric parallel between our chiral problem and the Yang-Mills Instanton, we make use of Witten's ansatz [15] to parametrize the soliton solutions. Mainly, we show that near the origin, the center of the soliton field configuration, the spherically symmetric 1-soliton solution to the field equations is exactly given by the standard stereographic projection from S^3 to $R^3 \cup \{\infty\}$, of degree 1. In other words, the nontrivial topological structure is localized arbitrarily near the origin where the pion fields have a singularity, a simple zero. This Cayley map of degree 1 is the key ingredient in Williams' proof of spinor structure for the quantized 1-soliton sector [16]. For completeness and logical cogency of our work, we give a compact reformulation of this topological proof. Two other developments also constitute new contributions by us. As a supplement to Skyrme's collective coordinate treatment of the point singularity [17] we deduce from the topological dynamics an effective invariance group $K = SU(2)_J \otimes SU(2)_I$ and its associated isobaric spectrum for the quantized 1-soliton sector. Thus on the one hand, Skyrme's model describes the dynamics of Nambu-Goldstone bosons, the pions, as they locally restore chiral symmetry, the nucleons emerge as classical soliton bound states with homotopic baryon number and dynamical fermionic spin. On the other hand, this model can equally be viewed as a remarkable canonical realization of the Sugawara-Sommerfield dynamical theory of currents [18]. We derive the associated current algebra and show the key difficulties of the Sugawara model to be resolved. Mainly from the soliton generation mechanism <u>and</u> the SU(2) nature of the model, the quantum soliton has baryon number and its associated half-integral spin and isospin states with the usual connection between spin and statistics. One witnesses here the distinct <u>signal</u> of a field theoretical bootstrap dynamics among hadrons. This fermionization mechanism is seen as the obverse of Nambu and Jona-Lasinio's [12].

Our work is organized as follows: In Section II the elements of chiral geometry and topology are written down for later use. In Section III Skyrme's model is formulated in a chiral invariant manner, the essential motivations for its choice are given. An ideal form for the soliton "singularity" of degree 1 is justified via a Witten-type ansatz applied to the field equations. In

Section IV we include Williams' proof of admittance of fermionic spin for the quantized 1-soliton sector. An effective strong coupling dynamics is deduced. In Section V the current-algebra of the model is derived and its implications discussed. In Section VI we comment on the connections between our work and the algebraic approach to quantum field theory. In closing we define several directions for further research.

II. CHIRAL GEOMETRY AND TOPOLOGY

We recall that chiral dynamics is in essence the dynamics of Nambu-Goldstone bosons, e.g. pions ($m_\pi = 0$). As there exists no 3-dimensional linear representation of the chiral group SU(2) × SU(2), the pion fields were taken to transform as the 3-dimensional nonlinear realization which is a representation of the group in a curved Riemannian space [4]. This means that chiral symmetry is an interaction symmetry not shared by the asymptotic fields. Generally then the mathematical framework of chiral dynamics consists in the coordinatization of some homogeneous space M by the Nambu-Goldstone fields ϕ^μ ($\mu = 1,2,\ldots,n$), $n = \dim M$. The Riemannian manifold M has a transformation group G and is realized as a coset space $M = G/H$. H is a maximum subgroup of G and leaves the vacuum invariant. Cartan's differential forms provide a manifestly chiral invariant formulation, independent of the coordinate basis ϕ^μ. It turns out that they also form the most natural framework for the discussion of the topology of chiral solitons as well as for a proper quantum soliton expansion by the background field method. To prepare the ground for the latter procedure, it will be useful to gather here the essential ingredients of the geometric approach [5,6]. The latter may not be widely known and its particular suitability for a quantum theory of solitons may be a new and important point of this section.

In a Riemannian n-space M with local coordinate ϕ^μ ($\mu = 1,2,\ldots,n$), let any set of n linearly independent vectors $\underset{\sim}{e}_{(i)}(\phi^\mu)$ form a complete vector basis, a "repér frame" at each point. Greek indices refer to the coordinates ϕ^μ and tensor components with respect to them. Latin indices refer to the frame $\underset{\sim}{e}_{(i)}$ and transform like scalar under coordinate transformations. So $e^\alpha_{(i)}$ are the contravariant components of the vector $\underset{\sim}{e}_{(i)}$. The frame dependent components of the metric tensor are $g_{ij} \equiv g_{\alpha\beta}e^{\alpha}_{(i)}e^{\beta}_{(j)} = \underset{\sim}{e}_{(i)}\underset{\sim}{e}_{(j)}$ which has a symmetric inverse $g^{ij}, g^{ij}g_{ik} = \delta^{i}_{k}$. One also defines the dual basis $\underset{\sim}{e}^{(i)} \equiv g^{ij}e_{(j)}$ with $\underset{\sim}{e}^{(i)}e_{(j)} = \delta^{i}_{j}$. The key differential forms that can then be introduced are

1) the set of 1-forms

$$\omega^{i}(d\underset{\sim}{\phi}) = e^{(i)}_{\alpha}d\phi^{\alpha} \tag{2.1}$$

associated with the $e^{(\alpha)}$. Solving for $d\phi^{\alpha}$ gives $d\phi^{\alpha} = e^{\alpha}_{(i)}\omega^{i}$, a "translation"

2) The n^2 connection 1-forms

$$\omega^{i}_{j} = \Gamma^{i}_{jk}\omega^{k} \tag{2.2}$$

the Γ^{i}_{jk} called Ricci coefficients are the n^3 numbers, scalars under coordinate transformation

$$\Gamma^{i}_{bc} \equiv -e^{(i)}_{\beta|\gamma}\, e^{\beta}_{(j)}\, e^{\gamma}_{(k)} = +e^{\beta}_{(j)|\gamma}\, e^{(i)}_{\beta}\, e^{\gamma}_{(k)} \tag{2.3}$$

where

$$e^{\alpha}_{(j)|\gamma}\, d\phi^{\gamma} = \left[D\underset{\sim}{e}_{(j)}\right]^{\beta}$$

and the covariant differential $D\underset{\sim}{e}_{(j)} = \underset{\sim}{e}_{(j)}(x + dx) - e_{(j)}(x)$. So Γ^{i}_{jk} yield the components of the covariant derivatives of $\underset{\sim}{e}^{(\alpha)}$ w.r.t. the reper frame. Furthermore

$$D\underset{\sim}{e}^{(i)} = -\omega^{i}_{j}e^{(j)} \quad , \quad D\underset{\sim}{e}_{(j)} = \underset{\sim}{e}_{(i)}\omega^{i}_{j} \tag{2.4}$$

which are "rotations."

3) Given any 1-form $E = A_{\alpha}d\phi^{\alpha} = A_{\alpha}\omega^{\alpha}$ the Riemann tensor $R^{\alpha}_{\beta\gamma\delta}$ is defined by

$$A_{\beta|\gamma\delta} - A_{\beta|\delta\gamma} = A_{\alpha}R^{\alpha}_{\beta\gamma\delta} \tag{2.5}$$

and the Ricci tensor is $R_{\alpha\beta} = R^{\mu}_{\alpha\beta\mu}$. The curvature 2-forms are

$$\Omega^{i}_{j} \equiv \frac{1}{2}\, R^{i}_{jk\ell}\,\omega^{k}\wedge\omega^{\ell} \quad , \tag{2.6a}$$

the torsion form is

$$\Omega^i = \frac{1}{2} S^i_{jk} \omega^j \wedge \omega^k \tag{2.6b}$$

$\wedge$ denotes the wedge or exterior product. The Cartan equations are

$$ds^2 = g_{\alpha\beta} d\phi^\alpha d\phi^\beta = g_{ij} \omega^i \omega^j \tag{2.7}$$

for the metric, with d denoting exterior differentiation and

$$d\omega^i + \omega^i_j \wedge \omega^j = \Omega^i \tag{2.8}$$

$$d\omega^i_j + \omega^i_k \wedge \omega^k_j = \Omega^i_j \tag{2.9}$$

are the first and second structure equations respectively. They summarize the essence of Riemannian geometry and form the roots of the chiral geometric description [4,5]. In a space coordinatized by a^α in a certain basis depending continuously on the parameters b^i, the forms ω^i define a vector da^α, $\omega^i = \omega^i(a,b,da)$ while $\omega^i_j = \omega^i_j(a,b,da,db)$ defines a transformation of a vector under a basis change $dA^i(a,b) = -\omega^i_j(a,b,0,db)A^i$ and the covariant derivative

$$DA^i = \frac{\partial A^i}{\partial a} da + \omega^i_j(a,b,da)A^j \tag{2.10}$$

Now if the holonomy group H of the space M is a subgroup of G then

$$\omega^i_j(a,b,da,db) = (\lambda_\alpha)^i_j \, \theta^\alpha(a,b,da,db) \tag{2.11}$$

are given by a linear combination of the generators λ_α of H. Greek indices are to distinguish entities in H. If furthermore M is a homogeneous space on which G acts transitively then the structure equations (2.8)-(2.9) become

$$d\omega^i = C^i_{k\alpha} \omega^k \wedge \theta^\alpha + \frac{1}{2} C^i_{k\ell} \omega^k \wedge \omega^\ell \tag{2.12}$$

$$d\theta^i = \frac{1}{2} C^\alpha_{\beta\gamma} \theta^\beta \wedge \theta^\gamma + \frac{1}{2} C^\alpha_{k\ell} \omega^k \wedge \omega^\ell \tag{2.13}$$

such that $C^{i}_{k\alpha}$, $C^{i}_{k\ell}$, $C^{\alpha}_{k\ell}$ and $C^{\alpha}_{\beta\gamma}$ are the structure constants of the group G with the algebra

$$\left[Y_\alpha, Y_\beta\right] = iC^{\gamma}_{\alpha\beta}Y_\gamma$$

$$\left[X_k, Y_\alpha\right] = iC^{i}_{k\alpha}X_i$$

$$\left[X_k, X_\ell\right] = iC^{\alpha}_{k\ell}Y_\alpha + iC^{m}_{k\ell}X_m \tag{2.14}$$

The Y_α are the generators of H and X_k those of the coset space M = G/H, taken in the adjoint representation. When G(a,b) = K(a)H(b), the form ω^i and θ^α are defined through

$$G^{-1}(a,b)dG(a,b) = i\omega^j(a,b,da,db)X_j + i\theta^\alpha(a,b,da,db)Y_\alpha \tag{2.15}$$

For the manifold of any semi-simple group, M is a <u>symmetric</u> space $R^{j}_{ik\ell|m} = 0$, one has $C^{\ell}_{ik} = 0$ so that from Eqs. (2.8) and (2.9), one gets $\omega^i_k = C^{i}_{k\alpha}\theta^\alpha$ and $S^{i}_{jk} = 0$ and $R^{j}_{ik\ell} = -C^{j}_{i\alpha}C^{\alpha}_{k\ell}$.

To compute the Cartan forms ω^i and θ^α which constitute the basic building blocks of chiral dynamics [5], it will turn out that for a quantum theory of chiral solitons the coordinate system to expand the fields is the <u>normal frame</u> given by the geodesic parametrization of M, $G^{(N)}_a = \exp(ix_k a^k)$. Let $x^k \rightarrow a^k t$. The structure equations (2.12) and (2.14) become

$$\partial_t \bar{\omega}^i = da^i + a^k \bar{\omega}^i_k$$

$$\partial_t \bar{\omega}^i_e = -R^{i}_{ejk} a^j \bar{\omega}^k \tag{2.16}$$

where

$$\omega^i = a^i dt + \bar{\omega}^i \quad , \quad \bar{\omega}^i\left(t, a^j, da^k\right)_{t=0} = \omega^i_e\left(a^j, da^k\right)_{|t=0} = 0 \ .$$

Then the solution to Eq. (2.16) is

$$\omega^i{}_{|t=0} = \sum \frac{(-1)^n}{2n+1} \left(\mathcal{M}^n\right)^i_k \, da^k$$

$$\omega^i_{k\,|t=0} = \sum \frac{(-1)^n}{(2n+2)} R^i_{kj\ell} a^i \left(\mathcal{M}^n\right)^\ell_\mu da^\mu \qquad (2.17)$$

where $\mathcal{M}^i_\ell = R^i_{kn\ell} a^k a^n$, $R^i_{kn\ell}$ being constants.

For $G = O(4) \approx SU(2) \times SU(2)$, of interest to us here, the above series sums up

$$\omega^i = \left(da^i + \left(\delta_{i\ell} - \frac{a_{i\ell}}{a^2}\right)\left(\frac{\sin a}{a} - 1\right) da^\ell\right) \qquad (2.18)$$

and

$$\theta^\beta = -a^j da^k \varepsilon_{\beta jk} \left(\frac{\cos a - 1}{a^2}\right) \qquad (2.19)$$

$$\left(a = \sqrt{a^i a^i}\right)$$

We shall have occasion to encounter these explicit forms.

So far our formalism applies to the classical chiral theory. To go over to a quantum theory [5,2] the key object is the generating functional for the S-matrix

$$W\left(\pi^{(in)}\right) = \frac{1}{N}\int \Pi\, d\mu(a) \; \exp\left\{i\int d^4x [\mathcal{L}(\omega(a,da),\theta(a,da)) - d^i a d\pi^{in}\right\} \quad (2.20)$$

written in a form with sources. N is the normalization, $\Pi\, d\mu(a)$, the invariant group measure and $\pi^{(in)}$ the asymptotic fields.

To have a proper soliton expansion, the geometry of the Riemannian space again enters crucially. Indeed one needs in any semi-classical expansion of $W\left(\pi^{(in)}\right)$ to properly separate out the "classical" fields ϕ_c from their "quantum" fluctuations ϕ over which integrations are done. Namely, if one wishes for a compact chiral invariant perturbation theory in which there are no reductions, then the usual shift of fields $a \to \phi_c + \phi$ should be understood as a vector addition in the curved space $M = G/H$ of the Nambu-Goldstone fields, i.e.,

$$G_a^{(N)} \to G_{\phi_c}^{(N)} \; G_\phi^{(N)} \tag{2.21}$$

which has an easy geometrical interpretation: it determines a normal coordinate system with the origin at the point ϕ_c such that the coordinates of the classical field ϕ_c are themselves normal coordinates.

We now comment on the topology of chiral solitons in general. We seek finite energy, static, topological solutions to some chiral model built on a symmetric space M with transformation group G. The first step should be the homotopic classification of the solutions allowed by the field equations and boundary conditions. Typically the Lagrangian has the geometrical structure of the line element Eq. (2.7)

$$\mathcal{L} = g_{ij}(\phi)\partial_\mu\phi^i\partial_\mu\phi^j \tag{2.22}$$

The energy finiteness condition then requires that the fields ϕ^i be constant at spatial infinity or equivalently that $U \xrightarrow[|\vec{x}|\to\infty]{} I$ (see Eq. (2.24)). This condition implies that the soliton fields are classified according to the homotopy classes of the maps from $S^3 \sim R^3 \cup \{\infty\}$ into M. While we will specifically deal with the more familiar instance of $M = S^3$ the general classification problem and that of finding the topological currents for an arbitrary manifold M has recently been studied by the powerful method of de Rham cohomology [20]. It turns out that in the case of maps into a Lie group the cohomological information is the same and not less than that provided by homotopy theory. Since it is not possible for us to comprehensively yet briefly summarize the results, we urge the reader to consult Isham's work as an ideal topological complement to the general geometric formalism given here.

Having specified the general geometric and topological formalisms, we now mention two explicit chiral systems. One example of a nonlinear manifold M is the 2-sphere $S^2 \approx SO(3)/SO(2)$, the coset space of the two dimensional continuum classical Heisenberg ferromagnet [21]. The nonlinear field on S^2 is the unit vector field $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$, $\vec{\phi}^2 = 1$ describing the spin waves or magnons. Its 3-dimensional counterpart is the manifold $S^3 \approx SO(4)/SO(3) \approx SU(2) \times SU(2)/SU(2)$, the 3-sphere coordinatized by the unit quaternion $\phi = (\vec{\phi}, \phi^o)$, $\phi^2 = \mathbb{1}$ with $\vec{\phi}$ being, say, the triplet of pion fields [22]. Skyrme's model is of this type [10], and will be discussed next.

Specifically, consider at each space-time point x_μ a quaternionic field $U(x) = \phi^o + i\vec{\tau}\cdot\vec{\phi}$ which takes value on the nonlinear manifold $M = SU(2) \times SU(2)/SU(2) \approx SO(4)/SO(3) \approx S^3$. Indeed U(x) is represented here by a unitary, unimodular 2×2 matrix taken in a doublet spinor representation of SU(2), τ being the Pauli matrices; M is parametrized on a unit sphere S^3, the natural habitat of unit quaternions. Since S^3 is embedded in a four dimensional Euclidean space E_4, the group O(4) carries this hypersphere into itself and is the largest group of isometrices of S^3. $O(4) \approx SU(2) \times SU(2)$ acts naturally on S^3 by the left and right shifts

$$\delta_L g = \delta\eta g \quad , \quad \delta_R g = g\delta\xi \tag{2.23}$$

g being a point in the group manifold of SU(2) and $\delta\eta$, $\delta\xi$ infinitesimal elements of the SU(2) algebra.

The above shifts correspond to the left and right screw motions, the two kinds of absolute parallelisms which by Adams' lemma exist on 4 remarkable spheres, one for each Hurwitz's algebra [23]. They are the spheres S^n, $n = 0,1,3$, and 7 coordinatized by the unit real, complex, quaternionic and octonionic numbers respectively.

For fixed x, U(x) is a group element of SU(2), hence it obeys the same equation as g in Eq. (2.23). Similarly the counterparts of the Cartan 1-forms in Eqs. (2.1) and (2.15) are the coordinate invariant left local group currents

$$L_\mu(x) = U^{-1}\partial_\mu U \tag{2.24}$$

also written as

$$L_{\mu(a)}(x) = e^{\alpha}_{(a)}(\phi)\partial_\mu\phi^\alpha$$

$\phi^\alpha(x)$ $(\alpha = 1,2,3)$ being the local coordinating field.

The L_μ remain invariant under the left-shift $U \to GU$ and transform as $L_\mu \to G^{-1}L_\mu G$ under the right-shift $U \to UG$, G is a global isotopic SU(2) rotation. Of course, the converse is true for the right group current $R_\mu = (\partial_\mu U)U^{-1}$, which could equally well be used as basic objects with which to construct chiral dynamics.

Now if of all possible fields U(x), we only select the particular subset which obeys the would-be energy finiteness condition

$$U(x) \xrightarrow[|\vec{x}|\to\infty]{} I \tag{2.25}$$

true at all times t. Then at any fixed t, U(x) or $L_\mu(x)$ maps the physical space R^3 into the group SU(2). The condition Eq. (2.25) implies that all points at infinity of R^3 are identified with a single point and mapped into the unit element; R^3 can be continuously deformed onto $S^3 \sim R^3 \cup \{\infty\}$, it is compactified onto S^3. Hence we have the equivalent mappings

$$L_i : S^3 \to S^3 \tag{2.26}$$

In consequence, the phase space of the $L_i(x)$ with condition Eq. (2.25) is split into an infinite set of topologically disconnected components, the Chern [24] classes of $\Pi_3(SU(2)) \approx \Pi_3(S^3) \approx Z_\infty$ · Z_∞ denotes the additive group of the integers which are the Brouwer degrees of the mappings. Any two mappings of the same class can be continuously deformed into one another while two maps belonging to different classes cannot. Examples of such deformations or homotopies are a global SU(2) rotation or a time evolution. So the degree of a mapping is a homotopic invariant, hence a conserved entity irrespective of the dynamics of the system. It depends solely on the periodicity of the field U(x) which arises from the compactness of SU(2) and the condition Eq. (2.25).

We proceed to write the degree of the mapping in terms of the group currents $L_\mu(x)$. From the Minkowskian and chiral geometries, it follows that a trivially conserved topological current is [10,13]

$$B_\mu = \frac{i\mathrm{Tr}}{4\pi^2}\, \varepsilon_{\mu\nu\rho\sigma}\left(\left[L_\nu, L_\rho\right] L_\sigma\right) \tag{2.27}$$

$\partial^\mu B_\mu = 0$ follows since $\partial^\mu L_\rho = \partial^\mu \partial_\rho \ln U$ is symmetric in μ and ρ, etc. The mapping degree is

$$B = \frac{i\mathrm{Tr}}{4\pi^2}\int d^3x\; \varepsilon_{ijk}\left(\left[L_i, L_j\right] L_k\right) \tag{2.28}$$

To see the explicit meaning of B, we use the geodesic parametrization of S^3

$$G^{(N)} \equiv U(x) = e^{i/2\, \vec{n}\cdot\vec{\tau}\psi} \quad , \quad \vec{n} \equiv \vec{\phi}/\sqrt{\vec{\phi}^2} \tag{2.29}$$

varies within the sphere $0 \le \psi \equiv \sqrt{\vec{\phi}^2} \le 2\pi$, then

$$B = \frac{1}{2\pi^2}\int \frac{\sin^2\alpha}{\alpha^2}\, d^3\alpha \tag{2.30}$$

where $d^3\alpha = (\alpha)^2 \sin\theta\, d\alpha\, d\theta\, d\chi$ is just the 3-dimensional volume element in spherical coordinates, the radius being $\alpha \equiv \psi/2$, the angles $0 \le \theta < \pi$ and $0 \le \chi \le 2\pi$. The integral Eq. (2.30) is then proportional to the surface of S^3; the factor $1/2\pi^2$ is the proper normalization allowed by the compactness of SU(2) so that B takes integer values. The Chern number B just measures the number of times S^3 is covered in the course of the mapping Eq. (2.26).

Let us remark that contrary to the case of vortices and monopoles where the topological current is axial, here B^μ is a vector current, a consequence of ϕ's being pseudoscalar fields, and B has no corresponding dual charge. The most remarkable feature of B^o is that it is not a total divergence and therefore the charge can be localized arbitrarily in space. This is again to be contrasted with the monopole charge whose detection involves Gauss' law, i.e. the flux through a large sphere at infinity. All this makes Eq. (2.27) an ideal baryon current whose nature is correlated with the existence of three pseudoscalar pions in nature. A topological origin of baryon number is very appealing; while the electromagnetic charge distribution of a particle can be measured through their form factor, the baryon charge is not coupled to any long range field and its measurement is performed through simple counting. Its law of combination is exactly that of the additivity of homotopy classes in the group $\Pi_3(S^3)$.

That the topological charge density can be arbitrarily localized has another crucial implication. It means that the topologically nontrivial structure of the soliton is concentrated at a singularity, e.g. a zero of the pion fields. Chiral dynamics has its exact counterpart in liquid crystals and nematics or in He^3 [7,8]. In these systems, Brinkman spheres or hedgehog singularities are solutions of the Landau-Ginzburg equations and have actually been experimentally observed. We are here seeking their hadronic analogs in a phenomenological chiral system.

Indeed a salient feature of chiral solitons lies in the field U(x) being for any fixed x a group element of SU(2). Hence any solution U(x) to the field equation can always be factorized as

$$U(x) = U_1(x)U_0(x) \tag{2.31}$$

by means of the group composition law. $U_1(x)$ can be a suitably selected singularity to carry the topological charge B while $U_0(x)$ is topologically trivial

(B = 0) and could correspond to the pion cloud which dresses the bare singularity $U_1(x)$. It can be verified that if U(x) is given by the product Eq. (2.31), the baryon number is additive $B = B_{(1)} + B_{(0)}$. This follows from the topological current [10]

$$B_\mu = B_{(1)\mu} + B_{(0)\mu} + \frac{i}{4\pi^2} \partial_\lambda \Omega_{\mu\lambda} \tag{2.32}$$

where

$$\Omega_{\mu\lambda} = \frac{1}{2} \varepsilon_{\mu\rho\lambda\nu} \left[L_{(1)\rho}^{\beta} U \tau^{\beta} U^{+} \tau^{\alpha} L_{(0)\nu}^{\alpha} \right]$$

This factorizability of the field means that any topological singularity can be isolated for study and the localizability property implies the possibility of associating a local field operator to the singularity in analogy to the correspondence between the chiral-Sine-Gordon soliton and the massive fermion of the Thirring model in two dimensions. Of course, the separation (2.31) with (2.29) for U_1 is the same as the prescription for the splitting of the field into its classical and quantum components in Eq. (2.21). While exact analytic solutions to three-dimensional chiral systems are available [14], they belong to models which have no phenomenological basis. In the physical model we choose to examine, no analytic solution has been found. Yet the preceding discussion guarantees that no essential topological information is lost in only analyzing the "singularity" separated from the soliton solution. We shall see that much can already be extracted at the level of our semi-quantitative study of this singularity.

In the next section, we shall motivate and analyze Skyrme's chiral model making use of the concepts and properties just laid out.

III. SKYRMIONS

We now write down Skyrme's chiral Lagrangian [10]. It is composed of two pieces

$$\mathcal{L} = \mathcal{L}_{(2)} + \mathcal{L}_{(4)} \tag{3.1}$$

$$\mathcal{L}_{(2)} = \frac{1}{2c^2}\,\mathrm{Tr}(L_\mu L_\mu) \quad , \quad \mathcal{L}_{(4)} = \frac{\varepsilon^2}{4}\,\mathrm{Tr}\left([L_\mu, L_\nu]^2\right)$$

the static Hamiltonian is

$$H_s = \int d^3\chi\,\mathrm{Tr}\left(\frac{1}{2c^2}L_i^2 - \frac{\varepsilon^2}{4}\left[L_i, L_j\right]^2\right) \tag{3.2}$$

where c is a length, for example the inverse of the pion decay constant f_π^{-1} and ε is a dimensionless coupling. $\mathcal{L}_{(2)}$ is just the usual pion chiral Lagrangian of Gursey et al. [22] written in the Sugawara form [25]. As to the additional quartic term, the key motivations for its inclusion are the following:

a) A simple application of Derrick's scaling argument [14] shows that $\mathcal{L}_{(2)}$ can only have stable, finite energy static solutions in two space dimensions, such as vortices with finite energy per unit length. To allow for truly three dimensional solitons one must introduce either higher spin gauge fields or at least additional terms quartic in the currents L_μ [10] or both [13]. In the second case, the most general chiral invariant interactions involving only four derivatives in the field are

$$\tilde{\mathcal{L}}_{(4)} = \alpha\left(L_\mu^2 L_\mu^2\right) + \beta\left(L_\mu L_\nu\right)^2 + \gamma\left(\partial_\mu L_\nu\right)^2 \tag{3.3}$$

α, β and γ are free dimensional parameters. For a compact notation, the trace symbol Tr is and will be omitted. Any other invariants can be cast into linear combinations of the above terms by use of the Maurer-Cartan identities

$$\partial_\mu L_\nu - \partial_\nu L_\mu = i[L_\mu, L_\nu] \tag{3.4}$$

Skyrme's model is a special case of $\tilde{\mathcal{L}} = \mathcal{L}_{(2)} + \tilde{\mathcal{L}}_{(4)}$. Then the same scaling argument applied to $\mathcal{L}$ and $\tilde{\mathcal{L}}$ shows that finite energy static solutions are possible in <u>only</u> three space dimensions.

b) Secondly, an argument of much physical importance is the following. The quartic terms of Eq. (3.3) should be viewed as a specific choice of counterterms at the one loop level in Slavnov's superpropagator regularized Lagrangian [5]

$$\mathcal{L}_R^{\text{Ren}} = \frac{1}{2}\left(1 + Z_1\lambda^2\right)L_\mu L_\mu + \lambda^4 Z_2\left(L_\mu L_\mu\right)^2 + \lambda^3 Z_3\left(\partial_\mu L_\mu \partial_\nu L_\nu\right)$$
$$+ \lambda^4 Z_4\left(\left[L_\mu, L_\nu\right]^2\right) + \frac{1}{2}\Lambda^{-4}\partial_\mu L_\mu \Box \partial_\nu L_\nu \qquad (3.5)$$

which leads to a divergence free S-matrix. Z_1 diverges quadratically and the others logarithmically. The last term is the regulator removing the divergences in Z_2, Z_3 and Z_4. With Z_1 and Z_3 being the wavefunction renormalization and the renormalization for the regulator $\Box$, there remain only two free parameters Z_2 and Z_4. In the limit of the cut-off $\Lambda \to \infty$, the Z_3 term drops out. Then the selection of $Z_4 = -Z_2 = \varepsilon^2/4$ gives us the model Eq. (3.1). The system (3.5) has been shown to satisfy current algebra constraints, PCAC, it predicts the correct threshold behavior and is unitary up to reasonable energies below a cut-off. As to the higher loop counterterms a general algorithm for their construction is available in the Cartan form approach to chiral theories [5,6].

A related argument for a model of this type comes from a lattice treatment of $\mathcal{L}_{(2)}$. Latticization is necessary in the definition of functional integrals in any case. On the other hand it is also the functional formalism which allows for a description of the quantum theory in terms of the classical one and hence most suitable for any quantum theory of solitons. Here the $\mathcal{L}_{(4)}$

type terms naturally emerge from the dynamics of $\mathcal{L}_{(2)}$ upon integrating out the field components between two frequencies Λ_0 and Λ_1 in the renormalization group scaling procedure [26]. So the quartic term is seen to have a quantum origin and solving for the solitons say in Eq. (3.1) is to classically simulate intrinsically quantum effects up to the one loop level [27]. We will see that by giving rise to an effective interaction the radiative corrections in the SU(2) $\otimes$ SU(2) invariant model not only make possible solitons with topological baryon number but also fermionic spin states.

From the above arguments the effective nature of the model becomes explicit. As a cut-off theory its form has no deep significance lest the higher order loop contributions turn out somehow to be negligible.

c) Finally from the soliton approach, the most appealing argument for the specific choice of the commutator as the quartic term is rooted in the particular form of the topological current Eq. (2.24). It allows for a lower bound to the soliton energy as we can recast Eq. (3.2) into

$$H_s = \frac{1}{2}\int d^3x \left(\frac{1}{c} L_i - \frac{\varepsilon}{2}{}^*L_i\right)^2 + \frac{\varepsilon}{2c}\int d^3x\, L_i {}^*L_i \geq \frac{2\pi^2\varepsilon}{c}|B| \tag{3.6}$$

where ${}^*L_i^a \equiv \varepsilon_{ijk}\varepsilon^{abc} L_j^b L_k^c$. Therefore Eq. (3.6) yields the lower bound for the kink-mass in each homotopy class. The would-be nucleon classical bound states are all massive in accord with the nonlinear realization scheme. Moreover the existence of a lower bound to the energy is the best guarantee for the stability of the small wave expansion about the classical solitons. We now observe that this bound would be saturated if the first term in Eq. (3.6) identically vanishes when

$${}^*L_i^a = \frac{2}{\varepsilon c} L_i^a \tag{3.7}$$

a condition analogous to the self-duality $F^a_{\mu\nu} = \pm F^{a+}_{\mu\nu}$ of the Instanton solution [3]. The resemblance stops here however. At first sight it looks as if the simpler first order equations Eq. (3.7) could replace the highly nonlinear second order field equations, which result from the Heisenberg equation

$$\partial_\mu L_\nu = -i\left[\int \Theta_{\mu 0}\, d^3x \, , L_\nu\right] \tag{3.8}$$

with the energy momentum tensor

$$\Theta_{\mu\nu} = L^a_\mu \left(G^{-1}\right)^{ab} L^b_\nu - g_{\mu\nu}\mathcal{L} \tag{3.9}$$

and

$$G^{ab} = \frac{\varepsilon^2}{2}\left[\left(L^c_\nu L^c_\nu + \frac{2}{\varepsilon^2 c^2}\right)\delta^{ab} - L^a_\nu L^b_\nu\right] \quad . \tag{3.10}$$

The equations of motion are then Eq. (3.4), the Maurer-Cartan equations and

$$\partial_\mu\left[\left(\partial^{ab} + c\varepsilon^2\, \Lambda^{ab}\right)L^b_\mu\right] = 0 \tag{3.11}$$

where

$$\Lambda^{ab} \equiv L^a_i L^b_i - \left(L_i\right)^2 \delta^{ab} \tag{3.12}$$

Equation (2.24) gives Eq. (3.4) as the integrability condition which is purely geometrical in character. It is simple to observe that Eq. (3.7) cannot be reconciled with the Maurer-Cartan Eq. (3.4). So the lower bound for the static energy cannot be saturated by the topological structures which give rise to this nonzero lower bound. Since we cannot get simpler equations to replace the Eq. (3.11) via the "saturation mechanism" [3], we must try solving it directly if explicit solutions are desired.

We shall look for these solutions by use of the following general Witten [15] ansatz

$$L^a_i = \frac{\phi_1}{r}\left(\delta_{ai} - \frac{x_a x_i}{r^2}\right) + \frac{\phi_2}{r}\,\varepsilon_{iaj}\frac{x_j}{r} + A_1\,\frac{x_i x_a}{r^2} \tag{3.13}$$

which upon substitution in Eq. (3.12) yields

$$\Lambda^{ab} = \left(\frac{\phi_1^2 + \phi_2^2}{r^2} - A_1^2\right)\left(\delta^{ab} - \frac{x_a x_b}{r^2}\right) \tag{3.14}$$

and the equation of motion Eq. (3.11) simplifies to

$$A_1' + \frac{2}{r} A_1 - \frac{2\phi_1}{r^2}\left(1 + c^2\varepsilon^2\left[\frac{\phi_1^2 + \phi_2^2}{r^2} - A_1^2\right]\right) = 0 \quad . \tag{3.15}$$

Equation (3.15) contains three unknowns, which implies that the solution to Eq. (3.11) with Eq. (3.13) can be very general. Here we shall restrict ourselves to a spherical hedgehog type solution [10,9]. The latter are obtained by setting in Eq. (3.15)

$$A_1 = \frac{1}{2}\psi'(r) \quad , \quad \left(\psi' = \frac{\partial\psi}{\partial r}\right)$$

$$\phi_1 = \frac{1}{2}\sin\psi(r) \quad ,$$

$$\phi_2 = -1 - \frac{1}{2}\left(1 - \cos\psi(r)\right) \quad . \tag{3.16}$$

Hence

$$L_i^a = \frac{1}{2r}\left[\left(\delta_{ai} - \frac{x_a x_i}{r^2}\right)\sin\psi + \frac{x_a x_i}{r}(r\psi') - \varepsilon_{aij}\frac{x_j}{r}(1 - \cos\psi)\right] \tag{3.17}$$

which should be compared with Eqs. (2.18) and (2.19). For these restricted configurations the equation of motion Eq. (3.11) becomes

$$\psi'' + \frac{2}{r}\psi' - \frac{2\sin\psi}{r^2}\left[1 + \frac{c^2\varepsilon^2}{r^2}\left(4\sin^2\frac{\psi}{2} - r^2\psi'^2\right)\right] = 0 \tag{3.18}$$

Since ψ is an angular variable, the topology of the problem imposes the ensuing boundary conditions: $U(x) \underset{r\to\infty}{\to} I \Rightarrow \psi(r) \underset{r\to\infty}{\to} 0$ and $U(x) \underset{r\to 0}{\to} -I \Rightarrow \psi(r) \underset{r\to 0}{\to} 2\pi$ modulo 2π.

It is amusing to rewrite Eq. (3.18) into the form

$$\ddot{\alpha} + \dot{\alpha} - \sin 2\alpha\left(1 - f(\alpha,\dot{\alpha})\right) = 0 \tag{3.19}$$

where we again use $\alpha \equiv \frac{\psi}{2}$ the radius of S^3 and $\dot{\alpha} \equiv \frac{d\alpha}{d\tau}$ with $\tau = \ln \frac{r}{r_0}$, a fictitious time variable

$$f(\alpha,\dot{\alpha}) \equiv 4c^2\varepsilon^2 r_0^{-2} e^{-2\tau}\left(\dot{\alpha}^2 - \sin^2\alpha\right) \tag{3.20}$$

For $r \to \infty$ or $\tau \to \infty$ we can clearly drop terms of order $e^{-2\tau}$ i.e. $f(\tau) \to 0$. So in the near asymptotic region, we have a pendulum equation with friction

$$\ddot{\alpha} + \dot{\alpha} - \sin 2\alpha = 0 \qquad r \gg a \tag{3.21}$$

which further linearizes to $\ddot{\alpha} + \dot{\alpha} - 2\alpha = 0$, $r \to \infty$. Its solution has the behavior

$$\alpha(r) \sim \frac{1}{r^2} \qquad r \gg 1 \tag{3.22}$$

For $r \to 0$ or $\tau \to -\infty$, f dominates as it grows exponentially with τ. Hence the behavior of near the origin $r \sim 0$ is approximately determined by solving for $f(\alpha,\dot{\alpha}) = 0$, which gives

$$\dot{\alpha} = \pm \sin\alpha \tag{3.23}$$

Its two solutions are

$$\alpha = 2\tan^{-1}\left(\frac{r_0}{r}\right) \tag{3.24a}$$

$$\alpha = 2\tan^{-1}\left(\frac{r}{r_0}\right) \quad , \qquad r \sim r_0 \to 0 \tag{3.24b}$$

respectively. The length r_0 is a constant of integration giving the size of our 'singularity'.

As the above expressions are only good near $r \approx 0$, we cannot select one solely on the basis of the asymptotic boundary conditions. But fortunately we have another boundary condition just right for this purpose, namely

$U(\vec{x}) \underset{r\to 0}{\longrightarrow} -I$. The solution (3.24b) fails to satisfy this, as it gives $U \underset{r\to 0}{\longrightarrow} I$. Hence we pick (3.24a) as the acceptable solution for $r \approx 0$.

So the mapping $U_{(1)}(x)$ and the group current L_i^a about the particle singularity (as $r_0 \to 0$) are respectively

$$U_{(1)}(x) = \frac{1 + i\vec{\tau}\cdot\vec{x}\left(\frac{r_0}{r^2}\right)}{1 - i\vec{\tau}\cdot\vec{x}\left(\frac{r_0}{r^2}\right)} \tag{3.25}$$

and

$$L_i^a = \frac{-2r_0}{\left(r^2 + r_0^2\right)}\left[\left(r_0^2 - r^2\right)\delta_{ai} + 2x_a x_i + 2r_0\varepsilon_{aij}x_j\right]$$

We note that the group current L_i^a is identical to Eq. (4) for the 1-instanton potential A_i^a in Ref. 28. The connection between chiral dynamics and massive gauge fields is of course known [25]; the interaction of the most singular longitudinal polarization of the latter gauge fields is exactly given by $\mathcal{L}_{(2)}$ [26]. The key difference with the Instanton is that the kink (3.25) has a length scale r_0 fixed to be arbitrarily small to give a singularity and it it <u>not</u> a solution of the field equations.

The full solution to the equations of motion which gives Eq. (3.22) and Eq. (3.24a) respectively, as $r\to\infty$ and $r\to 0$ respectively, can be sought for either numerically, or if one is clever, analytically. But since we have previously noticed that our topological structure is truly localized, and Eq. (3.24a) represents the desired solution, within a very good approximation, in the immediate vicinity of the particle center, it would not be worthwhile here to seek the exact expression for the static energy which we define to be the mass of the kink. Indeed again on the basis of true localization the mass of the kink is approximately given by the lower bound for H_S, Eq. (3.2). The corrections to this mass can be found by numerical methods; they are not attempted here.

We see that written in terms of the pion fields, the kink solution around the particle-like core is

$$\vec{\phi} = \frac{2\, r_0 \vec{x}}{r^2 + r_0^2}$$

$$\phi^0 = \frac{r^2 - r_0^2}{r^2 + r_0^2} \qquad (3.26)$$

They are readily recognized as the stereographic mapping of the Einstein space S^3 onto compactified Euclidian 3-dimensional space $R^3 \cup \{\infty\} \approx S^3$. We verify that the solution (3.23a) is a map of degree + 1. This comes out as a result of the boundary conditions and the particular form of the hedgehog ansatz

$$B = \frac{1}{2\pi} \int_0^{2\pi} d\psi (1 - \cos\psi) \qquad (3.27)$$

The anti-kink solution with B = -1 is given by U^+. We can reinforce our argument that our kink is truly localized by explicitly calculating the topological charge density about the kink singularity.

$$B_0 = -\frac{1}{8\pi^2 r^2}(1 - \cos\psi)\psi' = \frac{4r_0^2}{\pi^2} \frac{1}{\left(r^2 + r_0^2\right)^3} \qquad (3.28)$$

Finally, we remark that the similarity between chiral dynamics and general relativity becomes even more striking in the soliton sectors. At any fixed time t, the soliton field U(x) effectively "sees" a closed physical space S^3 because of the boundary condition Eq. (2.25). So its physics is reminiscent of that of closed Einstein universes with hadronic size radii [24]. Due to the correlation between the internal and spacetime symmetries, the curved geometry of chiral dynamics induces an effective curvature in space felt only by the soliton field. We can describe the situation equivalently that there is a kink structure near the origin given by Eq. (3.25) or that the pion fields see a closed Einstein universe of radius $r \sim r_0$, as the metric of space is given there by

$$ds^2 = \frac{d\vec{x} \cdot d\vec{x}}{\left(1 + \frac{r^2}{r_0^2}\right)^2} \tag{3.29}$$

generated by the dynamical geometrization of the field U(x) via the mechansim of spontaneous symmetry breakdown. This connection merits a more detailed study.

IV. SPINOR STRUCTURE

A. Topology of Fermionization

As an intriguing bypass to the current reliance on the proliferating quark fields in hadron models, one somewhat unconventional notion deserves serious attention. Can the usual spinor field be actually not fundamental but emerge as a point limit to operators creating and annihilating specific states of a bosonic soliton field? That this fermionization mechanism is possible in 4-dimensional theories generating their own superselection rule sectors has been rigorously proved for dyons [30]. In the latter systems, the existence of quantized fermionic states is tied in a one to one way with the topological magnetic charge whose existence necessitates a long range gauge field. Here we are seeking the realistic analog of the Sine-Gordon-Thirring correspondence [10,31]. The long range fields are the Nambu-Goldstone fields, however the chiral topological charge has no dual charge such as an electric charge. Moreover its homotopic charge density being arbitrarily localized [13] allows for the possibility of constructing a local fermionic operator for the bare soliton [17,32]. This is not possible for dyons which are nonlocal entities.

We begin with the remark that, being classical, a soliton field configuration must be single-valued under the action of the Poincaré group, e.g. under the SO(3) group of spatial rotations. On the other hand, spinors are characterized by their quantum mechanical double-valuedness under a 2π rotation. The connection between half-integral spins and the group SO(3) is most directly seen in the path integral formalism [33]. There different homotopy classes of paths enter into the sum over paths with arbitrary relative phases. The ray representations of SO(3) come about from its double connectivity: $\pi_1(SO(3)) \approx Z_2$, the additive group of integers modulo 2. To the two classes of closed paths in SO(3) corresponding to the elements I and -I of Z_2 are associated the integral and half-integral spin representations respectively.

In general relativity, it is an important problem to find the conditions under which various space-times manifolds admit a spinor structure [34]. Analogously, from the above remarks, the possibility of fermionic states emerging from the canonical quantization of purely bosonic chiral solitons is also related to the global topology of the arena for chiral dynamics, the phase space Ω of the fields. The necessity and sufficient conditions for a soliton theory to admit spin have been formulated by Finkelstein et al. [16]; they are satisfied for Skyrme's model. For completeness, logical continuity and cogency of our paper, we will reformulate compactly the topological proof of admittance of spin. Essentially our treatment attempts a summary of the few papers on this topic [16,35].

Consider the functional phase space Ω whose points are the chiral fields S(x). This bundle $\Omega = \prod_x M_x$ is given by the topological product of all the "internal tops" $M = S^3$, the manifold of SU(2), the fiber, one at each space-time point x. Due to the boundary condition $U(x) \xrightarrow[|\vec{x}|\to\infty]{} I$, Ω is split into an infinite set of topologically disconnected components $\{\Omega_{-\infty} \ldots \Omega_{\infty}\}$ each labelled by the Chern index B, Eq. (2.28), the homotopic invariant of $\Pi_3(SU(2)) \approx Z_{\infty}$. While the structure of Ω is relevant to the topology of solitons in Section II, we now must analyze the homotopic structure of the chiral field propagator within each component Ω_i of Ω [33]. In fact it suffices to consider the 1-soliton sector Ω_1. Following Ref. [16], all fields in the basic Lagrangian are to obey commutation relations, as in canonical quantization, but the state functionals on which these fields act are allowed to be <u>multivalued</u>. Such a multivalued quantization of a classical system Σ is a canonical quantization of the <u>covering</u> system $\tilde{\Sigma}$ which is defined in close analogy to that of the universal covering space in topology [24]. Then a soliton theory is said to "admit spin" if it is possible to <u>define</u> on Ω_1, a wave-functional $\Psi_1(U(x),t)$ which is doublevalued under a 2π spatial rotation. More precisely we should work in the universal covering $\tilde{\Omega}_1$ of $\Omega_1[\Omega_1 \approx \tilde{\Omega}_1/\pi_1(\Omega_1)]$ and define the functional $\tilde{\Psi}_1(\tilde{U}(x),t)$ which is singlevalued in $\tilde{\Omega}_1$ such that

$$\Psi_1[U(x)] = \tilde{\Psi}_1[\tilde{U}_0(x)\Gamma] = a_1(\Gamma)\tilde{\Psi}_1[\tilde{U}_0(x)] \tag{4.1}$$

gives the branches of that functional in Ω_1. U_0 is a base point in a typical fiber such that any point $\tilde{U}(x)$ in $\tilde{\Sigma}$ is represented by $\tilde{U}(x) = \tilde{U}_0(x)\Gamma$ where Γ ranges over the elements of the Poincaré group $\Pi_1\ (\Omega_1)$, the $a_1(\Gamma)$ are the unimodular phases. The functional propagator of the chiral field in the connected component Ω_1 of Ω is given by

$$K_1[U(x)|U(x)] = \sum_{\Gamma} \tilde{K}_1[\tilde{U}_0(x)|U(x)_0\Gamma]a_1(\Gamma) \tag{4.2}$$

$a_1(\Gamma)$ is a unidimensional unitary representation of the fundamental Abelian group $\Pi_1(\Omega_1)$ still to be computed for the theory.

Specifically, let us consider a path generated by a rotation of the soliton field configuration. Being continuous deformations complete rotations can only induce closed loops within Ω_1, so the existence of half integral spins is possible if

$$\Pi_1(\Omega_1) \approx Z_2 \tag{4.3}$$

Since there then exist two kinds of propagators corresponding to the two homotopy classes of paths, they correspond to the integral and half-integral spin in the instance of the trivial and nontrivial loops in Ω_1 respectively. For this double connectivity of Ω_1 to be uniquely linked with the manifestation of spinor structure, it must be shown that $\Pi_1(\Omega_1) = \Pi_1(SO(3))$, that the non-trivial loops of Ω_1 are in fact generated by actual 2Π spatial rotations of the soliton field.

If Eq. (4.3) holds, then it follows that $\Pi_1(\Omega_i) \approx Z_2$ as a consequence of the isomorphism $\Pi_1(\Omega_i) \approx \Pi_1(\Omega_j)$ for any Ω_i, Ω_j ε Ω, which is a topological space. This is why it suffices to analyze the 1-soliton sector Ω_1. In particular we have $\Pi_i(\Omega_0) \approx \Pi_1(\Omega_1)$; which reduces the problem to the simpler one of seeking the path structure of the trivial sector Ω_0. Now it can be proved that [36].

$$\Pi_1(\Omega_0) \approx \Pi_4(M) \tag{4.4}$$

due to the homeomorphism between the space of closed loops in Ω_0 and the mapping space $R_4 \to G$ with suitable boundary conditions. On the other hand, it is known [24] that $\Pi_4(S_3) \approx Z_2$. Hence the Skyrme model passes the first test, it obeys $\Pi_1(\Omega_1) \approx Z_2$. It only remains to prove that nontrivial closed paths in Ω_1 based at some point $Q(x)$ actually correspond to a 2Π spatial rotation. This is done next.

We shall define a closed path in Ω_1, based at a particular "point" $\tilde{Q}_1 \in \Omega_1$

$$Q_1 : \tilde{S}^3 \to S^3$$

where we have chosen explicitly

$$\tilde{Q}_1 = \phi_1^0 + i\, \vec{\tau} \cdot \vec{\phi}_1$$

$$\phi_1^0(x) = X^0 \quad , \quad \phi_1^a(x) = X^a \tag{4.5}$$

here X^0, X^a spherical projective coordinates defined in the usual way vis. Eq. (3.25).

$$X^0 = \frac{r^2 - r_0^2}{r^2 + r_0^2} \quad , \quad X^a = \frac{2r_0 x^a}{r^2 + r_0^2} \tag{4.6}$$

Such a closed path is in fact a continuous sequence of mappings of $\tilde{S}^3 \to S^3$ defined by

$$\tilde{Q}_1[\lambda] = \phi_1^0(\lambda) + i\,\vec{\tau}\cdot\vec{\phi}_1\,(\lambda) \tag{4.7}$$

where

$$\phi_1^0(\lambda) = X^0$$

$$\phi_1^a(\lambda) = R^{ab}(\lambda)X^b \tag{4.8}$$

$R^{ab}(\lambda)$ is a rotation matrix to be defined.

As Q_1^+ creates an antisoliton, it annihilates Q_1, so $Q_1^+Q_1$ is an automorphism which transfers one from the 1-soliton to the 0-soliton sector. Since we need a parameter to describe the paths induced by rotation, the paths in Ω_0 can be described by the paths in Ω_1 as

$$Q_o[\lambda] = Q_1^{-1}\,[0]\;Q_1\;[\lambda] \tag{4.9}$$

Using Eq. (4.6) and its degree-zero counterpart we get

$$\phi_0^0[\lambda] = \phi_1^0(0)\phi_1^0(\lambda) + \phi_1^a(0)\phi_1^a(\lambda)$$

$$= X_0^2 + R^{ab}(\lambda)X^aX^b$$

$$\phi_0^a[\lambda] = \phi_1^0(o)\phi_1^a(\lambda) - \phi_1^0(\lambda)\phi_1^a(0) + \varepsilon^{abc}\phi_1^b(0)\phi_1^c(\lambda)$$

$$= X^0[-\,\delta^{ab} + R^{ab}(\lambda)]X^b + \varepsilon^{abc}R^{cd}(\lambda)X^bX^d$$

Let us choose a spatial rotation of $2\pi\lambda$ to be performed around the third axis, then

$$R^{ab}(\lambda) = \begin{pmatrix} \cos 2\pi\lambda & \sin 2\pi\lambda & 0 \\ -\sin 2\pi\lambda & \cos 2\pi\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.10}$$

The rotated configuration becomes

$$\begin{aligned}
\phi_0^0(\lambda) &= 1 - (1 - \cos 2\pi\lambda)\ (X_1^2 + X_2^2) \\
\phi_0^1(\lambda) &= X_0[X_1(\cos 2\pi\lambda - 1) + X_2(\cos 2\pi\lambda - 1)] \\
&\quad + X_3(X_1(\cos 2\pi\lambda - 1) + X_2 \sin 2\pi\lambda)] \\
\phi_0^2(\lambda) &= X_0[-X_1 \sin 2\pi\lambda + X_2(\cos 2\pi\lambda - 1)] \\
&\quad + X_3[X_1(\cos 2\pi\lambda - 1) + X_2 \sin 2\pi\lambda)] \\
\phi_0^3(\lambda) &= -\sin 2\pi\lambda[X_1^2 + X_2^2]
\end{aligned} \tag{4.11}$$

specifying thus a path in Q_0. We can now associate a mapping $\tilde{f}_0$ with path:

$$\tilde{f}_0 : S^4 \to S^3 \tag{4.12}$$

The added structure reflects the role of the parameter λ labelling the path in the covering space. So Ω_0 is nontrivial if and only if $\tilde{f}_0$ is nontrivial. The essential element of William's proof comes from the observation that for $\lambda = 1/2$, one has

$$\begin{aligned}
\phi_0^0 &= 1 - 2\ (X_1^2 + X_2^2) \\
\phi_0^1 &= -2\ (X_0X_1 - X_3X_2) \\
\phi_0^2 &= -2\ (X_0X_2 - X_3X_1)
\end{aligned} \tag{4.13}$$

which is nothing but the celebrated Hopf map $S^3 \to S^2$ [37]. Furthermore $\tilde{f}_0$ is known as a <u>suspension</u> of the Hopf map, hence nontrivial. The proof is thus completed.

Finally, it has also been shown for models of Skyrme's type that another defining property of a multi-fermion state, the doublevaluedness of the wavefunctional under field interchange implies doublevaluedness under a 2π rotation. It was also shown that every quantized fermionic soliton will always differ from a bosonic soliton in at least one particle number, such as B in our case, as well as in spin; that the usual spin and statistics connection follows from the same topology invoked in the above proof, parastatistics being excluded [16].

We note that the existence of fermionic spin demonstrated topologically in this section is consistent with our quantum mechanical interpretation of the origin of quartic term in Section III and is further collaborated by the implications of the current algebra of Skyrme's model.

B. Effective Dynamics

From the foregoing sections, a general physical picture underlying Skyrme's model is apparent. The topological kernel of the soliton resides in the occurence of a simple zero in the pion fields. This singularity marks a local restoration of chiral symmetry at the center of the soliton due to the infra-red effects of the Nambu-Goldstone pions. This process is the obverse of that proposed by Nambu and Jona Lasinio. Through this soliton creation the chiral theory then generates in a self-consistent manner its own sources. We recall that because the chiral field U(x) is also a group element for any fixed x, a formal structural separation of the soliton field is possible, it provides a useful alternative description when no exact solution of the field equations is available, which is the case in the model Eq. (3.1). This group property is responsible for the very localized character of the soliton bound states and thus allows the soliton field to be factored into two pieces, an idealized point singularity and its coherent pion cloud. Indeed while the interactions are simple conceptually in non-linear field theories, the definitions as to what constitute the particles on the one hand and their residual interactions on the other are correspondingly ambiguous. So the mathematical splitting through the procedure Eq. (2.31) of a soliton field into source and field is necessarily arbitrary. To different sleections of the $U_1(x)$ must then correspond the appropriately different coupling constants between the sources and the residual fields.

From Section III, the most natural and extreme means of lifting much of the above arbitrariness is provided by an $U_{(1)}$ given by Eq. (2.25) in the point-like limit of $r_0 \to 0$. It is in this limit that we may hope to make contact with the standard linearized field theory of pions and nucleons, the latter being put in by hand ab initio into the Lagrangian. As is done in the passage from the Sine-Gordon to the massive Thirring models [31,32], Skyrme has studied the problem of deriving quantum mechanically an effective Lagrangian between point sources and residual pion fields by the method of collective coordinates [17]. We refer the reader to his remarkably compact paper. To establish a smooth connection between our work and Ref. [17], two remarks are in order. First, Skyrme selects to parametrize the particle singularity by the solution Eq. (3.24b). We have explained the inappropriateness of this ansatz. However we have verified explicitly that none of his results are modified in the physical solution, Eq. (3.24a), is used. Secondly, Skyrme's main objective is to obtain a Dirac equation for the description of the singularity in the specific projected state with spin 1/2 and isospin 1/2. This derivation is in our opinion at most heuristic as it does not readily follow from his nonrelativistic collective coordinate analysis. For that reason, we deem it unenlightening to display his method of calculations as applied to the singularity Eq. (3.24a). Instead, we shall argue that without invoking any detail of the dynamics, one can yet deduce the essential properties of the spectrum of the quantized 1-soliton sector.

In particle physics, we know generally little about the strong interaction Hamiltonian. And even when we have a system like (3.1), it is no easy task to uncover its nonperturbative spectrum by a frontal attack. However, it is often possible to introduce some kind of group structure and argue for a choice of noninvariance group which will pin down the dynamics [38]. So we take this dynamical group approach in order to distill some structure out of Skyrme's model. Only the topology of the dynamics rooted in the boundary condition Eq. (2.25) is of essential importance along with the point-like limit of the singularity. Furthermore, our conclusions will only hold for the static case.

First we observe that the static Hamiltonian H_S, Eq. (3.2), is only a function of the current J_i^a. In the topologically nontrival 1-soliton sector, the indices a and i are identified as manifest in the hedgehog ansatz, Eq. (3.25). In other words, the rotation group which is O(3) in the trivial $B = 0$ sector has been compactified onto SU(2) as R^3 gets compactified onto S^3. As is stated in the previous subsection, this is equivalent to the following statement: because the wavefunctional of the singularity is doublevalued under a 2π spatial rotation, one must do canonical quantization on the covering space where the wavefunctional is singlevalued and the Hamiltonian is self-adjoint. This is the well known problem of self-adjoint extensions of the Hamiltonian [39], which contain new physics, that of the superselection rule sectors which are baryons. Specifically in the point limit where the singularity is represented by Eq. (3.24a) Skyrme [17] has shown that the spin s^i and isospin t^α of the kink is connected via $s^i = e_i^{\ \alpha}\, t^\alpha$, $e_i^{\ \alpha}$

transforming in the covering functional space $\tilde{\Sigma}$ as a vector with respect to the two SU(2) groups of a dynamically induced invariance group of the Hamiltonian Eq. (3.2) $K = SU(2)_J \times SU(2)_I$ which arises from the chiral symmetry SU(2) × SU(2). This is clear since dynamical fermionic spin is induced from the isospin by the compactification condition Eq. (2.25) and the SU(2) ⊗ SU(2) nature of the chiral symmetry. When we note that it can be shown [10] that the effective interaction between the kink singularities and the pion is the usual derivative coupling, i.e. that we are considering a p-wave pion isotriplet the identification with the usual symmetric pseudoscalar static model is complete [40].

On the basis of the latter, we deduce that the quantized Skyrme Hamiltonian in the point-source approximation Eq. (3.25) has an infinite sequence of bound states (isobar levels), the latter have I = J = 1/2, 3/2, ... in the form of the unitary irreducible representation of the type (I,I) of the dynamically derived invariance group $SU(2)_J \times SU(2)_I$. Patterning our reasoning after the dynamical group approach [38], one further argues for an associated noninvariance group SU(4) which upon reduction with respect to O(4) lead to the above class of representations. All this is consistent with our intuition that the point nucleon limit should correspond to a regime where the pion-nucleon coupling constant tends to infinity and hence the well known infinite sequence of the bound states. Clearly only the low lying isobars are expected to be physically realized; the infinite coupling approximation breaks down at higher energies. In the soliton language, the extension of the source becomes important and is correlated to the intermediate coupling regime. While the way is open we shall not pursue further here the dynamical group approach. It suffices to note that the latter does provide a most suitable algebraic method to handle the rich spectrum of the quantized solitons. Our further comments on the formulation of the effective dynamics will be deferred until the final section.

V. CURRENT ALGEBRA

Before the advent of QCD, there has been much interest in formulating complete dynamical theories based solely on current densities taken as basic dynamical variables [41]. The motivations are many [42]: They are the spectacular successes of various current algebra sum rules, the sharing of the same currents by all the basic interactions, the seemingly democratic status of all hadrons. All these features call for a deep dynamical understanding and possible extension of Gell-Mann's current algebra hypothesis.

The basic idea in dynamical theories of currents is to postulate along with the commutation relations between an assumed complete set of local current densities the dynamics by giving an energy momentum tensor made up solely with current. Specifically new methods were being sought to solve such models and to uncover how fermionic representations can emerge from a structure composed entirely of bose type operators. A survey of past literature shows that no concrete answer to this key problem was available in any realistic theory. As will become clear, a very appealing solution is provided by Skyrme when his model is seen from the perspective of chiral dynamics and current algebra. The solution is also consistent with algebraic quantum field theory ideas [43].

The Sugawara-Sommerfield theory of currents has become the prototype relativistic model. It is well known [25] that its canonical field theory realization is nothing but the Gursey <u>et al</u>. [22] system $\mathscr{L}_2$ in Eq. (3.1). In this case the currents are simply the conserved left and right invariant group currents:

$$L_\mu^a = \frac{1}{2i}\,\mathrm{Tr}(\tau^a U^+\,\partial_\mu U)\,,\quad R_\mu^a = \frac{1}{2i}\,\mathrm{Tr}(\tau^a \partial_\mu U U^+) \qquad (5.1)$$

with their zeroth components being the generators of the corresponding local transformations:

$$[L_0^a(x),\ U(y)] = -\frac{\tau^a}{2}\, c^2 U(x)\ \delta^3(x-y)$$

$$[R_0^a(x),\ U(y)] = c^2 U(x)\,\frac{\tau^a}{2}\ \delta^3(x-y) \qquad (5.2)$$

Defining as usual $L_\mu = V_\mu + A_\mu$ and $R_\mu = V_\mu - A_\mu$ all having dimension-1, we readily work out the Sugawara-Sommerfield algebra for the group currents

$$[V_0^a(x),\ V_0^b(y)] = [A_0^a(x),\ A_0^b(y)] = ic^2\ \varepsilon^{abc}\ V_0^c\ (x)\ \delta^3(x-y)$$

$$[V_0^a(x),\ V_i^b(y)] = [A_0^a(x),\ A_i^b(y)] = ic^2\ \varepsilon^{abc}\ V_i^c\ (x)\ \delta^3(x-y)$$

$$+\ ic^2\ \delta^{ab}\ \partial_i^{(x)}\ \delta^3(x-y) \qquad (5.3)$$

$$[V_0^a(x),\ A_i^b(y)] = [A_0^a\,(x)\ ,\ V_i^b(y)] = ic^2\ \varepsilon^{abc}\ A_i^c(x)\ \delta^3(x-y)$$

$$[V_i^a(x),\ V_j^b(y)] = [V_i^a(x),\ A_j^b(y)] = [A_i^a(x),\ A_j^b(y)] = 0$$

We note that the Schwinger terms are finite c-numbers, and that the appearance of the dimensional constants c arises from V and A having dimension -1.

We proceed to the more complex algebra of currents for the full Skyrme's model and analyze the modifications introduced by the quartic term. First we define the left and right isospin currents

$$\mathscr{J}^{\mu,a} = G^{ab}(L^2)L^{\mu,b}$$

$$\tilde{\mathscr{J}}^{\mu,a} = G^{ab}(R^2)R^{\mu,b} \qquad (5.4)$$

where G^{ab} is given by Eq. (3.10). These physical currents obey the conservation laws

$$\partial_\mu \mathscr{J}^{\mu,a} = \partial_\mu\ \tilde{\mathscr{J}}^{\mu,a} = 0 \qquad (5.5)$$

as they should.

Their fourth components are the generators of the corresponding left and right transformations

$$\left[\mathscr{J}_0^a(x),\ U(y)\right] = -\frac{\tau^a}{2} U(x)\delta^3(x-y)$$

$$\left[\tilde{\mathscr{J}}_0^a(x),\ U(y)\right] = U(x)\frac{\tau^a}{2}\delta^3(x-y) \qquad (5.6)$$

From the above relations we derive

$$\left[\mathscr{J}_0^a(x),\ L_0^b(y)\right] = i\,\varepsilon^{abc} L_0{}^c(x)\delta^3(x-y)$$

$$\left[\mathscr{J}_0^a(x),\ L_i^b(y)\right] = i\,\varepsilon^{abc} L_i{}^c(x)\delta^3(x-y) + \frac{i}{2} c\delta^{ab}\,\partial_i^{(x)}\delta^3(x-y) \qquad (5.7)$$

By use of Eq. (5.7) and after some tedious algebra, the universal current algebra of Skyrme's model is found to be

$$\left[\mathscr{J}_0^a(x),\mathscr{J}_0^b(y)\right] = i\,\varepsilon^{abc}\mathscr{J}_0^c(x)\delta^3(x-y) + \frac{i}{2} C_i{}^{ab}(x,y)\partial_i\delta^3(x-y) \qquad (5.8)$$

$$\left[\mathscr{J}_0^a(x),\mathscr{J}_i^b(y)\right] = i\,\varepsilon^{abc}\mathscr{J}_i^c(x)\delta^3(x-y) + \frac{i}{2} C^{ab}(x,y)\partial_i\delta^3(x-y) \qquad (5.9)$$

$$\left[\mathscr{J}_0^a(x),\mathscr{J}_j^b(y)\right] = i\left[A_{ij}^{ab}(x,y)\delta^3(x-y) + C_i^{ab}(x,y)\partial_j\delta^3(x-y) - C_j^{ba}\,\partial_i\delta^3(x-y)\right] \qquad (5.10)$$

where

$$C_i^{ab}(x,y) = \frac{\varepsilon^2}{4}\left[L_0^a(x)L_i^b(y) + \delta^{ab}L_0^c(x)L_i^c(y)\right]$$

$$C^{ab}(x,y) = \frac{1}{2c^2}\,\delta^{ab} + \frac{\varepsilon^2}{4}\left[(L_0^c)^2\,\delta^{ab} - L_0^aL_0^b + 2L_i^aL_i^b\right] \tag{5.11}$$

$$A_{ij}^{ab}(x,y) = \frac{\varepsilon^2}{4}\left[\varepsilon^{abc}\,\Delta_i^{cd}(x,y)L_j^b(y) - \varepsilon^{bcd}\,\Delta_j^{cd}(x,y)L_i^a(y)\right]$$

and

$$\Delta_i^{ab}(x,y) = L_0^a(x)L_i^b(y) + L_i^b(x)L_0^a(y)$$

And in terms of the vector and axial currents, as defined by $\mathscr{V}_\mu = \mathscr{J}_\mu + \tilde{\mathscr{J}}_\mu$ $\mathscr{A}_\mu = \mathscr{J}_\mu - \tilde{\mathscr{J}}_\mu$, we finally obtain the following modified Sugawara-Sommerfield algebra after extensive use of $\mathrm{Tr}(\mathscr{G}^2) = \mathrm{Tr}(\tilde{\mathscr{G}}^2)$

$$\left[\mathscr{V}_0^a(x),\mathscr{V}_0^b(y)\right] = \left[\mathscr{A}_0^a(x),\mathscr{A}_0^b(y)\right]$$

$$= i\,\varepsilon^{abc}\mathscr{V}_0^c(x)\delta^3(x-y) + iC_i^{ab}(x,y)\partial_i\delta^3(x-y)$$

$$\left[\mathscr{V}_0^a(x),\mathscr{V}_i^b(y)\right] = \left[\mathscr{A}_0^a(x),\mathscr{A}_i^b(y)\right]$$

$$= i\,\varepsilon^{abc}\mathscr{V}_0^c(x)\delta^3(x-y) + iC^{ab}(x,y)\partial_i\delta^3(x-y)$$

$$\left[\mathscr{V}_0^a(x),\mathscr{A}_i^b(y)\right] = \left[\mathscr{A}_0^a(x),\mathscr{V}_i^b(y)\right] = i\varepsilon^{abc}\mathscr{A}_i^c(x)\delta^3(x-y) \tag{5.12}$$

$$\left[\mathscr{V}_i^a(x),\mathscr{V}_i^b(y)\right] = \left[\mathscr{V}_i^a(x),\mathscr{A}_j^b(y)\right] = \left[\mathscr{A}_i^a(x),\mathscr{A}_j^b(y)\right] = 0(\varepsilon^2)$$

It is not our purpose in the case of the space-space commutators to do an explicit computation say for experimental comparison with polarization asymmetry in electroproduction. So we do not display here the intricate and unenlightening looking expressions for the r.h.s. of these commutators. It suffices to observe that the r.h.s. is nonvanishing and of order ε^2 so it vanishes in the limit $\varepsilon \to 0$, as it should.

The remarkable properties of the system Eq. (5.11) which make it a much improved algebra over Eqs. (5.3) are

a) In contrast to the Sugawara model [44,45], there is no problem with the dimensions of the currents. Here because of the quartic term, the $\mathcal{V}_\mu$ and $\mathcal{A}_\mu$ are bona fide currents endowed with canonical dimensions, both the space and time components have dimension -3.

b) The schwinger terms are being promoted from c-number to q-numbers [44]. So the trouble with they being c-number carrying dimension -2 is resolved. All this is of course consistent with the interpretation of the quartic term as part of quantum mechanical renormalization effects.

c) Dashen and Frishman [46] have proved that the energy momentum tensor is composed of two separated commuting conserved tensors interchangeable under the parity operation P

$$\Theta^{(s)}_{\mu\nu} = \Theta^{+}_{\mu\nu} + \Theta^{-}_{\mu\nu}$$

$$\partial^\mu \Theta^{\pm}_{\mu\nu} = 0 \tag{5.13}$$

and

$$P\, \Theta^{\pm}_{\mu\nu}\, P^{-1} = \Theta^{\mp}_{\mu\nu}$$

Then the SU(2) × SU(2) symmetry entails a new symmetry under the product of two independent Poincaré groups $\mathcal{P}_+ \otimes \mathcal{P}_-$. This results in a parity doubling of the states. It was later shown [47] that $\Theta_{\mu\nu}$ can no longer be so decomposed if quartic terms for instance are added to $\mathcal{L}_{(2)}$ and that the Dashen-Frishman theorem no longer applies. The same conclusion applies to Skyrme's model. Inspection of its $\Theta_{\mu\nu}$ shows that it cannot be partitioned as in Eq. (5.13), so the proclaimed parity degeneracy is lifted in the present case.

d) As noted briefly before, another desirable feature of the system Eq. (5.12) is the <u>nonzero</u> nature of the space-space commutators which are known to be very model dependent. The property is required if theory stands a chance to account for the polarization asymmetry experiments in electroproduction [47].

All in all, the quartic addition to $\mathscr{L}_2$ has resulted in a more physically realistic current algebra. Of course these positive features should definitely be seen in the context of the solution to another key issue in any theory of currents. Namely, where are the fermions in a scheme having only boson operators? The developments in the preceding sections provide an obvious answer in the soliton generation mechanism coupled to the topological existence proof of a spinor structure definable in the topologically nontrivial sectors of the phase space of fields. Due to its SU(2) <u>quaternionic</u> structure, Skyrme's model allows for dynamical baryonic number as well as fermionic spin generation at the quantum level. Thus one expects that Hilbert space of states to be splitted up into superselection rule sectors labelled by a Casimir invariant, the Chern-index B. B labels the inequivalent fermionic representations of current algebra. The analysis done on Skyrme's model seen as a canonical realization of the algebra Eq. (5.12) shows that baryons can be identified with the soliton bound states of Nambu-Goldstone bosons. This remarkable dynamical mechanism is dual to that of Nambu and Jona-Lasinio's. We have here exhibited a simple three dimensional analog of the bootstrapped duality which exists between the soliton of the Sine-Gordon model and the massive fermion of the Thirring model [31]. The latter connection was first studied by Skyrme [10] and has been rigorously established recently [49].

Due to the technical difficulties inherent to our realistic model, our analysis when supplemented by Skyrme's extensive study only circumscribes semi-classically this mechanism of spontaneous generation of baryon number and fermionic spin from the dynamics of flavors. To pin it down rigorously necessitates a more frontal and a fully quantum mechanical attack of the problem. All this leads us to make some final comments next.

VI. FINAL REMARKS

Since most conclusions are stated in the appropriate sections, our final remarks will be relatively brief.

It has been widely assumed that a basic theory of matter must involve fermion fields but needs not contain boson fields. The latter are to emerge as bound states of the former. While this may well be so in nature, other theoretical alternatives exist. Thus supersymmetries testify to the realization of fermions and bosons as equal partners in some supermultiplet. In this work, we focus on a third possibility, fermionization or the soliton generation of dynamical fermion states from boson fields.

In two dimensions where there is no spin, the equivalence between the quantum soliton of the Sine-Gordon model and the fermion in the Massive Thirring model has become an ideal showcase for the axiomatic field theorists [50]. This is so because such emergence of fermionic states from a structure endowed solely with commutation relations has a natural place in the algebraic approach to quantum field theory. Very succinctly, we recall that in the latter method a quantum system is completely specified by its algebra of local observables $A(\Omega)$. This algebra corresponds to all measurements performed locally in a bounded domain Ω. Along with obeying the usual axioms of relativistic quantum physics, the observables and states also decompose into superselection rules sectors labelled by absolutely conserved quantities such as baryon or lepton numbers. This superselection rule structure plays a central role since through it other derivative concepts such as a quantized field, its charge, its statistics can be introduced solely in terms of observables. For example if we wish to describe some entity measured in a domain Ω, we must introduce the notion of a local change in the observable, one which reflects the existence of the object in question as compared to observations, performed in its absence, upon the vacuum. Such a "localized morphism," a quantum analog of a canonical transformation, transforms each observable in Ω into a different one causally related to it in the same Ω. It is in this manner that all observables and hence all states in any superselection rule sector can be obtained from those of the vacuum sector via such localized morphisms. An example is the fermion field of Skyrme [10] and Mandelstam [32] which carries charge from the zero-charge sector to the charge one sector in the quantized Sine-Gordon model. The statistics is also uniquely determined by the observables and is related to the types of particle involved; it is given by the permutation symmetry of the state vector of spacelike separated identical particles. Since all charges are uniquely linked with the superselection rule sectors and all sectors can be reached from the vacuum, we must be able to predict in principle all types of particles, their charges, masses and other quantum numbers of a theory from its algebra of observables alone. We just have to solve the problem of classifying its representations! In actual practice to single out of all the representations the physically relevant ones, which actually occur phenomenologically, we need to assume that the algebra A (Ω) is derived from a field theory. This was done rigorously in two dimensions in the cases of the massless then massive Thirring models [50]. We have initiated here a modest attempt in four dimensions leaning heavily on Skyrme's remarkable works by taking his model as a prototype dynamical theory of observable currents. Compared to QCD, the system Eq. (3.1) either seen as a chiral model for pions or a dynamical model of currents has no basic significance. It could at best parametrize at the outset the effective dynamics of the strangeness zero sector of the hydrodynamic behavior of QCD in its PCAC phase.

Yet the topological chiral dynamics already embodies a striking self-consistent structure, a field theoretical bootstrap mechanism.

Indeed the relation of chiral dynamics to a fundamental theory of hadrons may be analogous to that of the Landau-Ginzburg to the BCS theories of superconductivity, of hydrodynamics to a microscopic theory of liquids. In this outlook, the chiral soliton generation mechanism is seen as dual to that proposed by Nambu and Jona-Lasinio [12]. In the latter approach, in consequence of spontaneous breakdown of γ_5-symmetry, massive fermions arise from massless ones provided Nambu-Goldstone pions are also generated as bound states and thus restore the symmetry. Now we recall that nonlinear chiral dynamics was conceived to give a well-defined meaning to the idea of symmetry restoration. One does not inquire into how strong interactions attain its PCAC phase but simply posits a dynamical symmetry, an interaction symmetry among observed hadrons. In the soliton generation, we witness a self-consistent mechanism at work, one which induces the nonlinear dynamics of the Nambu-Goldstone bosons which in turn produce superselection rule sectors, the baryonic bound states. The solitons arise as local restoration of chiral symmetry due to the infrared effects of the Nambu-Goldstone bosons. This apparent signature of a bootstrap dynamics emerges from the collaborative implications of the various facets of the systems analysed in Sections II to V.

For maximum benefit, our paper should be supplemented with Skyrme's works. Due to the technical difficulties inherent to a four dimensional nonrenormalizable model, our analysis lacks the needed mathematical penetration to rigorously uncover the full nonpertubative structure of Skyrme's model. However our purpose has only been to clarify and to bridge the various formerly disparate physical problems by way of the soliton concept. We have confined ourselves to correlate exact conclusions which can be reached at the semi-classical level and from the topology of the dynamics. By so doing, we hope to at least bring a more up to date focus on Skyrme's work in the general context of chiral dynamics, to stimulate further and more rigorous investigations. Thus we close by pointing out some inviting directions for research.

a) To seek a more derivation of Lagrangians portraying the effective dynamics between solitons, Skyrme's collective coordinate analysis should be much improved. For this task, use should be made of the general collective coordinate method formulated recently and applied to analogous physical problems [51]. The formalism of pseudomechanics [52] may be very appropriate to coordinatize spinor and internal degrees of freedom.

b) The splitting of the soliton field into a point singularity plus cloud is presumably good only at very low energies and in the strong-coupling regime. For intermediate couplings the extension of the soliton plays an important role. It is then imperative to seek analytic solutions to (3.1) or to some suitably generalized form such as (3.5) [14]. One should divorce oneself entirely with the point-limit theory, i.e. with any contact with the standard chiral field theory of point nucleons and pions. The corresponding quantum theory of solitons should be undertaken along the lines drawn by Faddeev and Korepin in their comprehensive review of the quantized Sine-Gordon theory [2]. In this respect, we observe that the "zero soliton" sector has been vigorously investigated by Soviet workers [6]. It would be most interesting to supplement their method with that of Faddeev and perform computations in the soliton sectors. The necessary basic ingredients for such a program have been laid down in our Section III. A local gauge extension should be studied as well [13].

c) As the quantized soliton leads naturally to towers of states with spin and isospin, the connection between the quantized soliton theory in its various coupling regime approximations (i.e. point-limit approximation, etc.) and dynamical noninvariance groups should be sought [40]. This is an important link as it would clarify the connection between chiral symmetry, which is an interaction symmetry, and the dynamical group, which is a spectrum generating symmetry.

d) Regarding extension to the soliton sectors of chiral theories invariant under $SU(N) \otimes SU(N)$, the special role of the $SU(2) \otimes SU(2)$ subgroup is apparent since the topological possibility of spinor structure induced by the correlation between space-time and internal symmetries depend crucially on $G = SU(2)$. Mathematically the problem is then one of embedding $SU(2) \otimes SU(2)$ in the larger groups in analogy to the problem of seeking monopole or Instanton solutions in a general Lie group [53]. The privileged role of $SU(2) \otimes SU(2)$ may lead in soliton sectors to a dynamical understanding of the symmetry breaking of a $SU(3) \otimes SU(3)$ theory.

e) As to the finite mass of the physical pions, a soliton theory which takes the chiral geometry seriously implies that it should arise through some deformation of the geometry of the manifold of $SU(2) \approx S^3$. This very kind of geometric approach to produce the desired symmetry breakings, be they weak, electromagnetic or strong, had in fact been proposed in connection with the dynamical theory of currents [54].

f) Finally it is clear that a latticization of Skyrme's model should be pursued in the manner of Polyakov's work on the 2-dimensional Heisenberg ferromagnet [26]. Only through the renormalization group method can one truly track down the rich non-perturbative structure of this model and to generate in a systematic and reliable way its various truncated Hamiltonian extensions, if one wishes to enlarge the applicability of the model to other energy scales.

Having satisfied ourselves of the dynamical consistency and richness of chiral theories we are pursuing our investigations along the above directions.

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