

QUARK CONFINEMENT\*

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These lectures are concerned with the problem of "where are the quarks?" Almost 10 years ago the classic MIT-SLAC experiments on deep inelastic electron scattering<sup>1</sup> revealed the existence of localized bits and pieces of electromagnetic charge and current within the proton. These more elementary parts of the proton--or partons--were inferred from the Bjorken scaling behavior.<sup>2</sup> Furthermore, the subsequent studies with electron, muon, and neutrino beams as well as the spectacular results of electron-positron collisions have strengthened the interpretation of partons as quarks.<sup>3</sup> In fact today there is no persuasive alternative to the physical picture of hadrons as rather loosely bound aggregates of relatively light quarks. However beyond the impressive successes of the quark model in its diverse applications to spectroscopy<sup>4</sup> (both old and charmed), scaling behavior, and quark line counting rules for large  $p_T$  processes,<sup>5</sup> there remains a dilemma: Why are individual quarks apparently trapped within the hadron?

As you well know the actual experimental nature and extent of quark trapping or confinement is not completely settled by experiment. Although many searches have failed to find fractionally charged particles to a very low density in matter,<sup>6</sup> LaRue, Fairbank and Hebard<sup>7</sup> have recently reported observation of fractional charges on two superconducting Ni spheres weighing

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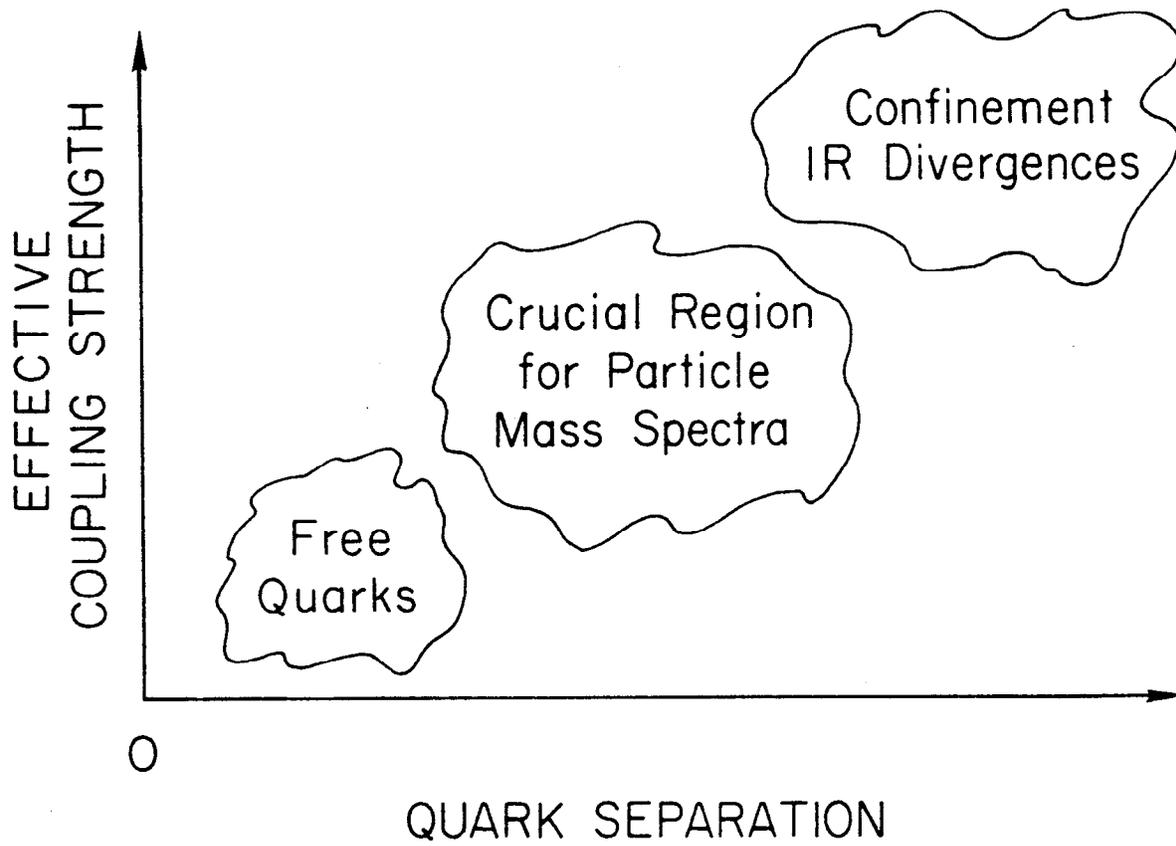
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$10^{-4}$  grams and magnetically levitated between condenser plates. Their response to a periodic electric force is interpreted as evidence of charge  $1/3$  on the spheres. If supported and confirmed by subsequent experiments this result will imply that quark confinement is not permanent but that quarks may in fact be paroled from their hadronic prisons by sufficiently high energies, perhaps one or more orders of magnitude greater than the hadron mass.

The purer the field theorist, the more unacceptable is this alternative to permanent confinement. In these lectures I won't take sides. I simply want to describe, explain, and analyze some theoretical efforts to understand the novel behavior of quark constituents in the hadron. In contrast to atomic, molecular, and nuclear forces that grow stronger the smaller the distance between the interacting particles, the forces between quarks grow weaker at small distances--that is how we understand Bjorken scaling and the almost-free quark dynamics, or asymptotic freedom, for large momentum transfer processes. On the other hand, these forces grow very strong--perhaps without limit if the confinement is indeed permanent, corresponding to "infra-red slavery"--as the separation increases corresponding to the dynamics of strongly interacting quarks at low momenta. (See Fig. 1.)

Such novel, unconventional physical behavior translates into new and different mathematical behavior in the fundamental underlying theory. This behavior is opposite from what we are familiar with in QED, our only true and tested friend in the theoretical world.

It is well known that in quantum field theories we must renormalize the coupling constants. In quantum electrodynamics the charge is the dimensionless coupling constant; and the renormalized charge  $e_R$  is smaller in magnitude than the bare one  $e_0$  introduced in defining the Lagrangian and Hamiltonian of the



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Fig. 1

theory.  $e_R$  is the charge of the physical electron dressed by quantum fluctuations. The vacuum, or ground state, of the quantized electromagnetic field is bubbling everywhere with virtual electron-positron pairs. Their presence originates in the zero point quantum fluctuations of the electromagnetic field strengths. Since all components of E and B don't commute we cannot set them identically to zero; only their average strength can vanish. The virtual pairs in the vacuum are physically polarized in the neighborhood of an electron by the electrostatic forces.<sup>8</sup> The effect of this polarized cloud of pairs in the vacuum sea is to partially shield and reduce the strength of the electron charge when it is viewed from a distance greater than the range characteristic of the polarization, i.e., the electron Compton wavelength

$$R \sim c\Delta t \sim \frac{\hbar}{mc} .$$

Thus the electron charge as seen by a macroscopic applied field, or by a test charge at distances  $> R$ , corresponding to scattering with  $q < mc$ , is

$$e_R = \int d^3r [\rho_0(r) + \rho_p(r)]$$

where  $\rho_0(r) = e_0 \delta^3(r)$  for an electron at the origin; and  $\rho_p(r)$  is illustrated in Fig. 2. When however we probe the electron at small distances we see inside the polarization cloud and eventually, for  $q \gg mc$ , probe the bare charge  $e_0 > e_R$ . This physical picture is important in assuring us that in QED, the simplest of gauge field theories, the exact force law grows stronger more rapidly than  $1/r$  at smaller distances, and decreases for large  $r$ . This is precisely opposite to the desired behavior for explaining confinement and asymptotic freedom; i.e., a charge that grows weaker when probed at small distances but grows stronger at large separation.

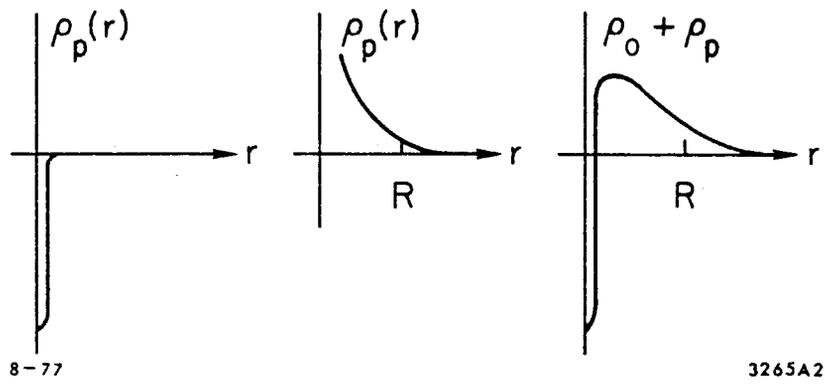


Fig. 2

From the above discussion there is a clue as to how field theory must be changed from QED if we are to construct a theory that incorporates asymptotic freedom. Spurred by the successes of QED as well as by their mathematical beauty we work in the context of gauge theories--such as QED--i.e., theories with space-time dependent symmetry transformations under which the theory is invariant, viz. in QED

$$A_{\mu}(x) \rightarrow A_{\mu}(x) + \frac{\partial \delta \chi}{\partial x^{\mu}}$$

$$\psi(x) \rightarrow (1 + ie\delta\chi(x))\psi(x) .$$

Evidently we will require a more complex gauge theory than QED for which the gauge group is U(1): i.e., the group of space-time dependent phase transitions on one complex or two real fields, representing a charged particle. The U(1) symmetry leaves the charge residing on the electron. In contrast if the forces are to weaken at short distances the coupling strength must weaken for large q and this can happen only if the polarization cloud can acquire at least some of the charge of the particle itself. This in turn means that the quanta must be able to remove charge from the sources in contrast to the electrically neutral photons of QED. This is a possibility for non-abelian gauge theories<sup>9</sup> when other special requirements are also satisfied. A non-abelian gauge theory is one with internal symmetry transformations that act on the particle labels themselves, such as turning protons into neutrons in the example of SU(2), in addition to depending on space-time coordinates, viz.

$$V_{\mu}^{\alpha} \rightarrow V_{\mu}^{\alpha} + \frac{\partial \delta \chi^{\alpha}}{\partial x^{\mu}} - g \epsilon^{\alpha\beta\gamma} \delta \chi^{\beta} V_{\mu}^{\gamma}$$

$$\psi \rightarrow (1 + ig \frac{\tau^{\alpha}}{2} \delta \chi^{\alpha}) \psi .$$

It has been shown for example that a renormalizable SU(3) gauge theory of strong interactions formulated with massless gluons that are color octets-- this is a renormalizable theory so long as the gluon fields are massless-- has the property of asymptotic freedom if there are not too many quarks whose masses are "light" on the scale of momentum transfers exhibiting the weak coupling limiting behavior and hence scaling.<sup>10</sup>

However the fact that the theory includes massless gluons introduces its own difficulties. With non-abelian theories the method for removing infra-red divergences is more complex and delicate than that of Bloch-Nordsieck since the ordering of vertices matters and there is no simple classical current behavior in the soft gluon limit.<sup>11</sup> This very difficulty has been conjectured into a virtue in order to explain quark trapping. The untamed singularities in the infra-red or soft gluon limit are presumed<sup>12</sup> to be the source of an interaction strength that grows beyond bounds as the separation between quarks increases. There is no mathematical proof of this conjecture; what is more quarks may have in fact been observed.<sup>7</sup> So this is a questionable conjecture on all grounds. What is known from the constructive analysis of Cheng, Eichten, and Li<sup>13</sup> is that the requirements of asymptotic freedom and of freedom from IR divergences are mutually incompatible in renormalizable gauge theories. Hence we must have IR divergences if we are to insist on asymptotic freedom and, therefore, it is necessary for us to appeal to confinement as a way of removing the IR difficulty. It is asserted that we cannot radiate massless gluons and consequently we avoid the infinities associated with their massless propagation.

This framework of ideas has aesthetic appeal although lacking hard calculational foundations. The problem we face as theorists is one dating to Year 1 of modern physics--how to solve relativistic field theories without

resorting to weak coupling iterative expansions for implementing the renormalization program. In other words, how can we test our theoretical ideas when we cannot solve our equations. That is today's problem and explains why I must proceed by asking very limited questions in search of modest progress and encouragement; and why I must resort to simple models that to many may appear to be little more than elementary school exercises. We ask:

What do we need, phenomenologically, of a field theory for confinement?

What methods can we bring to bear on field theory for constructing non-perturbative solutions? An iterative procedure starting with quark and gluon degrees of freedom can hardly be expected to give low-lying physical hadronic spectra in a reliable way in a finite number of practicable operations.

We take the following as the starting ingredients of a theory:<sup>14</sup>

- 1) There are three flavors of light "old" quarks. Massive new quarks (with "charm" and perhaps more varieties) are, by assumption, not essential in the study of confinement.
- 2) Quarks carry a hidden quantum number "color." Hadrons are always formed of combinations of quarks that are color neutral. The quarks come in three colors and the theory is invariant under SU(3) of color; then the configurations  $Q\bar{Q}$  and  $QQQ$  are the simplest colorless structures, corresponding to mesons and baryons respectively. Since these colorless--or color singlet--combinations of quarks have the quantum numbers of normal hadrons, the problem of quark confinement becomes that of color confinement.
- 3) Massless gauge gluons are the quanta carrying the force between quarks. The color quantum number plays the same role that electric charge does in QED; and the colored quarks interact with one another

via colored electric and colored magnetic fields. What we are describing here is a non-abelian gauge theory along the lines first introduced by Yang and Mills.<sup>15</sup> In contrast with abelian QED, where the quanta are electrically neutral and do not change the charge of their sources, the gluons are themselves colored. They must remain massless, as in QED, for the theory to be renormalizable in the usual fashion. However in order to maintain local color conservation (i.e. color conservation as a local, not simply a global, symmetry) the gluons must carry off, say, red-blueness upon changing a red to a blue quark. This means that such a theory with  $SU(3)$  of color must have 8 gluons corresponding to all  $3 \times 3 - 1 = 8$  color combinations, where we have subtracted the combination that is colorless and does not enter the theory. Another new feature of such a non-abelian gauge theory is that, being colored, gluons change the color charge of their sources. As we mentioned earlier such a theory can lead to forces that grow weaker at smaller and smaller distances, becoming asymptotically free at very short ones.

Such an asymptotically free gauge theory--henceforth called QCD--presents itself strongly as appropriate for a fundamental underlying theory of strong interactions if we can explain why we see neither quarks nor massless gluons--namely if we can explain "color confinement"! Furthermore for the theory as stated thus far the confinement must be permanent: no massless gluons are observed. In the highly successful theoretical tradition of turning adversity into achievement, or  $\infty$  to 0, we appeal to the infra-red divergences to trap the quarks and gluons as noted earlier. This is a hope! Color confinement has not been proved for such theories. We are dealing with strong forces

and looking for composite bound states of quarks that form the observed extended hadrons. But we have thus far lacked the mathematical methods for constructing such states from elementary quark and massless gluon constituents. The distance from starting point to physical matter is too great for perturbation theory.

In particular since we must avoid an iterative weak coupling expansion and the actual implementation of the renormalization procedure in terms of a Feynman graph expansion we must do something about the infinities buried in quantum field theory. In principle this can be done by introducing a cut-off and working with a theory that is finite at each stage. If the low-lying states and their excitation energies remain finite at the end in the limit of the cut-off increasing to  $\infty$  we may consider this approach satisfactory. One way to implement a cut-off procedure is to formulate the theory on a finite lattice with a finite number of degrees of freedom--in the same way that we can approximate a continuous string by a weighted one, or vice versa. In fact in their pioneering original paper on field quantization, Heisenberg and Pauli<sup>16</sup> started from a discrete theory in terms of discrete cells in space and proceeded to the continuum limit. We reverse this step with the lattice theory. This procedure, formulated initially by Wilson<sup>17</sup> in the language of path integrals and transcribed to Hamiltonian form by Kogut and Susskind<sup>18</sup> has stimulated a lot of work--a fair amount of it right here at SLAC.<sup>19</sup> Although no definitive solutions of physical gauge theories have been obtained, both physical insight and reliable and straightforward methods have been developed and I will describe them in my second lecture.

A very nice phenomenological approach that is less ambitious on fundamental theoretical grounds on one hand but which has been impressively

successful in reproducing known physical properties of low-lying hadronic states on the other is the approach of semi-classical "bag" theories; and in particular the MIT bag model.<sup>20</sup> In this approach the bag is posited by fiat in the form of boundary conditions imposed on a canonical field theory. Alternatively one can proceed by constructing exact solutions of the classical field equations in the interacting field theory that localize the energy in space and that cannot be found by iterative procedures starting from empty space vacua. Such solutions--known as solitons--occur<sup>21</sup> in field theories with degenerate vacua, in which case they are frequently referred to as topological solitons, an example of which is the 't Hooft-Polyakov monopole<sup>22</sup> in three dimensions; they also occur in field theories with conservation laws of charge, isospin, or some appropriate additive quantum number reflecting an internal symmetry, in which case they are referred to as non-topological solitons.<sup>21</sup> A well studied example is that of a charged bose or fermi field interacting with a neutral scalar Higgs field. The SLAC bag model<sup>23</sup> and other similar ones<sup>24</sup> which construct semi-classical extended hadrons from local field theory are of this latter class. Aside from being semi-classical solutions--i.e. solutions obtained by neglecting quantum fluctuations and zero point corrections to a one-particle treatment of the quarks, such approaches also have the presumed drawback of violating the requirements of asymptotic freedom since they contain Higgs scalars explicitly in the Hamiltonian. I view these Higgses as a phenomenological crutch for constructing low-lying colorless hadrons from QCD. They are useful because field theory has not yet fully demonstrated the ability to predict such hadrons from gluons and quarks alone.

The MIT bag model treats a hadron as a finite region in space to which almost free and light quarks are confined by fiat. This is accomplished formally by modifying the free quark field theory by two assumptions:

1. A constant energy density is added to the Hamiltonian within the hadron, i.e.:

$$T^{\mu\nu} \rightarrow T^{\mu\nu} + g^{\mu\nu} B$$

within the bag, or

$$H \rightarrow H + B$$

where the volume tension  $B$  acts to compress the bag against the outward pressure of the quark gas.

2. A boundary condition is imposed such that the colored fields of quarks and gluons are confined within a finite region of space-- i.e., the interior of the hadron:

$$n_{\mu} (T^{\mu\nu} + g^{\mu\nu} B) = 0 \quad \text{on the surface}$$

In this initial zeroth order formulation of the MIT bag model, if we treat the surface as that of a spherical static bag, or "cavity," of radius  $R$ , the energy of a quark in the bag is

$$E = \frac{c}{R} + B \frac{4\pi R^3}{3}$$

where  $c \approx 2$  is a constant characterizing the energy of a massless fermion in a spherical well with infinitely high walls. The minimum gives a ground state energy and bag radius of

$$E_0 = \frac{4}{3} \frac{c}{R_0} ; \quad R_0 = \left( \frac{c}{4\pi B} \right)^{1/4} \quad (1.1)$$

Adding massless color gauge fields, or gluons, to the bag Hamiltonian, with the assumption that the SU(3) color symmetry is unbroken, leads to the result that all hadronic states have conventional quantum numbers; i.e., they are color singlets. This follows from the boundary conditions for the color gauge fields which require that they vanish on the surface of the bag if the total energy-momentum are conserved within the bag world line--i.e., within the hypertube representing the space-time trajectory of the bag. This is the formal statement of gluon confinement and gives

$$n_{\mu} \mathcal{F}^{\mu\nu} = 0 \quad \text{on the surface}$$

where  $n_{\mu}$  is the space-like normal to the surface. Since by Gauss' law, the total color charge of a bag is given by a volume integral of the divergence of the color electric field, we have

$$Q_{\text{color}} = \int_{\substack{\text{bag} \\ \text{volume}}} d^3r \partial_k \mathcal{F}^{k0} = \int_{\substack{\text{bag} \\ \text{surface}}} dS n_{\mu} \mathcal{F}^{\mu 0} = 0 \quad (1.2)$$

Hence only color singlet bags can be formed.

With the inclusion of gluons and of interactions between the quarks via gluon exchange the bag model is reminiscent of nuclear shell theory which describes nuclear spectroscopy in terms of nucleons confined within a phenomenological nuclear potential and interacting with one another via a spin-orbit potential that shapes the shell structure. Here the bag is a relativistic representation of the dominant confining forces. There is a residual and relatively weak quark-quark interaction resulting from the exchange of an octet of massless colored gluons and which is expressed via the Yang-Mills mechanism. The quark-quark interaction is required to be weak at short

distances so that Bjorken scaling is not destroyed. It is also required to be weak at large distances, or near zero momentum, if we are to avoid large renormalizations of the impressively successful naive quark model estimates of transitions among low-lying baryonic states. The bag provides the infra-red cutoff so that the quark-gluon interaction is treated iteratively in a weak coupling expansion, only the lowest contribution of which has been analyzed.

In this approximation, the MIT bag model remains a static spherical "cavity" in which quarks and gluons and their first quantized modes of motion satisfy appropriate boundary conditions which insure that only color singlets occur. It is a four-parameter model since in addition to the volume tension  $B$ , there is also the quark-gluon coupling constant  $\alpha_c = g^2/4\pi$ , which characterizes the strength of the quark-quark interaction; the mass of the strange quark,  $m_s \neq 0$ , which characterizes the breaking of the SU(3) flavor symmetry; and a constant  $Z_0$  that characterizes the volume dependent zero point energy associated with the quantum modes of the bag.

There are four contributions to the energy of a hadron:

(1) The kinetic energy of the quarks confined in the bag; for  $N$  quarks in  $S$ -orbits within a spherical bag of radius  $R$  this is given by

$$E_{\text{kinetic}} = \frac{1}{R} \sum_{i=1}^N \sqrt{x_i^2(m_i R) + (m_i R)^2} \quad (1.3)$$

where  $x_i(m_i R)$  is the wave number of a quark in the static bag as fixed by boundary conditions, and is slowly varying with quark mass, with  $x_i(0) \approx 2$ . This term gives rise to roughly 3/4 of the mass of a typical hadron.

(2) The bag energy which expresses the fact that the confining potential, or pressure  $B$ , has a dynamical origin in an underlying relativistic field

theory and thus carries energy; in QCD it is the average gluon field energy within the bag,

$$E_{\text{vol}} = B \cdot \left(\frac{4\pi}{3}\right) R^3 \quad (1.4)$$

(3) The zero point energy of all the modes in the static bag, or cavity, enters because it changes with the volume of the cavity. In conventional field theory formulated over all space it is highly divergent but can be discarded as a constant, the same for all processes. Here we simply parametrize the finite part of the zero point energy by

$$E_0 = -Z_0/R \quad (1.5)$$

A divergent contribution proportional to  $V = 4\pi/3 R^3$  is submerged in a renormalization of  $B$ , and for simple slab-like and spherical models with spinor and vector field constituents there are no contributions to  $E_0$  proportional to  $R^2$ ,  $R$ , or constant.

(4) There is a contribution to the energy associated with the gluon mediated interactions between the quarks. The bag itself is assumed to be the expression of the strong, long-range confining forces whose energy is given by (1.4). The residual quark-quark interaction is treated perturbatively to lowest order in  $\alpha_c = g^2/4\pi$ . The two lowest order Born graphs are shown in Fig. 3. Since to this order no gluon self-couplings occur the gluons behave like 8 independent color fields  $\vec{E}^a$  and  $\vec{B}^a$ ,  $a = 1, \dots, 8$ , which must satisfy the boundary conditions for confinement. The assumption made in treating the self energy contributions (Fig. 3b) is that only those parts of it are retained that are required to enable  $\vec{E}^{(a)}$  and  $\vec{B}^a$  satisfy the required boundary conditions on the bag surface, i.e.,

$$\vec{n} \cdot \vec{E}^a (R) = 0 \quad (1.6)$$

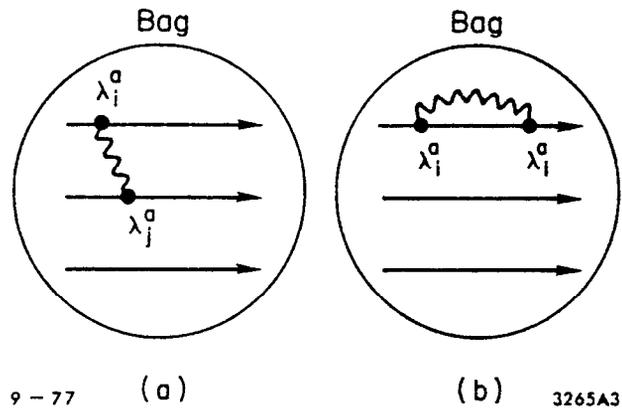


Fig. 3

$$\vec{n} \times \vec{B}^a (R) = 0 \quad (1.7)$$

Otherwise the self energy contributions are assumed to be included in the assigned phenomenological quark masses. The interaction energy from Fig. 3a is static and has the electrostatic and magnetostatic contributions of, respectively,

$$\Delta E_E^{(1)} = \frac{1}{2} g^2 \sum_{i \neq j} \sum_a \int_{\text{bag}} d^3 r \vec{E}_i^a (r) \cdot \vec{E}_j^a (r) \quad (1.8)$$

$$\Delta E_M^{(1)} = - \frac{1}{2} g^2 \sum_{i \neq j} \sum_a \int_{\text{bag}} d^3 r \vec{B}_i^a (r) \cdot \vec{B}_j^a (r) \quad (1.9)$$

To this must be added from Fig. 3b the contributions required to turn (1.8) and (1.9) into interaction energies of fields  $\vec{E}^a$  and  $\vec{B}^a$  for which the boundary conditions (1.6) and (1.7) are satisfied. This is no problem for the  $\vec{B}_i^a$  which, as constructed from the quark current distribution by Maxwell's equations, automatically satisfy (1.7), i.e., it follows from

$$\vec{\nabla} \cdot \vec{B}_i^a = 0$$

$$\vec{\nabla} \times \vec{B}_i^{(a)} = \vec{j}_i^a = q_i^+ \vec{\alpha} \lambda^a q_i = \frac{3}{4\pi} (\hat{r} \times \vec{\sigma}_i) \lambda_i^a \frac{\mu_i^a(r)}{r^3} ; \quad \hat{r} \equiv \frac{\vec{r}}{r} \quad (1.10)$$

where  $\mu_i^a(r)$  is the scalar magnetization density for a quark of mass  $m_i$  in the lowest cavity eigenstate, that  $\vec{B}_i^a$  satisfies (1.7) automatically. However, the  $\vec{E}_i^{(a)}$  do not satisfy the cavity boundary condition (1.6), a deficiency most easily remedied by adding the static electric self-energy from Fig. 3b to (1.8) leading to

$$\Delta E_E = \frac{g^2}{2} \sum_a \int_{\text{bag}} d^3r \left( \sum_i \vec{E}_i^a(r) \right) \cdot \left( \sum_j \vec{E}_j^a(r) \right) \quad (1.11)$$

Note that

$$\sum_j \vec{E}_j^a(r) = \sum_j \lambda_j^a \frac{\hat{r}}{4\pi r^2} \int_0^r d^3r' \rho_j(r')$$

and if we neglect quark mass differences  $\rho_j = \rho$ , independent of  $j$ . Hence

$$\sum_j \lambda_j^a | \text{color singlet} \rangle = 0 \quad (1.12)$$

so that the boundary condition (1.6) is satisfied since the color charge density vanishes locally.<sup>25</sup>

With these assumptions--in particular the assumption of zero local color separation in a spherical cavity--the electrostatic interaction energy vanishes, and the final expression for the magnetic interaction simplifies to a spin-spin term

$$\Delta E_M = 8\alpha_c \lambda \sum_{i>j} \left( \vec{\sigma}_i \cdot \vec{\sigma}_j \right) \frac{\mu(m_i, R) \mu(m_j, R)}{R^3} I(m_i, R, m_j, R) \quad (1.13)$$

where  $\lambda = 1$  for a baryon and  $= 2$  for a meson;  $\mu$  is the quark modes' magnetic moment, and  $I$  is a mass dependent constant. This interaction splits the  $\pi$  from the  $\rho$  mass, and the  $N$  from the  $\Delta$  in the right direction.

The sum of the four contributions to the hadron energy; i.e., (1.3), plus (1.4), plus (1.5), plus (1.13) evaluated in the appropriate spin and flavor state in the spherical static bag is then variationally minimized with respect to the radius  $R$  to determine the hadron mass. The spectrum of the ground state baryon [56] and meson [35] together with static properties of

these states are then compared<sup>20</sup> with the predictions based on four adjustable constants in Fig. 4 and Table I.

Fitting the data with these ingredients has been a considerable triumph of this MIT bag approach as seen in the next two figures. Note that these results emerge from a lowest order perturbation calculation of the quark-quark spin dependent interaction for which the expansion parameter is determined to be  $\alpha_c = 0.55$ .

Buoyed by these results, one can press on further to study low-lying excited hadrons for which one quark is in a P-wave excited state and the remainder are in the ground state. In such a study of  $1/2^-$  and  $3/2^-$  baryons and of  $0^+$  and  $1^+$  mesons there are no additional free parameters since all were determined in the fit to the ground state  $1/2^+$  and  $3/2^+$  baryons and  $0^-$  and  $1^-$  mesons. In this case the study is limited<sup>26</sup> to quarks in  $P_{1/2}$  states in a spherical cavity in order to satisfy the boundary conditions, which for the quark field become on the spherical surface

$$\hat{r} \cdot \nabla(\bar{q}q) = -2B$$

and can therefore not be satisfied locally for distributions, such as  $P_{3/2}$  states, that are not spherical. Since the calculations are carried out to lowest order in  $\alpha_c$ , the quarks are treated as non-interacting in forming the states.

However, there is a problem. If one makes hadrons out of quarks only in  $S_{1/2}$  and  $P_{1/2}$  modes, one finds that the resulting baryon spectrum contains states which do not exist in nature. These states turn out to be modes where the center of mass of the quarks moves relative to the surface of the bag—so-called translation modes. Such modes are spurious and result from use of

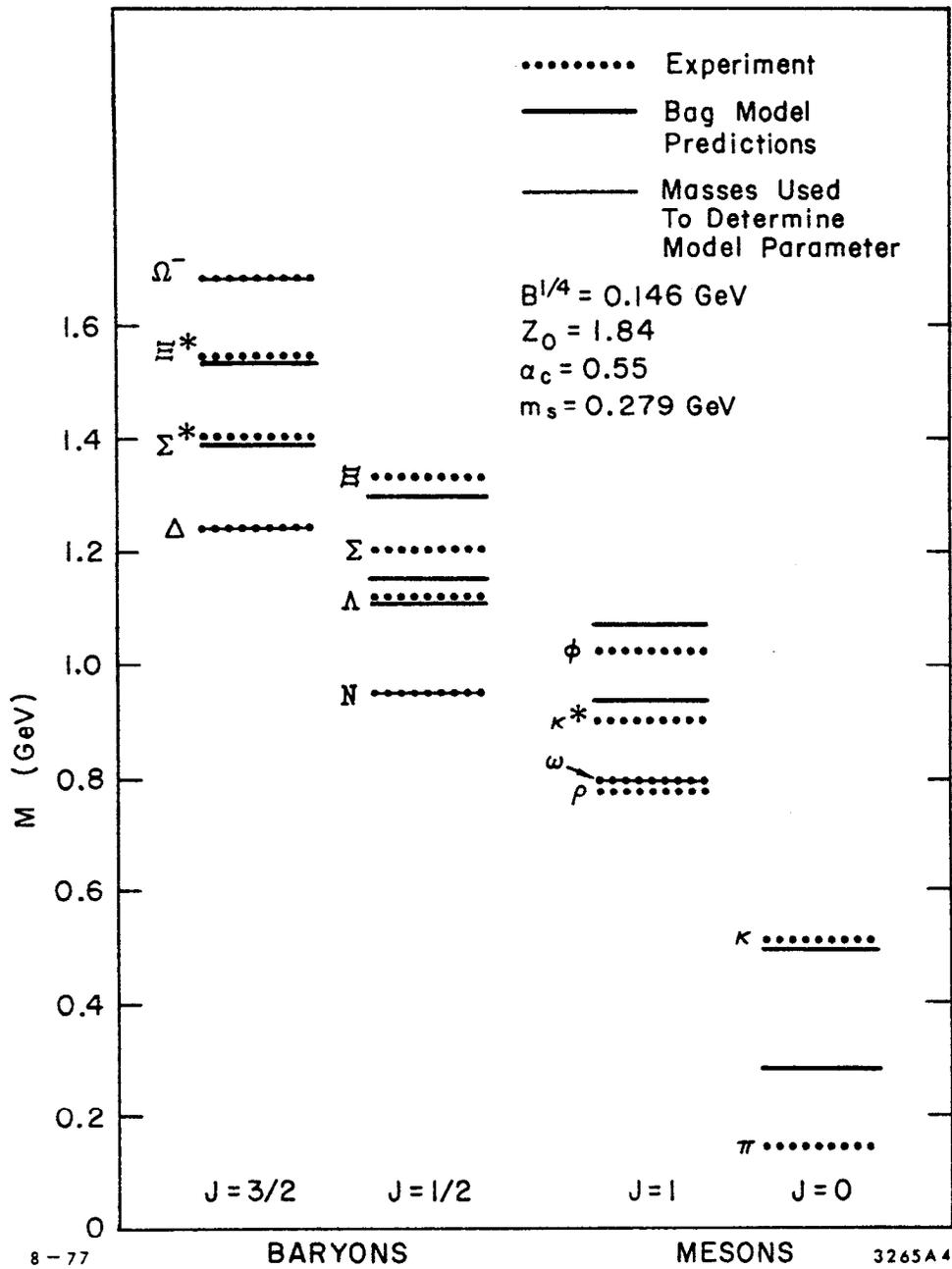


Fig. 4

Table I

		<u>Experiment</u>	<u>MIT Bag</u>	
Magnetic Moments	}	p	2.79	1.9
		n/p	-0.685	-2/3
		$\Lambda/p$	$-0.240 \pm 0.021$	-0.255
		$\Sigma^+/p$	$0.93 \pm 0.16$	0.97
		$\Sigma^0/p$		0.31
		$\Sigma^-/p$	$-0.53 \pm 0.13$	-0.36
		$\Xi^0/p$		-0.56
		$\Xi^-/p$	$-0.69 \pm 0.27$	-0.23
		<u>Experiment (Cabibbo)</u>	<u>MIT Bag</u>	
			$\Delta s = 0$ $\Delta s = 1$	
Axial-Vector Coupling Constant	}	F	$0.41 \pm 0.02$	0.44   0.47
		D	$0.83 \pm 0.02$	0.65   0.71
		<u>Experiment</u>	<u>MIT Bag</u>	
Charge Radii	}	$\langle r_p^2 \rangle^{1/2}$	$0.88 \pm 0.03$ fm	0.73 fm
		$\langle r_N^2 \rangle^{1/2}$	$-0.12 \pm 0.01$ fm	0 fm

a fixed cavity approximation for the bag. In reality, the quarks may move at will, and the surface of the bag will move to keep them confined: if the quark's center of mass moves, so will the "center" of the bag.

This problem has been discussed by Rebbi.<sup>27</sup> Its resolution involves including  $P_{3/2}$  quark modes in the spectroscopy, since the translation modes are eigenstates of orbital angular momentum. However,  $P_{3/2}$  modes cannot satisfy the bag boundary condition locally over the surface of the bag. In order to perform the calculations, the boundary condition has to be composed only to achieve a global pressure balance and spherical cavities are used for confinement. The resulting spectrum for the  $L = 1[70]$  baryon states of negative parity, built of two quarks in  $1S_{1/2}$  cavity eigen modes and the third quark in a linear combination of  $1P_{1/2}$  and  $1P_{3/2}$  modes is too low as found by deGrand<sup>28</sup> and shown in Fig. 5. Much of the blame for this defect is attributed to the  $P_{3/2}$  states whose eigenfrequencies are found to be quite small. Resolution of this difficulty awaits future study that will require going beyond a static spherical cavity approximation.

Next I turn briefly to the problem of deriving bag models from local canonical field theory.<sup>23,24</sup> The fundamental idea is that the vacuum is highly polarized in the presence of quarks. The "bags" themselves are extended, coherent vacuum excitations to which the quarks, which may have an extremely large bare mass, are bound. The mechanism creating the bags in the published models is a scalar Higgs field that develops a non-vanishing expectation value. A non-abelian colored gauge interaction is introduced à la QCD so that the binding to form low mass bag states occurs only when the quarks form color singlet states which are the hadrons. Within the bag the interaction between the quarks is relatively small, i.e., negligible in

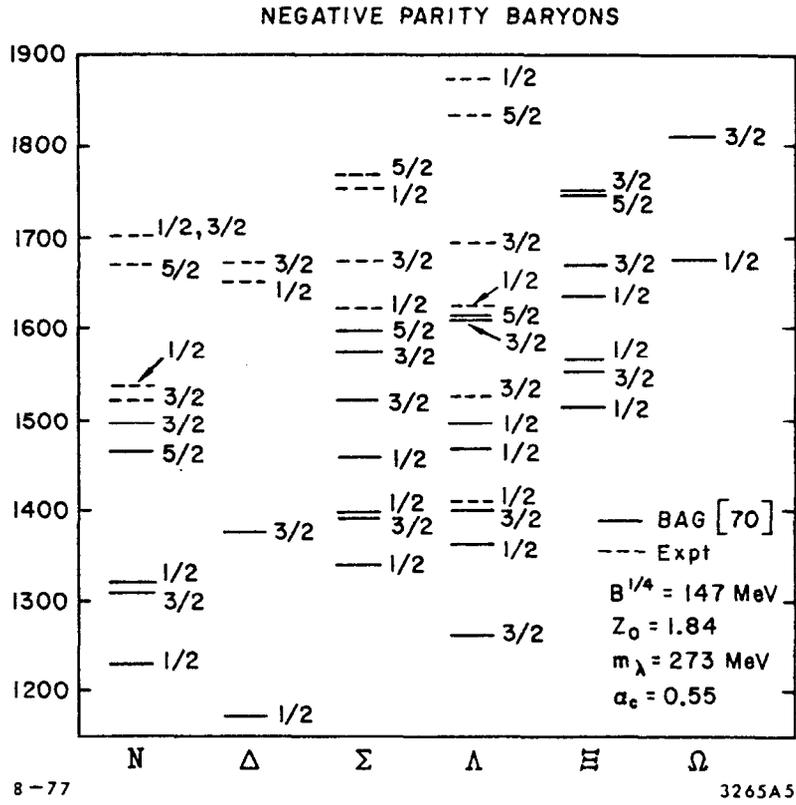


Fig. 5

comparison with the very strong interaction binding the quarks to the bag and forming their wave functions. This gives a picture of hadrons as bound states of two or three almost free quasi-particles from which they derive their SU(3) properties. However there is an energy associated with the size of the bag--or vacuum excitation--containing the quarks which prevents their being separated. In this picture quark trapping may be just an approximate state of affairs at low energy and massive individual colored quarks may be found at higher energies after all.

The mechanism for polarizing the vacuum and forming "bags" is the same one giving rise to the abnormal states in the Lee-Wick theory of uniform nuclear matter.<sup>29</sup> In their example with many particles, a classical treatment with neglect of surface effects is valid. To illustrate this idea, consider the following classical energy expression for a statistical ensemble of N nucleons, each of mass M, plus a scalar field with the self coupling illustrated in Fig. 6.

$$E = \int \rho dV [M + g\sigma(x)] + \int dV U(\sigma)$$

$\rho(x)$  represents the number density of nucleons,

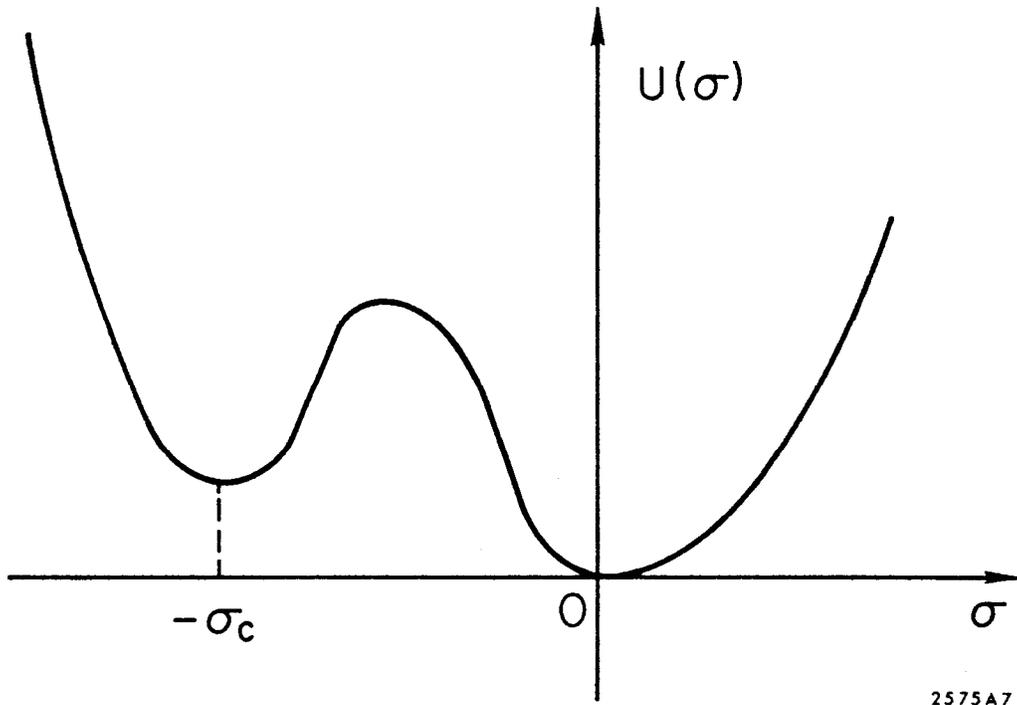
$$N \equiv \int \rho dV$$

and g is a measure of their interaction strength with the field strength  $\sigma$ .

The "normal state" of matter is described by setting  $\sigma = 0$  so that the field state is at the minimum energy  $U(0) = 0$  and the assemblage of nucleons, neglecting surface and kinetic effects (i.e., the height of the fermi surface relative to their mass energy), is

$$E_{\text{normal}} = MN = \rho_0 V_0 M$$

in terms of the uniform nucleon density  $\rho_0$  within the volume  $V_0$ .



2575A7

Fig. 6

The "abnormal state" is described by "polarizing the field vacuum" and exciting the field strength  $\sigma = -\sigma_c$ , the value at the local minimum in Fig. 6 within the volume  $V_0$  of nucleons, i.e.,

$$\begin{aligned} \sigma &= -\sigma_c && \text{inside } V_0 \\ \sigma &= 0 && \text{outside } V_0 \end{aligned} \quad (1.14)$$

The total energy is now given by

$$E_{\text{abnormal}} = \rho_0 V_0 (M - g\sigma_c) + U(-\sigma_c)V_0 = E_{\text{normal}} + V_0 \left[ -g\sigma_c \rho_0 + U(-\sigma_c) \right]$$

and evidently at large enough densities  $\rho_0$  the abnormal state (1.14) will be at a lower energy than the normal one.

As these models have been implemented so far the color interaction involving the gluons plays no direct role in the energies and structure of the color singlet states that are the hadrons. This is because the solutions have been constructed by treating the Higgs scalar field and the gauge fields classically and by ignoring the quantum fluctuations in treating the quark in terms of single particle Dirac theory. In particular the single particle energies of quarks trapped in bags vs. free particle quark states are considered but the shift in vacuum energies of the quarks due to the presence of the bag is ignored (i.e. we work in the tree approximation, and normal ordering terms are dropped). The canonical field theory approach to bags shares with the MIT bag model approach the defect that neither is as yet a fully quantized field theory.

Recently T. D. Lee and collaborators<sup>21,24</sup> (R. Friedberg and A. Sirlin) have given a thorough and systematic discussion of these quasiclassical soliton solutions for interacting spinor, scalar, and vector gluon fields. General solutions of the coupled field equations, including the

limiting forms reducing to both the MIT and SLAC bag solutions have been described. Friedberg and Lee have also studied the accuracy of the quasiclassical approximation in the two limiting regions where exact answers are known. For fully relativistic theories of interacting spinor-scalar fields in the weak coupling limit the ratio  $R$  of exact binding energies to those calculated quasiclassically is

$$R = 1 \quad \text{when } N \gg 1$$

$$R = 0.77 \quad \text{when } N = 2$$

where  $N$  is the number of fermions bound together. For non-relativistic quark fields, but relativistic scalar fields, they find

$$R = 1 \quad \text{when } N \text{ is arbitrary and the coupling is strong} \\ \text{or when } N \gg 1 \text{ and the coupling is arbitrary}$$

$$R = 0.77 \quad \text{when } N = 2 \text{ and the coupling is weak.}$$

It remains unknown however how good  $R$  is when  $N \sim 2$  or  $3$  as for mesons and baryons, and when the fields are relativistic and the coupling is strong. It is to this question that I now turn. In Appendix A, the quantum corrections are calculated for a special example in  $1x - 1t$  dimension and shown to be large for  $N = 1$  and strong coupling.

### QUANTUM FIELD THEORY ON A LATTICE

We shall now formulate quantum field theory on a lattice. Our first reason is to convert quantum field theory to a finite, well defined one without implementing a renormalization program which, in practice, requires an iterative weak coupling expansion in Feynman graphs. Such iterative steps cannot help us in calculating finite physical quantities in the presence of strong interactions. It is a deeper question of whether or not to take the lattice spacing  $a \rightarrow 0$  in the end, retaining the low lying spectra and long wavelength coherent phenomena that survive this limit. Perhaps there is a fundamental length or structure on a scale that is sufficiently small,  $\lambda \sim (\text{few}) \times 10^{-16}$  cm, that it has not yet been probed by experiments. This can remain an open question today since we are concerned only with excitations of a few GeV and with structures  $\lambda \sim 10^{-14}$  cm. Particularly in this lecture I will discuss models leading to coherent "soliton" solutions and phase transitions.

The second reason for going to a lattice is that phase transitions and coherent states have been extensively studied in statistical mechanics and solidstate physics which can provide us with powerful methods and deep insights. The alternative of working in momentum space, as we normally do with Feynman propagators, and simply cutting off the momenta at a  $k_{\text{max}} \sim 1/a$ , the reciprocal of the lattice spacing, is less rich in helpful insights. The question of phase transitions enters naturally into these discussions as shown in Fig. 7. A weak coupling, abelian gauge theory like QED exhibits and describes free electrons and massless quanta. However we seek a strong coupling, non-abelian gauge theory, like QCD, which confines its quarks and vector gluons,

STRONG COUPLING		Confined Quarks and Gluons  QCD?  Color Confinement
WEAK COUPLING	Free Electrons  Free Massless Quanta  QED!	
	ABELIAN	NON-ABELIAN

8-77

3265A6

Fig. 7

but which exhibits and describes physical hadrons. Is there a phase transition as we go from weak to strong coupling? or from abelian to non-abelian gauge theories? are both of the above required? or neither, i.e., is there a smooth transition with a typical quark mass characterized by, say,

$$m \sim \left(\frac{1}{a}\right) e^{-1/g^2}$$
$$\sim 10^{-3} \text{ GeV for } g^2 = \frac{1}{137}$$
$$\gg 10 \text{ GeV for } g^2 \gtrsim 1$$

To be useful, the procedure of going onto a lattice should not destroy symmetries of the theory. This means we require internal symmetries such as local gauge invariance and chiral symmetry to remain exact. Those derived from the homogeneity and isotropy of space, i.e., Lorentz and Poincaré invariance, will be lost because there are now only discrete symmetries on the lattice. But we require relativistic results for spectra, say, in the low energy region as well as in the  $a \rightarrow 0$  limit.

Let us first transcribe the gradient operator and then its gauge invariant form to a discrete lattice. For simplicity in writing I will use the notation of a one-dimensional spatial lattice; its generalization to three dimensions causes no problem. Time will be treated as a continuous variable, since I am constructing a Hamiltonian formalism, although formulations putting time on a discrete lattice have also been developed and give no special difficulties.

The continuum variable  $x$  is replaced by a discrete lattice of length  $L$  and spacing  $a \equiv 1/\Lambda$  defined so that there are  $(2N + 1)$  points:

$$L = (2N + 1)a \tag{2.1}$$

With periodic boundary conditions, the allowed momenta on the lattice are

$$k = \frac{2\pi}{L} n \quad n = 0, \pm 1, \dots, \pm N \quad (2.2)$$

and

$$k_{\max} = \frac{2\pi}{L} N. \quad (2.3)$$

The volume integral then becomes simply a sum over lattice points:

$$\int dx \rightarrow \frac{1}{\Lambda} \sum_{j=-N}^N \quad (2.4)$$

The fields and canonical momenta are defined at discrete lattice points and their fourier expansion includes a finite number of modes up to  $k_{\max}$ :

$$\begin{aligned} \psi(x) \rightarrow \psi_j &= \sum_{k=-k_{\max}}^{+k_{\max}} e^{ikj/\Lambda} \psi(k) \\ \psi_k &= \frac{1}{2N+1} \sum_j \psi_j e^{-ikj/\Lambda} \end{aligned} \quad (2.5)$$

The lattice version of the canonical (anti-) commutator relations is

$$\left[ \Pi_j, \psi_j \right] = -i\Lambda \delta_{jj} \quad (2.6)$$

The simplest and most direct definition of the gradient on the lattice is as a difference operator

$$\nabla \psi(x) \rightarrow \Lambda \left[ \psi_{j+1} - \psi_j \right] \quad (2.7)$$

This gives for the kinetic energy term in the Hamiltonian of a scalar field

$$\frac{1}{2} \int dx (\nabla \psi)^2 \rightarrow \frac{L}{2} \sum_k \psi^+(k) \psi(k) \left\{ 4\Lambda^2 \sin^2 \frac{k}{2\Lambda} \right\} \quad (2.8)$$

which reduces to the usual quadratic in  $k$  for  $k/2\Lambda \ll 1$ , and for all  $k$  gives the connection between energy and momentum for a free particle of mass  $m$ :

$$E(k) = \sqrt{m^2 + 4\Lambda^2 \sin^2 \frac{k}{2\Lambda}} \quad (2.9)$$

As illustrated this is a very good approximation to the correct dispersion relation for  $k/2\Lambda < 1$  and deviates up near the max where  $k_{\max}/2\Lambda = \pi/2$ , giving the familiar band structure of solids (Fig. 8).

There is however a problem<sup>19a,b</sup> in applying (2.7) to a theory of quarks described by spin 1/2 Dirac fields in that (2.8) is replaced by a first derivative expression which when properly hermitized becomes

$$\begin{aligned} -i \int dx \psi^\dagger(x) \alpha \nabla \psi(x) &= -\frac{i}{2} \int dx \left\{ \psi^\dagger(x) \alpha \nabla \psi(x) - \left[ \nabla \psi^\dagger(x) \right] \alpha \psi(x) \right\} \\ &\rightarrow L \sum_k \psi^\dagger(k) \alpha \psi(k) \cdot \left\{ \Lambda \sin \frac{k}{\Lambda} \right\} \end{aligned} \quad (2.10)$$

With this form for momentum given by (2.10) the energy-momentum dispersion relation becomes for a free Dirac particle, instead of (2.9),

$$E_D(k) = \sqrt{m^2 + \Lambda^2 \sin^2 \frac{k}{\Lambda}} \quad (2.11)$$

which has again the correct  $k/\Lambda \ll 1$  limit but which leads to a doubling of states of a given energy as illustrated by Fig. 8.

This difficulty is readily resolved by defining the gradient on the lattice to be the operator multiplying the fourier amplitude by  $ik$  and thereby assuring that in both cases of spin 0 and of spin  $\frac{1}{2}$  fermions (2.9) and (2.11) are replaced by the correct relativistic relation

$$E(k) = \sqrt{m^2 + k^2} \quad |k| < k_{\max} = \frac{2\pi N}{L}$$

On the lattice this definition reads

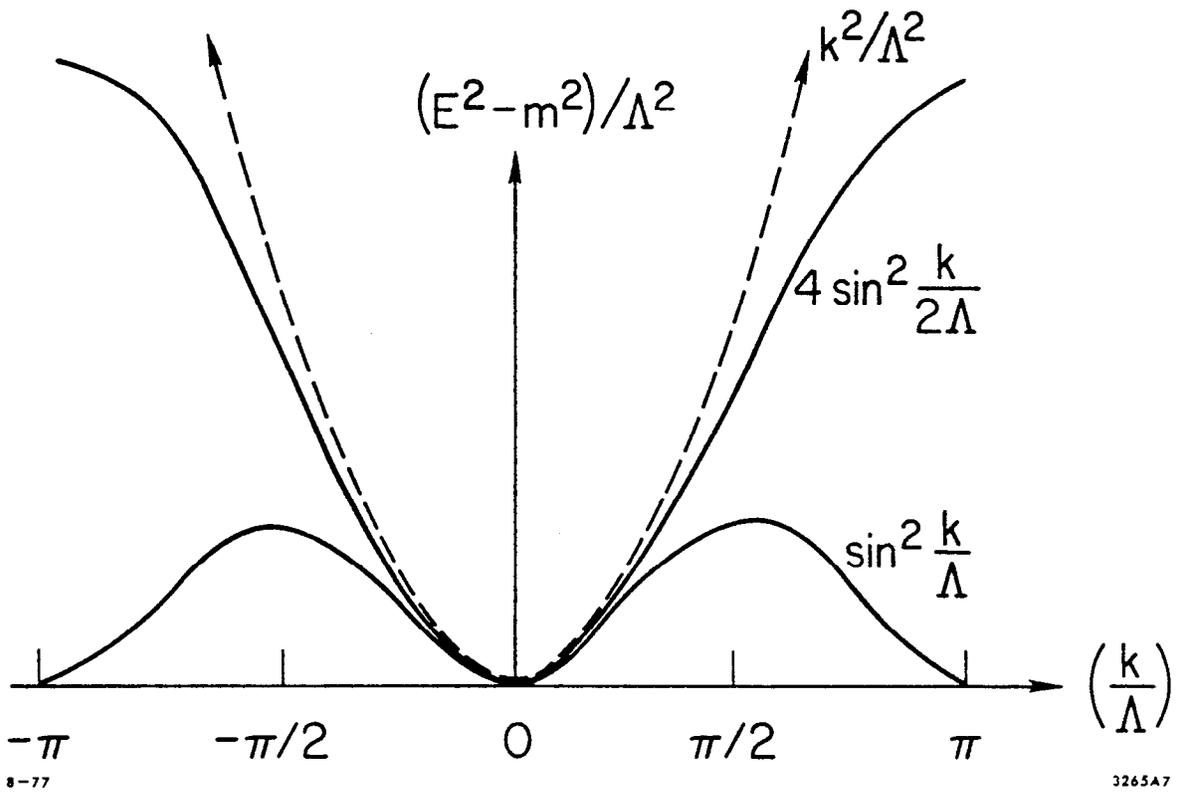


Fig. 8

$$\begin{aligned}
 (\nabla\psi)_j &\equiv \sum_k ik e^{ikj/\Lambda} \psi(k) \\
 &= \sum_{j'} \psi_{j'} \left\{ \frac{1}{2N+1} \sum_k e^{ik(j-j')/\Lambda} ik \right\} \\
 &\equiv \Lambda \sum_{j'} \psi_{j'} \left[ -\delta'(j-j') \right]
 \end{aligned} \tag{2.13}$$

As defined, for  $N \rightarrow \infty$

$$\begin{aligned}
 \delta'(j-j') &= \frac{-(-)^{j-j'}}{(j-j')} & j \neq j' \\
 &= 0 & j = j'
 \end{aligned} \tag{2.14}$$

and shows that the gradient couples sites all along the lattice. For completeness I record the analogous result in 3 dimensions

$$\delta'_x(\vec{j}-\vec{j}') = \delta'(j_x-j'_x) \delta_{j_y, j'_y} \delta_{j_z, j'_z} \tag{2.15}$$

to show that it remains simple.

Next we must form a gauge invariant derivative - i.e., introduce gauge fields while preserving local gauge invariance on the lattice. In continuum QED, local gauge invariance leads to a differential law of current conservation; i.e., locally at any point the time rate of change of the charge density equals the negative of the divergence of the current density from that point. The formal expression of the underlying local gauge invariance is this: in the Lagrangian or Hamiltonian of the theory there appear only the fields themselves

$$\text{viz. } \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{2.16}$$

or interaction terms formed gauge invariantly

$$\text{viz. } \frac{1}{i} \psi^\dagger(x) \vec{\alpha} \cdot (\vec{\nabla} - ie\vec{A}(x)) \psi(x)$$

which is invariant under

$$\begin{aligned} \psi(x) &\rightarrow e^{ie\chi(x)} \psi(x) \\ \vec{A} &\rightarrow \vec{A} + \vec{\nabla}\chi(x) \end{aligned} \tag{2.17}$$

On the lattice we have no local divergence of the current but can speak only of the (color) charge being moved in a discrete step from one lattice site to the next and so on. Indeed by (2.13) and (2.14) the lattice gradient moves a particle along the entire lattice chain (in the direction of the component of the gradient by (2.15)). To preserve local color charge conservation—and the asymptotic freedom idea—evidently the gauge field must be defined on the links joining the lattice sites if we are to avoid discontinuous steps of color charge disappearing from one lattice point and appearing on another one. This is the formulation of K. Wilson and of A. Polyakov,<sup>17</sup> who independently introduced the lattice theory with gauge fields thus defined as the oriented links or strings between nearest neighbor lattice points. In the language of abelian QED the operator representing a gauge field must transform like a field with charge +1 at the starting point of the link and with charge -1 at the final end point. If we start with a quark of charge +1 at lattice point  $j$  and wish to move it to the next point  $j+1$ , gauge invariance requires that a unit bit of string be created pointing from  $j$  to  $j+1$ . This is the operator  $U(\ell_{j, j+1})$  which transforms like a field with charge +1 at  $j$  and -1 at  $j+1$ . This process is illustrated as follows in Fig. 9. The simplest gauge invariant expression expressing this transition is

$$\bar{\psi}_j U(\ell_{j, j+1}) \psi_{j+1} \tag{2.18}$$

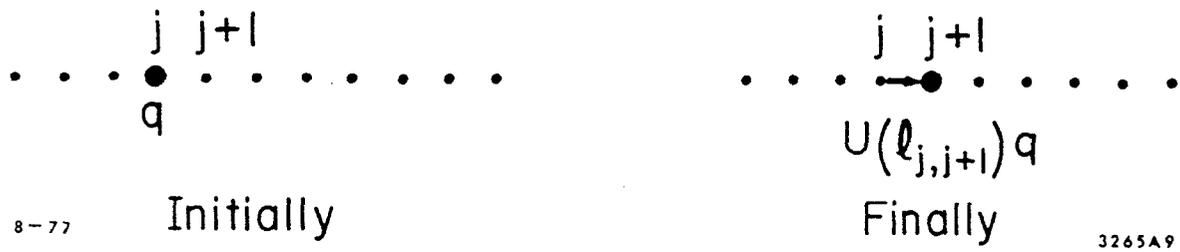


Fig. 9

if under the local gauge transformation,

$$\bar{\psi}_j \rightarrow \bar{\psi}_j e^{-ie\chi(j)} \quad (2.19)$$

$$\psi_{j+1} \rightarrow e^{ie\chi(j+1)} \psi_{j+1}$$

we also stipulate

$$U(\ell_{j,j+1}) \rightarrow e^{ie\chi(j)} U(\ell_{j,j+1}) e^{-ie\chi(j+1)} \quad (2.20)$$

Thus the form of  $U$  such that (2.17) carries over to the lattice for nearest neighbor points is

$$U(\ell_{j,j+1}) = e^{ieaA(\ell_{j,j+1})} \quad (2.21)$$

$$\text{with } A(\ell_{j,j+1}) \rightarrow A(\ell_{j,j+1}) + \frac{1}{a} \{ \chi(j) - \chi(j+1) \} \quad (2.22)$$

under a gauge transformation. Equation (2.21) is the lattice form of the exponential of a line integral from  $j$  to  $j+1$ . To transport charge over a succession of links we write the ordered product

$$U(j_1, j_2) = \prod_{j_1 < \ell < j_2} U(\ell) \quad (2.23)$$

and observe that with the convention

$$A(\ell_{1,2}) = -A(\ell_{2,1}) = -A(-\ell_{1,2})$$

we have

$$U(\ell_{1,2}) = U^+(\ell_{2,1}) = U^+(-\ell_{1,2}) \quad (2.24)$$

We can now write the gauge invariant extension of (2.16) to the lattice as

$$\left[ \sum_{j_1 < j_2} \psi_{j_1}^+ \alpha \left\{ \frac{i\delta'(j_1 - j_2)}{a} \right\} U(j_1, j_2) \psi_{j_2} + \sum_{j_1 > j_2} \psi_{j_1}^+ \alpha \left\{ \frac{i\delta'(j_1 - j_2)}{a} \right\} U^+(j_2, j_1) \psi_{j_2} \right] \quad (2.25)$$

and observe that in the continuum limit  $a \rightarrow 0$ , (2.25) reduces<sup>30</sup> to (2.17).

For a non-abelian gauge theory the analogue to (2.21) is

$$U(\vec{\ell}) = e^{ig \sum_{\alpha} \lambda_{\alpha} \vec{A}_{\alpha}(\ell) \cdot \vec{\ell}} \quad (2.26)$$

where the  $\lambda_{\alpha}$  are the c-number matrices belonging to a specific  $(\overline{N}N)$  representation of the gauge group, par ex. the 8 hermitian generators of  $SU(3)$ . The ordered product construction of (2.23) applies also in this case. In the lattice formulation the unitary operators (2.21) and (2.26) defined on lattice links (as illustrated in Fig. 9) replace the gauge field  $\vec{A}_{\alpha}$  appearing in (2.16) in the continuum theory.

Local gauge invariance requires that new string bits be created for each increased unit of separation of a particle  $q$  and anti-particle  $\bar{q}$  on the lattice. If each string bit has a finite mass (corresponding to finite field energy) the mass of a gauge invariant state increases without bound as the separation between  $q$  and  $\bar{q}$  grows; this behavior corresponds to confinement. What we must establish is that the lattice gauge theory we have formulated when applied to weak coupling abelian QED in 3x-1t dimensions does not confine; but does confine for non-abelian QCD and strong coupling.

Nothing like this has been accomplished yet. Calculations so far by Wilson<sup>17</sup> and by Kogut and Susskind and collaborators<sup>18</sup> have been iterative strong coupling treatments and contain approximations that make it impossible to judge the content of the theory independent of the validity of the calculational techniques. The problem at this time is to develop calculational methods that one can trust so that the result of calculating mass spectra in QCD will equally well destroy the theory if they fail to reproduce the observed spectra as they will confirm it if they do! No weaseling out! Wilson's 1975

result from his strong coupling expansion is a nucleon mass of 1720 MeV with a large lattice spacing of  $1/5 \times 10^{-13}$  cm. The Padé extension of strong coupling calculations by Kogut, Susskind and collaborators gives a  $\pi/N$  mass ratio of .82 instead of .15 and contains difficulties related to the use of (2.7) for the gradient together with a compensating point splitting prescription to avoid the difficulty in (2.11). These efforts and our local ones<sup>19</sup> are proceeding--it is slow and difficult work. Neither simplifying extreme of very weak or strong coupling approximations is adequate. I wish here to describe the methods we are exploiting and validating by solving simple models with known exact properties against which we can improve our understanding and approximations. We believe the basic problem that must be solved first is that of developing simple, understandable, intuitive techniques of calculation with which to arm oneself before undertaking the assault on the real physics problem.

#### TRANSVERSE ISING MODEL

The transverse Ising model in 1-space, 1-time dimension describes an array of spins with nearest neighbor interactions in the presence of a constant transverse magnetic field and is an interesting example on which to test methods for two reasons:

- 1) There are a finite number of states for the spin degree of freedom (up or down) at each lattice site in common with spin 1/2 quark theories (in 1-x dimension there are four states at each site for each color of quark: 0, q,  $\bar{q}$  and  $q\bar{q}$ ).
- 2) The exact solution of this model is known<sup>31</sup> and calculations can be compared with exactly computed critical indices and temperatures of the known phase transition.

The Ising model Hamiltonian represents an approximation to the  $\phi^4$  field theory in the 1x-1t dimension if we are far into the spontaneously broken symmetry region with strong coupling. To see this we write this theory on the lattice in terms of dimensionless canonical variables, and using the nearest neighbor gradient<sup>32</sup>

$$\frac{1}{\Lambda} H = \sum_{j=-N}^N \left[ \frac{1}{2} p_j^2 + \lambda_0 (x_j^2 - f_0^2)^2 \right] + \sum_{j=-N}^{N-1} \frac{1}{2} (x_{j+1} - x_j)^2$$

$$x_j \equiv \psi_j$$

$$p_j \equiv \frac{1}{\Lambda} \pi_j$$

$$[x_j, p_{j'}] = i \delta_{jj'}$$
(2.27)

The lowest two eigenlevels of the single-site Schrödinger problem (neglecting the coupling between two neighboring sites  $j$  and  $j + 1$  in the gradient term) lie deep in the potential well if the zero point energy is very small compared with the height of the center bump; i.e.,

$$\lambda_0^{\frac{1}{2}} f_0 \ll \lambda_0 f_0^4$$
(2.28)

These two low lying levels are, respectively, symmetric and antisymmetric under reflection as shown in Fig. 10. The energy gap between them is proportional to the tunneling between the two minima in the double-bottomed potential  $\lambda_0 (x_j^2 - f_0^2)^2$  at  $\pm f_0$ . Since (2.28) means that there is very little tunneling this gap is very small—i.e.,

$$\Delta \epsilon_{\text{gap}} \sim \lambda_0^{\frac{1}{2}} f_0 e^{-\lambda_0^{\frac{1}{2}} f_0^3} \ll \lambda_0^{\frac{1}{2}} f_0$$
(2.29)

if  $\lambda_0^{\frac{1}{2}} f_0^3 \gg 1$ . When conditions (2.28) and (2.29) are satisfied we can neglect higher excitations at each lattice site. The two states retained correspond to the spin down and up configurations in the Ising model. The gradient term

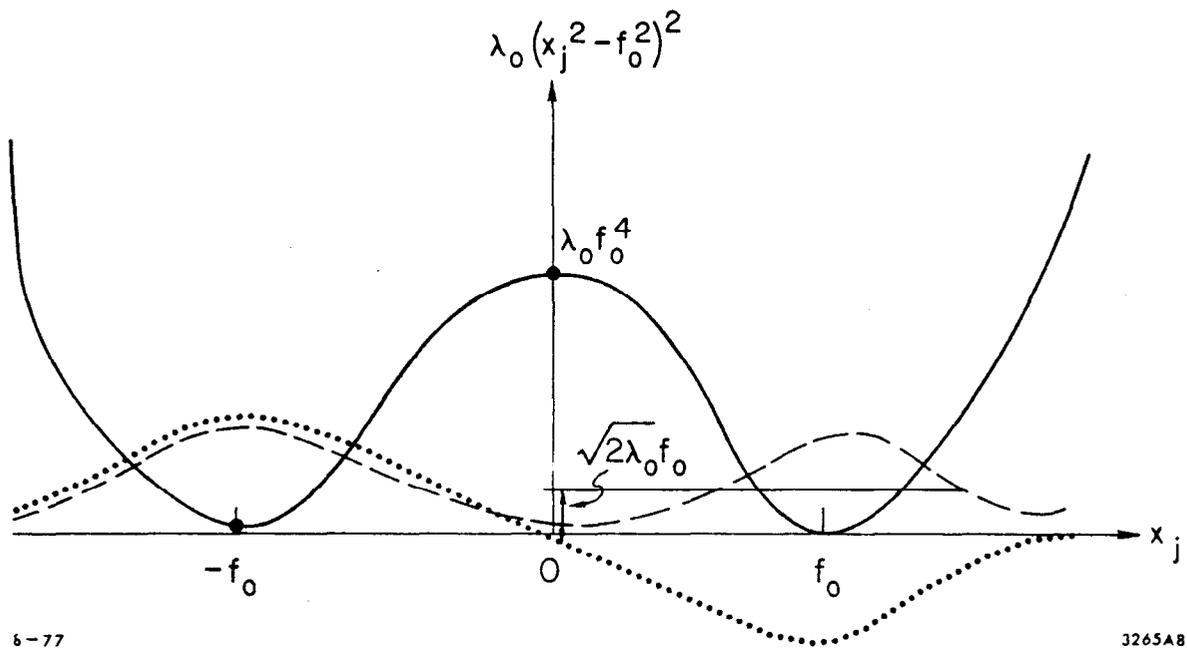


Fig. 10

induces mixing between the symmetric and antisymmetric solutions which is approximately given by

$$\langle \text{sym} | x_j | \text{antisym} \rangle^2 \approx f_0^2 \equiv \Delta_0 \quad (2.30)$$

When this mixing is comparable to the gap separating the levels—i.e., for

$$\lambda_0^{1/2} f_0 e^{-\lambda_0^{1/2} f_0^3} \approx \Delta_0 = f_0^2 \quad (2.31)$$

the gradient term is comparable to the single site terms and we can make neither a weak nor strong coupling limiting approximation. Condition (2.31) requires  $f_0^2 \ll 1$ ,  $\lambda_0 \gg 1$ , consistent with  $\lambda_0^{1/2} f_0^3 \gg 1$ .

The explicit form of the Hamiltonian in this approximation can be written in terms of Pauli matrices

$$\frac{1}{\Lambda} H = \sum_{j=-N}^N \left\{ \frac{1}{2} \epsilon_0 \sigma_z(j) \right\} - \sum_{j=-N}^{N-1} \Delta_0 \sigma_x(j) \sigma_x(j+1) \quad (2.32)$$

which is just the transverse Ising model. Looking back at (2.27) we see the value of configuration space methods in studying strong coupling problems in field theory. For large  $\lambda_0$  the character of the solutions is determined by the potential term and so it makes sense to start in a representation or language that diagonalizes this part of the Hamiltonian. This is what we do in configuration space by expressing H as a sum of single site Schrödinger problems, with the "relatively" small gradient term providing the coupling between different sites. Such couplings can be treated iteratively in this limit. The spectrum of eigenstates at each site is identical. The eigenstates of H are then characterized by specifying the different individual levels of excitation populated at each site. Barring additional degeneracies arising for specific values of  $\lambda_0$  and  $f_0^2$ , the first excited state will be

$(2N + 1)$ -fold degenerate corresponding to having the excited level at any one of the lattice sites. When the gradient terms are included in  $H$ , their effect is to lift this degeneracy; they also mix these states in general with the ground state and with the more-highly-excited spectrum. It is when these gradient-induced splittings are small relative to the spacing between the single-site excited states that the site basis is expected to provide a reasonable picture of the true ground state. In contrast for weak couplings, we want to treat the gradient exactly and this we accomplish by working in momentum space in terms of which the gradient is diagonal and a propagator formalism is then useful for iterating the relatively weak potential terms.

Important features of the solution to (2.32) can be learned from studying its limiting behavior for  $\epsilon_0/\Delta_0 \rightarrow \infty$  (strong coupling) and  $\epsilon_0/\Delta_0 \rightarrow 0$  (weak coupling). In the strong coupling limit,  $\epsilon_0/\Delta_0 \rightarrow \infty$ , (2.32) describes an assembly of noninteracting spins that all line up with spin down in the nondegenerate ground state

$$|0\rangle = \prod_j \begin{pmatrix} 0 \\ 1 \end{pmatrix}_j \quad (2.33)$$

of energy density (in units of  $\Lambda$ )  $E_0(\Delta_0/\epsilon_0 \rightarrow 0) = -\epsilon_0/2$ . The particle-like excitations lie  $+\epsilon_0$  above the ground state for each site excited to the spinup configuration,  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}_j$ . In the opposite, or weak coupling extreme,  $\epsilon_0/\Delta_0 \rightarrow 0$ , the eigenstates

$$|\Rightarrow\rangle_j \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}_j \quad (2.34)$$

and

$$|\Leftarrow\rangle \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}_j \quad (2.35)$$

diagonalize the Hamiltonian. The ground state is doubly degenerate, being formed as a product of states (2.34) at each site, or all states (2.35) at each site. For each "wall" between two adjacent sites, one formed as (2.34) and the other reversed as (2.35), there is an excitation of  $+2\Delta_0$  units of energy. In this extreme the excitations are kink-like as illustrated by Fig. 11. These low lying excitations in the weak coupling limit correspond to collective "kink" states rather than single particle excitations.

From a study of the exact H, (2.32), it is known<sup>31</sup> that a second order phase transition occurs between the nondegenerate ground state (2.33) and the degenerate configurations (2.34) and (2.35). The transition occurs when  $\epsilon_0 = 2\Delta_0$ . The behavior of the order parameter, or "magnetization," in this model is given by

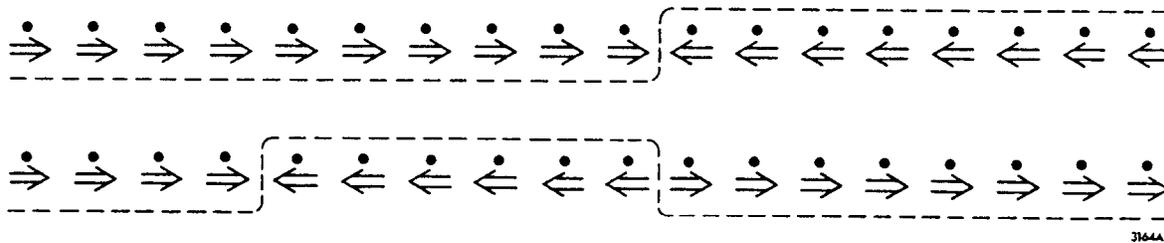
$$\langle \sigma_x \rangle = \begin{cases} \left[ 1 - (\epsilon_0/2\Delta_0)^2 \right]^{1/8} & \text{for } \frac{\epsilon_0}{2\Delta_0} \lesssim 1 \\ 0 & \text{for } \frac{\epsilon_0}{2\Delta_0} > 1 \end{cases} \quad (2.36)$$

It is these exact results that we must reproduce by a simple and direct calculational procedure that lends itself to generalization to 4-dimensional gauge theories. Our approach is as follows:

- 1) Divide the lattice into small blocks containing several adjacent sites, as illustrated in Fig. 12 for two-site blocks. Within each block diagonalize the two site Hamiltonian; viz. for (2.32)

$$H_{(1)} = \frac{\epsilon_0}{2} \left[ \sigma_z(1) + \sigma_z(2) \right] - \Delta_0 \sigma_x(1) \sigma_x(2) \quad (2.37)$$

which has  $2^2 = 4$  eigenstates<sup>33</sup> that can be determined exactly.



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Fig. 11

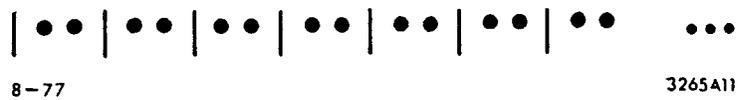


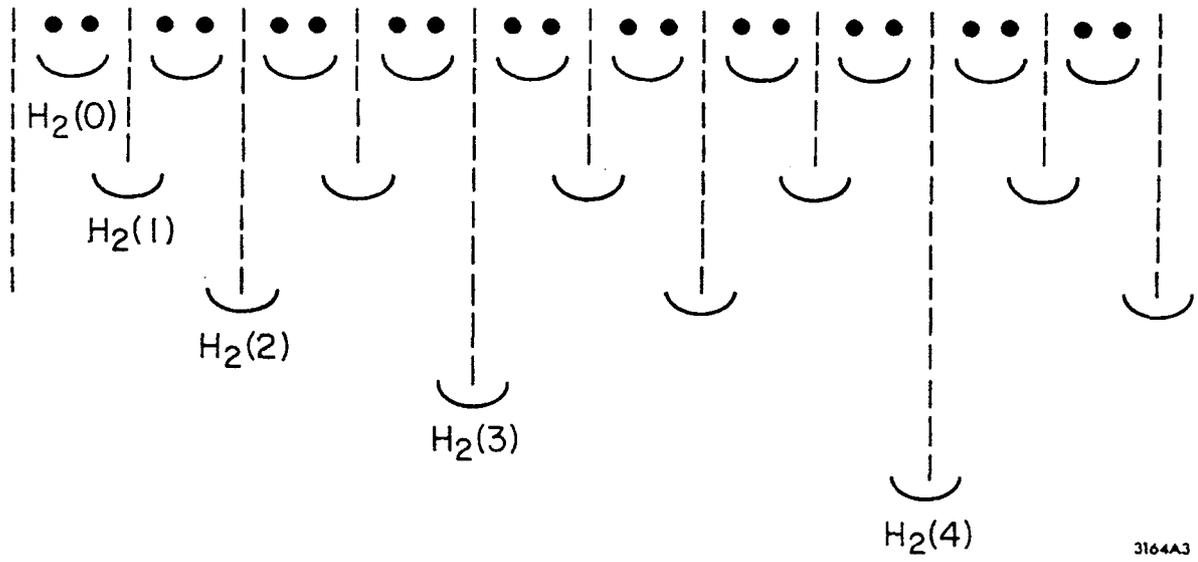
Fig. 12

2) "Thin" the degrees of freedom by a truncation procedure which amounts simply to keeping a suitable subset including the lowest lying of these eigenstates and discarding the higher excitations. This is equivalent to a variational solution using a trial form that spans a subset of the full Hilbert space and constructing an effective truncated Hamiltonian. Our simplest algorithm for (2.37) is simply to keep the two lowest of its four eigenstates. Part of the motivation of our work with this model is to learn by experience the ingredients of accurate algorithms.

3) We then iterate this procedure, including thereby more terms in the gradient that couple different lattice sites by forming neighboring blocks: two neighboring blocks into superblocks in the example in Fig. 13. The two retained eigenstates from the original blocking form a block basis in terms of which to express the Hamiltonian. In this simple example we again find a spin form since there are 2 eigenstates at each block site; but the effective coupling strengths, block spacings, and energy intervals are renormalized.

4) We repeat the same steps successively, continually eliminating the higher excitations. We can think of this procedure as successively eliminating higher momentum states in constructing the series of truncated Hamiltonians to describe the physics of the low momentum states alone.

5) The iterative steps are continued until the successive rescaling leads to a soluble problem, either in the weak or strong coupling regime, for the treatment of the remaining coupling terms



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Fig. 13

between different blocks contained in the gradient term.

Applying this procedure to (2.37) we note that within one block of two adjacent sites there are four independent states which we denote by  $|\uparrow\uparrow\rangle$ ,  $|\uparrow\downarrow\rangle$ ,  $|\downarrow\uparrow\rangle$ , and  $|\downarrow\downarrow\rangle$ , where  $|\uparrow\uparrow\rangle \equiv |\uparrow\rangle_1 |\uparrow\rangle_2$ , etc. The problem of diagonalizing the 2-site Hamiltonian reduces simply to one of diagonalizing two  $2 \times 2$  matrices, since  $|\downarrow\downarrow\rangle$  mixes only with  $|\uparrow\uparrow\rangle$ , and  $|\uparrow\downarrow\rangle$  with  $|\downarrow\uparrow\rangle$ . The eigenstates and eigenvalues are simply found and are given in Table II. Step (i) of our general procedure will be to choose this set of four eigenstates as the new orthonormal system which we will use to construct a basis for H. Step (ii), the thinning out procedure, is simply accomplished by retaining only the two lowest energy states in Table II for each box when we add back the terms linking different boxes in (2.37). It is reasonable to expect that the most important part of the true ground state will be in the subspace spanned by these two states in each box. In order to implement this approximation we need only construct the truncated or effective Hamiltonian for this choice of trial states and see if we can solve it.

To compute  $H^{(tr)}$  we label each 2-site box by an integer 'p' and divide the Hamiltonian into two parts,  $H_1$  and  $H_2$ .  $H_1$  contains only those terms in (2.37) which refer to single boxes and  $H_2$  contains the remaining interaction terms in (2.37) which couple sites in adjacent boxes; i.e.,

$$H_2 = -\Delta_0 \sum_p \sigma_x(p,1) \sigma_x(p+1,0) \quad (2.38)$$

where  $\sigma_x(p,\alpha)$  operates on the spin in box p and at site  $\alpha = 0, 1$  within each box. In keeping with our approximation of retaining only the two lowest states in each box, the truncated  $H_1^{(tr)}$  can be written as a sum of  $2 \times 2$  matrices operating on the two states we keep for each box. In particular

Table II

State	Energy	Energy Relative to Lowest State
$\frac{1}{\sqrt{1+a_0^2}}( \uparrow\uparrow\rangle + a_0 \uparrow\downarrow\rangle)^* \equiv  \psi_0\rangle$	$-\sqrt{\epsilon_0^2 + \Delta_0^2}$	0
$\frac{1}{\sqrt{2}}( \uparrow\uparrow\rangle +  \uparrow\downarrow\rangle) \equiv  \psi_1\rangle$	$-\Delta_0$	$\sqrt{\epsilon_0^2 + \Delta_0^2} - \Delta_0$
$\frac{1}{\sqrt{2}}( \uparrow\uparrow\rangle -  \uparrow\downarrow\rangle)$	$+\Delta_0$	$\sqrt{\epsilon_0^2 + \Delta_0^2} + \Delta_0$
$\frac{1}{\sqrt{1+a_0^2}}(-a_0 \uparrow\uparrow\rangle +  \uparrow\downarrow\rangle)$	$+\sqrt{\epsilon_0^2 + \Delta_0^2}$	$2\sqrt{\epsilon_0^2 + \Delta_0^2}$

$*a_0 = (\sqrt{\epsilon_0^2 + \Delta_0^2} - \epsilon_0) / \Delta_0$ .

referring to Table II we see that  $H_1^{(tr)}$  can be written as

$$\begin{aligned}
 H_1^{(tr)} &= \sum_p \begin{pmatrix} -\Delta_0 & 0 \\ 0 & -\sqrt{\epsilon_0^2 + \Delta_0^2} \end{pmatrix}_{(p)} \\
 &= \sum_p \left\{ -\frac{1}{2} \left[ \Delta_0 + \sqrt{\epsilon_0^2 + \Delta_0^2} \right] \mathbb{1}_{(p)} + \frac{1}{2} \left[ \sqrt{\epsilon_0^2 + \Delta_0^2} - \Delta_0 \right] \sigma_z(p) \right\} \quad (2.39)
 \end{aligned}$$

The eigenstates of (2.39) can be written as products over boxes of the two lowest eigenstates in Table II; i.e.,

$$|\Psi_1\rangle = \prod_{\text{boxes } p} |\psi(p)\rangle \quad (2.40)$$

Hence the interaction (2.38) can now be re-expressed in terms of the truncated basis (2.40) by evaluating its matrix elements for flipping one "spin" in each of two adjacent boxes. To compute this we take the matrix element

of  $\sigma_x(p,1)$  between the states

$$|\psi_0(p)\rangle = \frac{1}{\sqrt{1+a_0^2}} \left\{ |\uparrow\uparrow\rangle + a_0 |\uparrow\downarrow\rangle \right\} \quad (2.41)$$

and

$$|\psi_1(p)\rangle = \frac{1}{\sqrt{2}} \left\{ |\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle \right\} \quad (2.42)$$

The actual computation is quite trivial:

$$\sigma_x(p,1) |\psi_0(p)\rangle = \frac{1}{\sqrt{1+a_0^2}} \left[ |\uparrow\uparrow\rangle + a_0 |\uparrow\downarrow\rangle \right]_p \quad (2.43)$$

and so

$$\langle \psi_1(p) | \sigma_x(p,1) | \psi_0(p) \rangle = \frac{1+a_0}{\sqrt{2(1+a_0^2)}} \quad (2.44)$$

Similarly

$$\langle \psi_1(p+1) | \sigma_x(p+1,0) | \psi_0(p+1) \rangle = \frac{1+a_0}{\sqrt{2(1+a_0^2)}} \quad (2.45)$$

It follows from this that for 'j' in the  $p^{\text{th}}$  box, and for both cases  $\alpha = 0$

and 1,

$$\sigma_x^{(\text{tr})}(j) = \frac{1+a_0}{\sqrt{2(1+a_0^2)}} \sigma_x(p) \quad (2.46)$$

$$\sigma_x^{(\text{tr})}(j+1) = \frac{1+a_0}{\sqrt{2(1+a_0^2)}} \sigma_x(p+1)$$

We can now rewrite our effective Hamiltonian for the two site box. Since our truncation procedure retained just two states per box we again have a spin form for the truncated Hamiltonian.  $H^{(\text{tr})}$  has exactly the same form as the original Hamiltonian but different coefficients:

$$H^{(\text{tr})} = \sum_P \left[ c_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_P + \frac{1}{2} \varepsilon_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_P - \Delta_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_P \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{P+1} \right] \quad (2.47)$$

where

$$c_1 = -\frac{1}{2} \left( \Delta_0 + \sqrt{\varepsilon_0^2 + \Delta_0^2} \right)$$

$$\varepsilon_1 = \sqrt{\varepsilon_0^2 + \Delta_0^2} - \Delta_0 \quad (2.48)$$

and

$$\Delta_1 = \frac{\Delta_0(1 + a_0)^2}{2(1 + a_0^2)}$$

At this point we face one of two possibilities. Either the values of  $\varepsilon_1$  and  $\Delta_1$  are such that we can treat the resulting effective Hamiltonian,  $H^{(\text{tr})}(1)$  by perturbation theory for  $\varepsilon_1/\Delta_1 > 1$  or  $\varepsilon_1/\Delta_1 < 1$ ; or, we may repeat the same procedure that we just went through, but this time combining neighboring pairs of blocks  $p$  in the Hamiltonian  $H^{(\text{tr})}$  and thereby including additional interaction terms in a new basis to which we again apply the same state-thinning steps as in (2.38) to (2.47). One readily sees in the comparison of (2.47) with the original (2.37) that each successive restriction of our class of trial wave functions by this procedure leads us to a new effective Hamiltonian of the same form as the original Hamiltonian, and with the coefficients of the effective Hamiltonian given by (2.48) in terms of the coefficients found in the preceding step of the calculation.

The details of this procedure and its systematic improvement with more sophisticated and general, but nevertheless very simple, truncation steps are discussed in Ref. (19c). For these lectures I will simply summarize the more salient features: We define the ratio after  $n$  iterations

$$y_n \equiv (\epsilon/\Delta)_n$$

and study

$$R(y_n) \equiv y_{n+1} - y_n \tag{2.49}$$

in order to see whether with successive iterations this ratio decreases, driving us to the weak coupling or gradient-dominant limit, or whether it increases driving us to the strong coupling limit.  $R(y)$  is plotted schematically in Fig. 14. This figure shows that there are three fixed points in our procedure of generating new effective Hamiltonians in successive steps (called a "renormalization group transformation" by Wilson and Kadanoff); namely, at  $y = 0$ ,  $y = \infty$ , and  $y_c = 2.55\dots$ . At fixed points, the ratio  $y \equiv \epsilon/\Delta$  does not change as the iteration process is continued and so  $H^{\text{tr}}$  changes only by an overall scale. When  $R(y) = 0$  we have a fixed point. There is also a fixed point if  $y = \infty$  and  $R(\infty) > 0$  so that this value cannot be reduced. Whenever  $R(y) < 0$  the ratio  $(\epsilon/\Delta) = y$  decreases for that iteration and so the new  $(\epsilon'/\Delta')$  lies to the left of the  $y$  we started with. Since, as shown in Fig. 14,  $R(y)$  is negative for all  $y < y_c = 2.55$  we see that if we start at any point in this range, successive iterations of our truncation procedure will drive us to a form for the effective Hamiltonian which is the weak coupling perturbation theory limit. On the other hand, for  $y > y_c$  successive iterations drive us to  $y = \infty$  since, in this case,  $R(y) > 0$ . This implies  $\epsilon/\Delta \gg 1$  which is the strong coupling limit of the Hamiltonian. Hence those theories described by (2.37) for which the initial  $y < y_c$  are theories with a degenerate ground state and spontaneously broken symmetry as described earlier. On the other hand, for  $y > y_c$  we have a unique ground state. Clearly  $y_c$  is the point at which the nature of the ground state changes, and so  $y_c$  is the critical point of this theory.

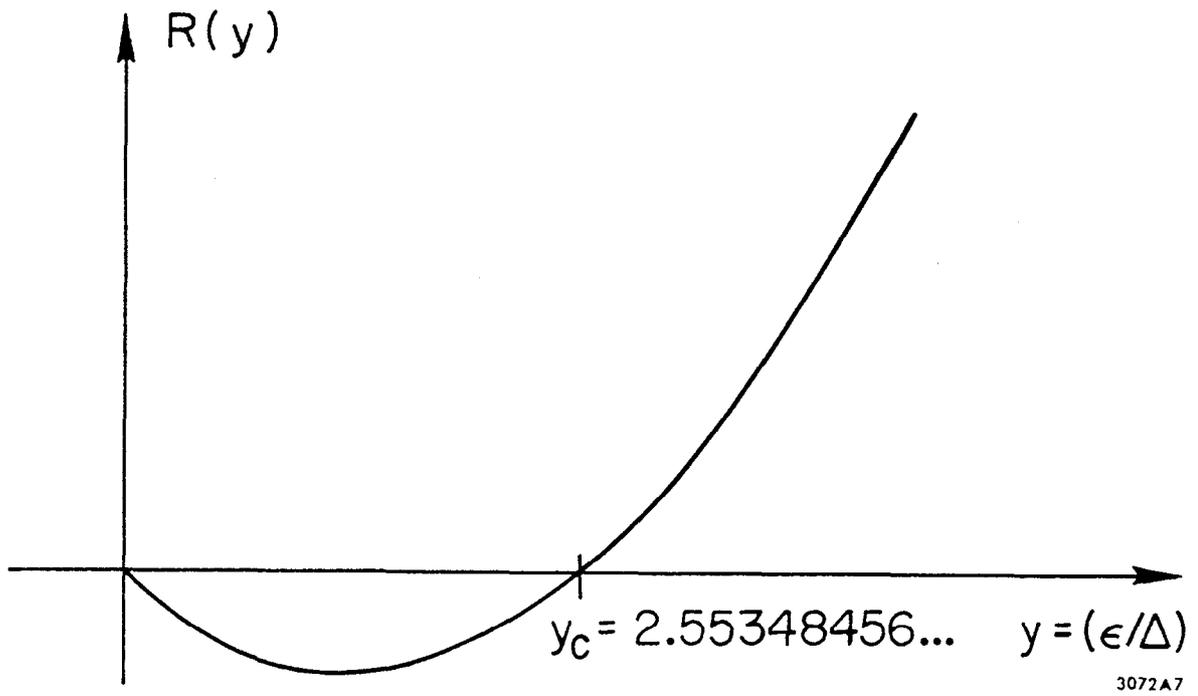
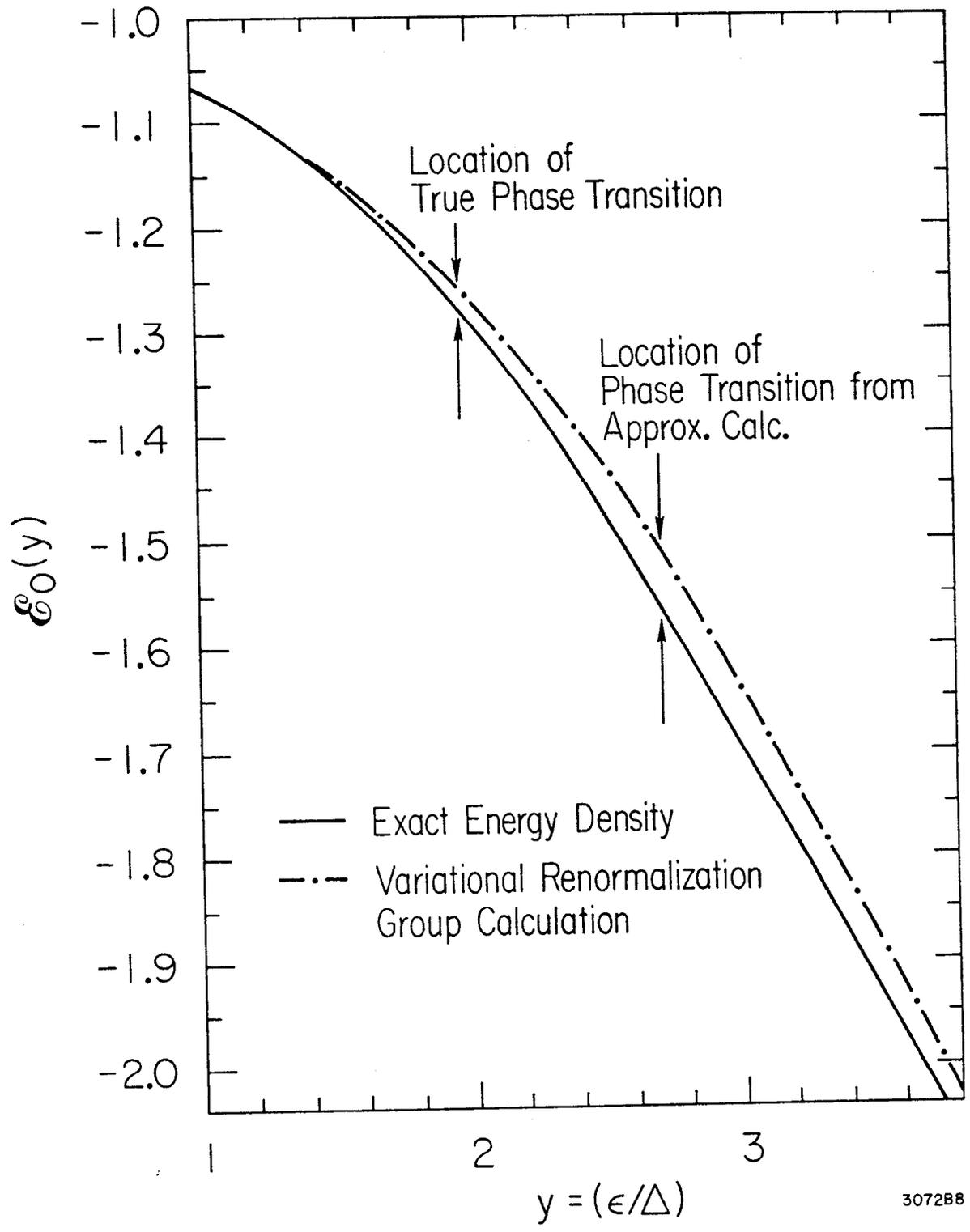


Fig. 14

The result  $y_c = 2.55 \dots$  which is obtained from our simple procedure is not far from the exact transition point  $y_c^{\text{exact}} = 2$  discussed above (2.36). The fixed points  $y = 0$  and  $y = \infty$  are the stable fixed points of this renormalization group transformation, and the fixed point at  $y = y_c$  is an unstable fixed point. The fact that at  $y = y_c$  the Hamiltonian continues to reproduce itself up to a scale factor says that at this critical point the physics going on at different length scales is essentially the same.

One can further discuss the order parameter (2.36) which we find with the above procedure to have a critical index of 0.39 instead of the exact result of 1/8. The simple modification of replacing  $a_0$ , as defined in Table II by our truncation procedure, by a variational parameter  $a(\epsilon/\Delta)$  determined at the end of the iteration to minimize the energy reduces the calculated value to 1/5. The comparison of the ground state energy density computed this way and the exact energy is shown in Fig. 15. The worst disagreement is 3%. This can be readily reduced further.<sup>34</sup> Discontinuities in the ground state energies are also well reproduced by this procedure as described elsewhere.<sup>19c</sup>

Abstracting from this model we see that study of  $R(y)$ , which gives the change in the ratio of the single site potential terms in the theory to the gradient terms coupling neighboring sites, teaches a great deal about what we want to learn from the study of gauge theories and confinement. Let us consider such theories with only one coupling constant--i.e., one single site potential term in the theory--which is the strength of the coupling between the quarks and gluons. Then with  $y$  defined as above as the ratio of the strength of the single-site coupling to the gradient term, we can plot the general form of the function  $R(y) = (\text{change of } y \text{ in finite number of iterations})$

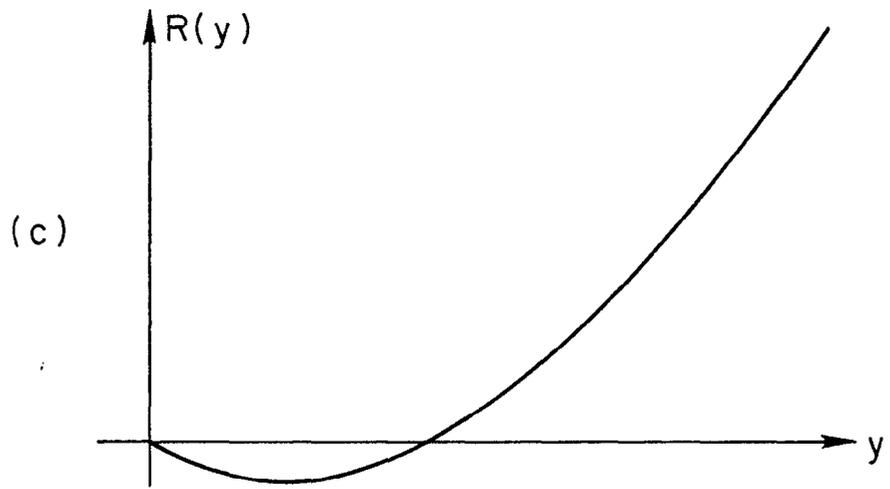
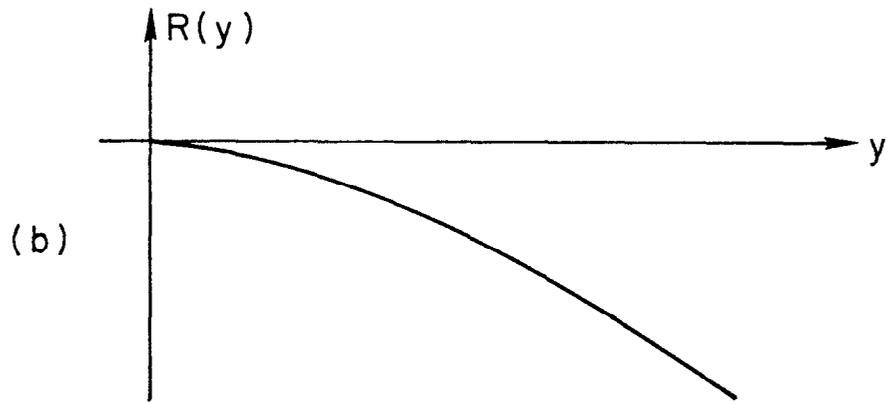
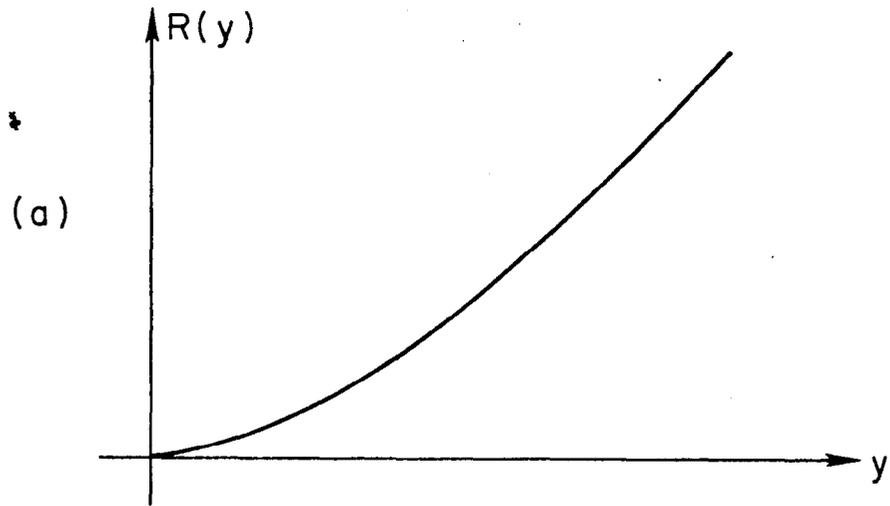


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Fig. 15

as in (2.49). A few examples of simple forms for  $R(y)$  are given in Figs. 16a-16c and lead to different conclusions about the theories they are assumed to characterize.

In Fig. 16a we see that  $R(y) \geq 0$  for all values of  $0 \leq y \leq \infty$ . If a theory has this form for  $R(y)$  we can conclude two things. First, the points  $y = 0$  and  $y = \infty$  are the only fixed points of the theory. The Hamiltonian at  $y = 0$ , i.e., zero coupling constant, is a "free field theory," and can presumably be solved exactly. The  $y = \infty$  Hamiltonian becomes the single site Schrödinger problem with neglect of the gradient terms. Second, we observe that if we start at some finite value of  $y$  successive iterations drive us to larger value of  $y$ ; i.e.,  $R(y) > 0$ . Eventually after a finite number of iterations our problem can be studied by treating the gradient terms as a perturbation on the single site terms. Hence, in any theory for which  $R(y) > 0$  we can conclude that the low energy (or long-wavelength) physics is described by an effectively strong-coupling constant Hamiltonian. It follows from this discussion that the mass gap in such a theory will be given by calculating the gap between the first two eigenstates of the effective single site Schrödinger problem. The gap is thus a function of the effective single site coupling  $g_\infty$ , where the subscript denotes the many iterations  $N \gg 1$  to reach the strong coupling behavior. In general, since the scale of  $H$  is set by the cutoff  $\Lambda$ , this means that the lowest mass gap in the theory will be  $\sim \Lambda g_\infty$ . However, the scale of physical masses should be negligible with respect to the maximum momentum  $\Lambda$  if we are to retain practical Lorentz invariance for the low-lying eigenstates in spite of our cutoff procedure. Therefore we are only interested in theories for which  $g_\infty \ll 1$ , or in other words,  $g_\infty \Lambda$  finite (and perhaps  $\sim 1$  GeV).



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Fig. 16

Generally the Hamiltonian at a fixed point reproduces itself up to a scale factor  $\rho$ , and after  $N$  iterations the overall scale of  $H_N$  is  $\Lambda \rho^N$ . Since this should be finite ( $\sim 1$  GeV) this suggests that the question of the practical relativistic invariance of a theory for which  $R(y)$  behaves as in Fig. 16a can be settled by computing the scale parameter  $\rho$  in the  $y = 0$  limit. If we find  $\rho < 1$  then we can take the cutoff  $\Lambda \rightarrow \infty$  and still keep the masses of the lowest states finite if we choose the original bare coupling constant  $g_0$  to tend appropriately to zero as a function of increasing  $\Lambda$ . This is an example of a theory whose short distance behavior is "free" but whose long wavelength behavior is not. Such a theory could describe asymptotically free quarks for high momenta and also give confinement.

If we next look at  $R(y)$  for Fig. 16b we come up with the opposite conclusion. If  $R(y) < 0$  each successive set of  $N$ -iterations will make it smaller. Hence the large wavelength or low energy physics of this theory is given by weak coupling perturbation theory, whereas the single-site or short distance behavior is governed by a strong coupling constant.

Fig. 16c tells us that the two different cases can occur depending upon the starting value for  $y$ , i.e., whether  $y_0 < y_c$  or  $y_0 > y_c$ . This is just the form of  $R(y)$  calculated for our Ising model in Fig. 14 and one can refer back to the exact solution of this theory<sup>31</sup> to see how an effectively relativistic theory emerges.

The use of the function  $R(y)$  to catalogue types of theories has its analogue in the study of the renormalization group equations in momentum space, where one encounters the well known  $\beta(g)$  function in terms of which the asymptotic behaviors of field theories are described. Both functions,  $\beta(g)$  and  $R(y)$ , describe the change in coupling constant ( $g$  or  $y$ ) as we change

the scale of distance in the theory. The two functions are complementary to one another in that we have introduced  $R(y)$  here in coordinate space, whereas  $\beta(g)$  normally appears in the momentum space analysis of the renormalization group equations. In our renormalization group procedure on a lattice we build larger and larger blocks at each state of the calculation so that we are studying the behavior of the theory at lower and lower momenta. When working in momentum space one normally studies the renormalization group equations by scaling up the momenta to higher and higher values at each stage, and correspondingly to smaller and smaller values of the underlying lattice spacing. In our approach Fig. 16a describes a theory which is asymptotically free (high momenta) and Fig. 16b describes one that is infrared stable. The  $\beta$  function has just the complementary behavior as illustrated by Fig. 17 for asymptotically free and infrared stable theories.

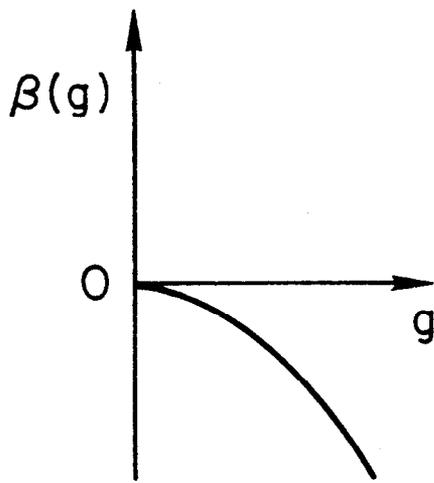
#### THIRRING MODEL

The Thirring model<sup>35</sup> of quartically self-coupled fermions, described by the Hamiltonian

$$H = \int dx \left[ -\bar{\psi} i \not{\partial} \psi - \frac{1}{2} g \left\{ (\bar{\psi} \psi)^2 - (\bar{\psi} \gamma_5 \psi)^2 \right\} \right] \quad (2.50)$$

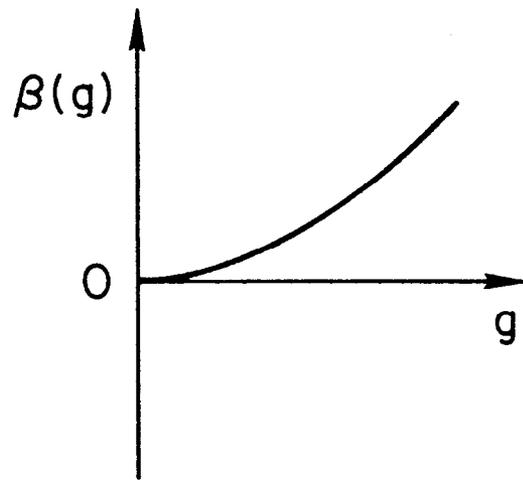
is a more interesting one for applying these methods for several important reasons:

- 1) It deals with fermions (à la quarks).
- 2) It is essential to use the definition (2.13) for the lattice gradient which couples distant lattice sites in order to avoid the deficiency in (2.11) of doubling the number of states as illustrated in Fig. 8 and at the same time preserve chiral symmetry.
- 3) In the continuum Thirring model the wave function renormalization  $Z_2$  vanishes when the strength of the coupling  $g$  exceeds a



5-77

(a)



(b)

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Fig. 17

finite critical value and it is important to reproduce this behavior indicating failure of the multiplicative renormalization procedure and the concomitant loss of the single charged fermion (and generally charged degrees of freedom) from the finite mass spectrum. Failure of such states to propagate is a form of confinement in this model.

Introducing dimensionless field variables  $\chi_j = \Lambda^{-1/2} \psi_j$  and a convenient matrix representation that diagonalizes the quartic term in (2.50); viz.

$$\alpha = \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\chi_j = \begin{pmatrix} b_j \\ d_j^+ \end{pmatrix} \quad \{b_j, b_{j'}^+\} = \{d_j, d_{j'}^+\} = \delta_{jj'}, \quad (2.51)$$

$$\{b_j, d_{j'}\} = 0 \quad \text{etc.}$$

the Hamiltonian can be written<sup>36</sup>

$$\frac{1}{\Lambda} H = \sum_{j_1, j_2} i\delta'(j_1 - j_2) \left\{ \begin{matrix} b_{j_1}^+ & b_{j_2} \\ -d_{j_1}^+ & d_{j_2} \end{matrix} \right\}$$

$$-g \sum_j (n_b(j) + n_d(j) - 1)^2 \quad (2.52)$$

where  $n_b(j) \equiv b_j^+ b_j$  and  $n_d(j) \equiv d_j^+ d_j$  are number operators. This theory possesses two conserved "charges"

$$Q = \sum_j (n_b(j) - n_d(j)) = \sum_j :X_j^+ X_j:$$

$$Q_5 = \sum_j (n_b(j) + n_d(j) - 1) = \sum_j X_j^+ \gamma_5 X_j \quad (2.53)$$

as well as discrete P, C, and T operations for classifying states. The weak coupling limit  $g = 0$  reduces to free massless fermions, the solution to which

is most readily recognized in momentum space which diagonalizes the gradient operator. In fact with

$$b(k) \equiv \frac{1}{\sqrt{2N+1}} \sum_j b_j e^{-ikj/\Lambda}$$

$$d(k) \equiv \frac{1}{\sqrt{2N+1}} \sum_j d_j e^{-ikj/\Lambda}$$
(2.54)

we have

$$H(g=0) = \sum_{k=-\pi\Lambda}^{\pi\Lambda} k \left( b^\dagger(k)b(k) - d^\dagger(k)d(k) \right)$$

$$= \sum_{k=-k_{\max}}^{k_{\max}} k \left( n_b(k) - n_d(k) \right)$$
(2.55)

The lowest eigenstate of (2.55) corresponds to filling all  $k < 0$  states with fermions (b-quanta) and all  $k > 0$  ones with antifermions (d-quanta), i.e.

$$n_b(k) = \theta(-k)$$

$$n_d(k) = \theta(k) .$$

This leads to a doubly degenerate ground state in the neutral  $Q = 0$  sector depending on whether the  $k = 0$  state is empty ( $Q_5 = -1$ ) or occupied by a pair ( $Q_5 = +1$ ) and to two states of the same energy and with charge  $Q = \pm 1$ ,  $Q_5 = 0$  corresponding to a fermion or an antifermion present in the  $k = 0$  state. The energy of these four-fold degenerate states is

$$E_0 = -2 \sum_{k=0}^{k_{\max}} k$$

In contrast the strong coupling limiting behavior  $g \gg 1$  is found simply from the quartic term in (2.52) which can be diagonalized on each site separately. There are only four states at each site corresponding to  $n_b(j) = 1, 0$  and  $n_d(j) = 1, 0$ . These site states, their quantum numbers and eigenenergies are shown in Table III, where  $|0\rangle$  is defined by  $b_j|0(j)\rangle = d_j|0(j)\rangle = 0$ .

We see therefore that at each site the  $Q = 0$  states, corresponding to nothing or a bound pair at a site, are degenerate ground states, with the single fermion states of charge  $Q = \pm 1$  pushed up in energy by  $g\Lambda$ . Therefore any neutral state which contains an unbound pair with a fermion and an anti-fermion split to different lattice sites will lie higher in energy by  $2g\Lambda$  for each such split pair. One can of course study how the gradient term splits the  $2(2N + 1)$ -fold degeneracy of the ground states with  $Q = 0$ . However it is clear that we are seeing here a form of quark confinement in the strong coupling limit with  $g \gg 1$  since the gap to propagating single quark states is  $\sim 2g\Lambda \gg \Lambda$ , the lattice cutoff.

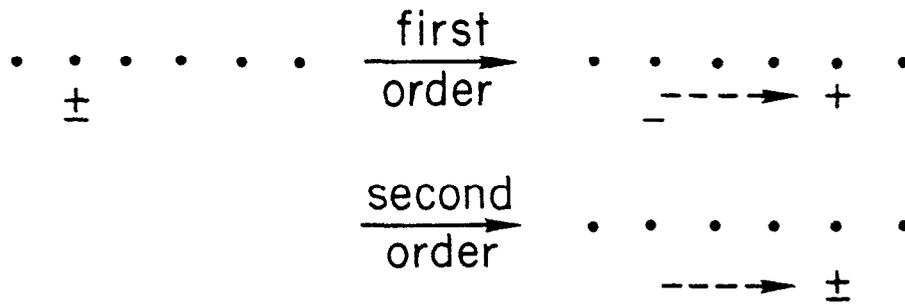
The perturbation treatment of the gradient term in the strong coupling limit has been given.<sup>19b</sup> It is clear from the form of (2.52) that the gradient moves single fermions or antifermions from one lattice site to another and gives no first order energy shift to the low lying states with  $Q = 0$  and  $n_b(j) = n_d(j)$ . In second order either a fermion and antifermion are both moved from a common initial lattice site to a common final one, as illustrated (Fig. 18a), or the (anti-) fermion is moved away and returns to its initial site (Fig. 18b).

It is not surprising that there exists a spin formalism in the sector of low lying states in which each lattice site is either empty (spin down)

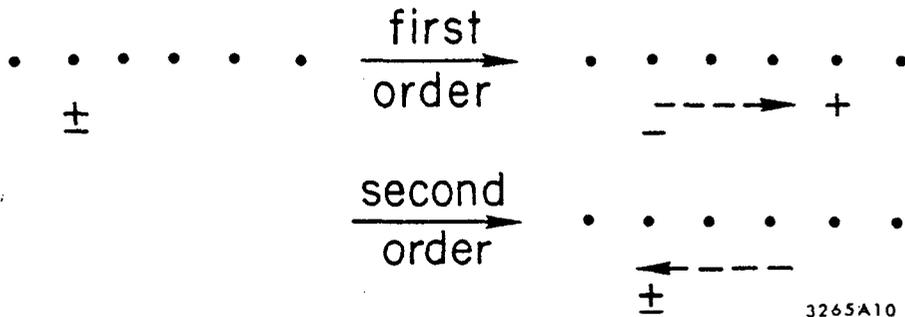
Table III

State	$Q$	$Q_5$	$-g [n_b(j) + n_d(j) - 1]^2$
$ 0\rangle$	0	-1	-g
$ +\rangle = b^+ 0\rangle$	+1	0	0
$ -\rangle = d^+ 0\rangle$	-1	0	0
$ \pm\rangle = b^+d^+ 0\rangle$	0	+1	-g

(a)



(b)



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Fig. 18

or occupied by a pair (spin up). In fact the second order treatment of the gradient gives an effective Hamiltonian in order  $1/g^2$  that is identical to the Heisenberg antiferromagnetic chain about which a great deal has been known since the work of Bethe<sup>37</sup> in 1931. In particular such a system is known to have a doubly degenerate ground state for a chain with an odd number,  $2N + 1$ , of lattice sites corresponding to the odd site being either spin up ( $S_3 = +\frac{1}{2}$ ) or down ( $S_3 = -\frac{1}{2}$ ) and to have a low lying excitation spectrum of spin waves that is linear in  $k$ , without a mass gap, for an infinitely long chain.

Our interest, as in the Ising model, is mainly in the intermediate coupling region where the Thirring model changes from free fermion to no single fermion propagation. Can we understand what is going on, as learned from the study of the continuum model, by our lattice truncation methods?

Aside from detailed technical issues associated with the fact that fermions give us two eigenstates at each lattice site for both particles and antiparticles and with the use of the gradient operator  $\delta'$  coupling distant lattice sites, (2.32) for the Ising model and (2.52) are very similar. Therefore so is the formal treatment via the lattice blocking procedure--and not surprisingly the success of these methods as well as the character of the results that are obtained. Already we have noted the strong coupling limiting behavior that charged fermions move high up to mass  $\sim g\Lambda$  and fail to propagate. In fact these methods show<sup>19d</sup> that there is a critical coupling strength such that for  $g > g_{cr} \approx 1.1$  we are driven to the strong coupling limiting behavior; whereas for  $g < g_{cr}$ , the Hamiltonian converges to the same fixed form that the free  $g = 0$  Hamiltonian iterates to. What happens is that the wave function renormalization constant defined as the amplitude to create a  $Q = 1, Q_5 = 0$  state at rest from the vacuum, i.e.,

$$\sqrt{Z_2(g)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \langle 0 | \chi(k=0) | + \rangle$$

vanishes for finite  $g \geq g_{cr}$  even in the presence of a finite cutoff  $\Lambda$ . When one discusses the continuum Thirring model it is necessary to give singular quantities in the theory well defined meanings. For example Johnson introduced a point splitting prescription

$$j^\mu(x) = \frac{1}{2} \lim_{\epsilon \rightarrow 0} \left\langle \left[ \bar{\psi}(x + \epsilon), \gamma^\mu \psi(x - \epsilon) \right] \right\rangle$$

to define the current, where the bracket indicates a suitable averaging over directions of  $\epsilon$ . This procedure also leads to a wave function renormalization  $Z_2(g)$  that vanishes at a finite value of  $g$ , similar to the lattice result. In both formulations the theory cannot be multiplicatively renormalized for  $g$  greater than some finite critical value  $g_{cr}$  and the Schwinger terms are similar.

The picture emerging from this truncation procedure is found to be consistent with the continuum model in that there is a finite critical value  $g_{cr}$  such for  $g < g_{cr}$  the theory has no mass gap; for  $g > g_{cr}$  the cutoff lattice theory cannot be multiplicatively renormalized in the usual fashion. The lattice theory still exists for  $g > g_{cr}$ ; in fact, for this region its behavior seems entirely sensible and is driven to the strong coupling limit which corresponds to a Heisenberg antiferromagnetic chain with nearest neighbor interactions only. This theory possesses a massless excitation spectrum as first found by Bethe. For  $g > g_{cr}$ , however, the "single particle" operator  $\frac{1}{\sqrt{L}} \int dx \psi^+(x)$  fails to create any finite energy states from the vacuum. In fact, the excitations of unit charge are found to lie an energy  $\sim g\Lambda$  above the ground state. This result shows that for a certain region of the parameter  $g$  the particles and low-lying excitation spectra found in finite cutoff lattice

theories are not simply related to the fundamental field introduced in the starting Lagrangian and Hamiltonian.

The breakdown of multiplicative renormalization and the concomitant loss of the charged degrees of freedom from the finite mass spectrum is interesting in that it seems to occur in several types of lattice theories for appropriate values of coupling constant. The strong coupling limit of the Schwinger model<sup>38</sup>--i.e., QED in  $1x-1t$  dimensions--is very similar to the Thirring model<sup>19b</sup> in that charged fermion states move up to high mass and fail to propagate. In particular each bit of gauge field adds an energy  $\approx g_0 \Lambda$ , where  $g_0$  is the dimensionless coupling constant and is related by  $g_0 \equiv e/\Lambda$  to the dimensional charge  $e$  as defined in (2.21) for a  $1x-1t$  dimensional gauge theory. Hence in this extreme limit each lattice site is "neutral." However the results of the blocking procedure with arbitrary coupling strength have not yet been found. Presumably if there is a qualitative difference between non-abelian gauge theories to which we look for an explanation of quark confinement, and these non-gauge models, it will be that for the gauge theories whenever  $g \neq 0$  we will lose the simple multiplicative renormalization procedure and, with it, propagating free fermion (quark) states.

This is where our program now stands. Having also shown that the iterative renormalization group techniques that I have been describing in this lecture satisfy Coleman's theorem<sup>39</sup> by not predicting false Goldstone bosons in  $1x-1t$  dimensions,<sup>40</sup> we are finally ready to tackle gauge theories in higher dimensions.